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This Course

STATICS: Force & Moment Balance
  Key skills: Vectors
  Free body diagrams
  Solving vector equations

STRENGTH of SOLIDS
  How long narrow things stretch, twist & bend.
  Concepts: strain & stress

VECTORS: Things represented with arrows.
  2 basic vectors: 1) Relative position \( \mathbf{r} \)
  2) Force \( \mathbf{F}_0 \)
  All other vectors are derived from these 2.

ADVERTISEMENT:
e.g. unit vectors (typically \( \mathbf{r}_0(\mathbf{r}_D) \)) & moment vectors

\[
\mathbf{r}_{BA} = \mathbf{r}_{AB} = \frac{4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} \cdot \mathbf{E}}{\sqrt{(4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})^2}} = \frac{4\mathbf{i} + \frac{7}{\sqrt{52}} \mathbf{j} + \frac{6}{\sqrt{52}} \mathbf{k}}{\sqrt{52}}
\]

check: \( \left| \mathbf{\hat{r}}_{AB} \right| = \sqrt{(\frac{4}{\sqrt{52}})^2 + (\frac{7}{\sqrt{52}})^2 + (\frac{6}{\sqrt{52}})^2} = 1 \)

unit vector has magnitude of 1

\( \mathbf{r}_{BA} \) reads as "r B with respect to A"  
\( \mathbf{r}_{AB} \) reads as "r A to B"

They mean the same thing: \( \mathbf{r}_{BA} = -\mathbf{r}_{AB} \)

But:  
\( \mathbf{r}_{BA} = -\mathbf{r}_{AB} \)  
\( \mathbf{r}_{AB} = -\mathbf{r}_{BA} \)
\[ F = F_\mathbf{\hat{a}} = (100N) \left( \frac{\text{tan} \theta}{\sqrt{2}} \right) \]
\[ = 100N \frac{\hat{i} + \hat{j}}{\sqrt{2}} \]

\[ F = F_\mathbf{\hat{i}} + F_\mathbf{\hat{j}} = \frac{100N}{\sqrt{2}} \hat{i} + \frac{100N}{\sqrt{2}} \hat{j} \]

\[ F_x, F_y \]

\[ \times \& \ y \ components \ of \ \mathbf{F} \]

**Vector Notation**

In equations: \( \mathbf{F}, \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}, (\mathbf{F}) \)

Unit vectors: \( \mathbf{\hat{i}}, \mathbf{\hat{j}}, \mathbf{\hat{j}}, \mathbf{\hat{i}}, \mathbf{\hat{j}}, \mathbf{\hat{i}}, \mathbf{\hat{j}} \)

In pictures: \( \mathbf{F} \)

**Lin needs to be clean for picture.**

\[ F_\mathbf{\hat{a}} = \text{unit vector in dir. of arrow drawn} \]

\[ 37^\circ \]

by slope, angle or dotted line between pts. w/ known locations
If you come early please sit towards middle of rows. (less climbing at start of class)

**TODAY**: Vectors, Vectors, Vectors  JAN 22 2003

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<td>Vector</td>
<td>$\vec{F}$, $\vec{F}$, $\vec{F}$</td>
<td>$F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$</td>
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<td></td>
<td>$F \triangleleft 62^\circ$</td>
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**Addition**

$A + B = C$

- "Triangle" rule
- "Parallelogram" rule, "tip to tail" rule

$C_x = A_x + B_x$
$C_y = A_y + B_y$
$C_z = A_z + B_z$

**Multiply by a scalar**

$a \cdot \vec{A} = \vec{C}$

$a$ times by $a$

than $\vec{A}$
\[ A \cdot B = A_x B_x + A_y B_y + A_z B_z \]

\[ C = A \times B \]

\[ C = A_x B_y - A_y B_x + A_z B_z \]

\[ A \times (B + D) = A \times B + A \times D \]

\[ i \cdot i = 1, \quad i \cdot j = k \]

\[ A = 2i + 4j, \quad B = i - 3k \]

\[ A \cdot B = 2 \cdot 1 + 4 \cdot (-3) = -10 \]

\[ C = C_x = A_y B_z + A_z B_y - A_x B_y + A_x B_z \]

\[ C = C_y = A_z B_x + A_x B_z - A_y B_z + A_y B_x \]

\[ C = C_z = A_x B_y - A_y B_x + A_x B_y + A_y B_x \]
ex) \[ \mathbf{A} = 3\hat{i} + 4\hat{j} + 6\hat{k} \]
\[ \mathbf{B} = \hat{i} - 2\hat{j} + 9\hat{k} \]
\[ \mathbf{A} \times \mathbf{B} = (3\hat{i} + 4\hat{j} + 6\hat{k}) \times (\hat{i} - 2\hat{j} + 9\hat{k}) \]
\[ = (7 \hat{i} - 9 \hat{j}) \hat{k} + (3 \hat{j} - 24 \hat{i}) \hat{k} \]
\[ + (3 \cdot 2 - 4 \cdot 1) \hat{i} \hat{j} \hat{k} \]

**Rules of Vector Algebra**

All distributive, associative, commutative rules that you know apply to \( \mathbf{A} + \mathbf{B} \), \( a \mathbf{A} \), \( \mathbf{B} \cdot \mathbf{A} \), \( \mathbf{B} \times \mathbf{A} \) except

\( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \).

*(Watch out: \(-\mathbf{A}\) is nonsense)*
2.\text{Y from book}

alt. approach:

\[ \text{comp. of } P \text{ in dir. } BC \]

\[ = P \cdot \frac{\hat{i} - \hat{j}}{\sqrt{2}} \]

\[ = P \left( \cos 15^\circ \left( \frac{\hat{i}}{\sqrt{2}} \right) + \sin 15^\circ \left( \frac{-\hat{j}}{\sqrt{2}} \right) \right) \]

\[ = P \left( \frac{\sin 15^\circ}{\sqrt{2}} + \frac{\cos 15^\circ}{\sqrt{2}} \right) \]

\[
\text{Trig aside: } \cos 30^\circ = \cos (45^\circ - 15^\circ) \\
= \cos 45^\circ \cos 15^\circ + \sin 45^\circ \sin 30^\circ \sin 15^\circ \\
= \frac{\cos 15^\circ}{\sqrt{2}} + \frac{\sin 15^\circ}{\sqrt{2}} \\
= \frac{\cos 15^\circ}{\sqrt{2}} + \frac{\sin 15^\circ}{\sqrt{2}} \\
\]

That is: With vectors we effectively derived the trig identity for \( \cos(\theta - \phi) \).
Moment of a Force (w.r.t. some pt.)

\[ M_{c} = r_{c} \times F \]

(\[ \sum \Delta \] = \( dF \))

Note: moment of a force is dependent on ref. pt.

Example: little arrow showing contact forces of B on A. Usually we don't want to know such detail but instead want the net effect.

SO ... (cont'd next lecture)
Today: Equiv. force systems, couples

(Please read Ruina & Pretap §2.6 online)

Force System: A collection of forces $F_i$ at locations $r_i = r_i(x_0 = y_0)$

Question: When are two force systems $I$ & $II$ equivalent?

$I$ & $II$ are equivalent iff $\Sigma F_i^I = \Sigma F_i^II$

C is any one point.

Forces & positions in syst $I$  \( \sum_{C} \frac{F_i}{r_i} \times F_i^I = \sum_{C} \frac{F_i}{r_i} \times F_i^II \)

Forces & positions in syst $II$. 

Jan 4, 2003
Why use this def? Because all that the 10 laws of mechanics know about a force system is the $\Sigma F_i$ & $\Sigma \vec{r}_{i0} \times \vec{F}_i$.

"Equivalent" = "statically equiv." = "mechanically equiv." = "Equivalent"

ex). one force

\[ \vec{r} \Rightarrow \vec{F}_i = \vec{F} \]

\[ \vec{F} = \vec{F}_i + \vec{r}_i \]

\[ \Sigma \vec{F}_i = \Sigma \vec{F} \]

\[ \vec{F}_i \times \vec{F} = \vec{F}_i \times (\vec{F}_i + \vec{r}_i) \]

\[ = \vec{r}_i \times \vec{F}_i + \vec{F}_i \times \vec{F}_i \]

\[ = \vec{r}_i \times \vec{F}_i + \vec{r}_i \times \vec{r}_i \]

\[ = r_i^2 \]

\[ \Sigma \vec{r}_i \times \vec{F}_i = \Sigma \vec{r}_i \times \vec{F} \]

\[ \vec{r}_i \times \vec{F}_i = (\vec{r}_i + \vec{r}_n) \times \vec{F}_i \]

\[ = \vec{r}_i \times \vec{F}_i + \vec{r}_n \times \vec{F}_i \]

\[ = \vec{r}_i \times \vec{F}_i + \vec{r}_n \times \vec{F}_i \]

\[ = \vec{r}_i \times \vec{F}_i \]

A force is equiv. to same force slid to a new location on its line of action.

Key Fact
ex) A collection of forces acts at one point.

\[ \sum F_i = \sum \vec{F}_i \]

\( \sum M_\alpha = \sum \vec{M}_\alpha \)

\( \sum \vec{r}_{i\alpha} \times \vec{F}_i = \sum \vec{r}_{i\alpha} \times \vec{F}_\alpha \)

\( \sum \vec{r}_{0\alpha} \times \vec{F}_\alpha = -\sum \vec{r}_{0\alpha} \times \vec{F}_\alpha \)

subex)

\[ \vec{F} \text{ equiv to} \]

A single force at that pt.

\( \sum \vec{F}_i = \vec{F} \)

\( \sum \vec{F}_i = \vec{F} \)

(Key Fact)
FACT: If Syst I is equiv. to Syst II for Ref. pt. C then it is necessarily equiv. for all pts. in the Universe.

If equiv. for C necessarily equiv. for D

You only need to check equivalence for one ref. pt. & then equiv. for all ref. pts. is assured.

\[ \sum F_i^I = \sum F_i^II \text{ ind. of ref. pt.} \]

\[ r_{i10} \times F_i = \sum r_{i10} \times F_i^II \]

\[ r_{i1b} = r_{i1c} + r_{i1d} \]

\[ \sum r_{i1c} \times F_i^I + r_{i10} \times \sum F_i^I = \sum r_{i1c} \times F_i^I + r_{i10} \times \sum F_i^I \]
FACT: If a force system has \( \sum F_i = 0 \) then
\[
\sum M_c = \sum M_0
\]
for all pairs of pts. \( c \in \text{D} \)

ex)

DEF: A couple is any force distr.
that adds to zero.

\[
\sum \vec{F} = \vec{0}
\]

\[
\sum \vec{M} = \sum \vec{M}_0 \times \vec{F}_i
\]

FACT: Any force syst. is equiv to
a \( \vec{F} \) and a \( \vec{C} \) at \( D \)

\( \vec{F} \) force \quad \vec{C} \) couple
Recall: Any force system is equivalent to a force and a couple. Force acts at \( C \), \( F_{\text{equiv}} = \Sigma F_i \).

You pick the pt. \( C \), \( M_{\text{equiv}} = \Sigma F_i \times r_i \).

Change the point \( C \) you change \( M_{\text{equiv}} \), but not \( F_{\text{equiv}} \).

\[ 2D: M_{\text{equiv}} = M_{\text{equiv}} \quad \text{and} \quad F_{\text{equiv}} = F_{\text{equiv}} \]

\[ \vec{F}_{\text{equiv}} = \vec{F}_{\text{equiv}} + \vec{F}_{\text{c}} \]

\[ \begin{bmatrix} F_{\text{equiv}} \\ \vec{M}_{\text{equiv}} \end{bmatrix} = \begin{bmatrix} 3 \hat{i} + 4 \hat{j} \end{bmatrix} \text{N} \] can be replaced by a force-couple sys.

"at origin that is equivalent!"
FACT: in 2D: Every force system is equivalent to either
   a) just a force at an appropriate location
      (8 that whole line of action)
      when \[ \sum F_i \neq 0 \]
or   b) just a couple applied any place
      when \[ \sum F_i = 0 \]

in 3D: Cannot generally reduce a force system to just a force,
[can reduce to a "wrench"]

We like this because:

[Diagrams of force systems, simplified vs. complicated]
What is equivalent Force-Couple system at 0?

$F_{\text{equiv}} = F = (3\hat{i} + 4\hat{j}) N$

$\begin{align*}
\Delta \tau_{\text{equiv}} &= \Delta \tau_{\text{free}} = \Delta \tau_{\text{free}} = (2m\hat{i} + 1m\hat{j}) \times (3\hat{i} + 4\hat{j}) N \\
&= (1m \cdot 15.0 \sin \theta) \quad \text{(too hard to use in this example)} \\
&= (2m \cdot 3N) \cdot \hat{i} + (1m \cdot 3N) \cdot \hat{k} \\
&= (5m N \hat{i} \cdot \hat{i} + (1m \cdot 3N) \cdot \hat{k}
\end{align*}$

$\begin{align*}
\Delta C_{\text{equiv}} &= 5mN \hat{i} \\
\Delta F_{\text{equiv}} &= 5mN \hat{i}
\end{align*}$

Given $F_{\text{equiv}}$ and $\Delta C_{\text{equiv}}$ is there a place where $\Delta \tau_{\text{equiv}} = 0$? Try $p(t) = 0$.

$\begin{align*}
\Delta \tau_{\text{equiv}} &= \Delta \tau_{\text{free}} = 5mN \hat{i} + \left[ (\hat{i} - 3\hat{k}) \times (3\hat{i} + 4\hat{j}) N \right] \\
&= [5 - 4 + 3]mN \hat{i} \\
&= 4mN \hat{i}
\end{align*}$

{as we knew had to work.
How to find locations where force system is equiv.
to just a force (w/o couple).  Find pt. D

$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_D \times \mathbf{r}_D$

so that

$0 = \sum \text{equiv.} = \sum \mathbf{F}_0 + \sum \mathbf{F}_D \times \mathbf{r}_D$

$0 = \sum \mathbf{F}_0 = \mathbf{F}_0 + \sum \mathbf{F}_D \times \mathbf{r}_D$

$3N \mathbf{i} + 4N \mathbf{j} = \mathbf{F}_0 = 6 \mathbf{N} \mathbf{m} \mathbf{\hat{r}}$

$\mathbf{F}_D \times \mathbf{r}_D = \mathbf{F}_D \times (3 \mathbf{N} \mathbf{i} + 4 \mathbf{N} \mathbf{j})$

$3 \mathbf{F}_D \times \mathbf{r}_D = 3 \mathbf{N} \mathbf{m}$

$\mathbf{r}_D = \frac{\mathbf{F}_D \times \mathbf{r}_D}{3 \mathbf{N} \mathbf{m}}$

$x_D = \frac{\mathbf{F}_0 \cdot \mathbf{y}}{\mathbf{F}_0 \cdot \mathbf{y}}

= \frac{4 \mathbf{N}}{6 \mathbf{N} \mathbf{m}}$

$= 1.5 \text{ m}$

Ans. one good pt. D is on x axis

4.5 m from origin.
Warning 1: In life, engineering & HW problems, geometry is often not handed to you on a silver platter. Don't let your geometry/trig issues get tangled with your mechanics issues.

\[ M_0 = \vec{F}_{\text{c10}} \times \vec{r} \]

\[ \vec{r}_{\text{c10}} = \vec{F}_{\text{c10}} + \frac{\vec{r}_{\text{A/A}}}{3} \]

\[ \vec{r}_{\text{A/A}} = 2m \left( \cos 50^\circ \hat{i} + \sin 50^\circ \hat{j} \right) \]

\[ \vec{r}_{\text{c10}} = 4m \left( -\cos 60^\circ \hat{i} + \sin 60^\circ \hat{j} \right) \]

\[ \hat{a} = \cos 25^\circ \hat{i} - \sin 25^\circ \hat{j} \]

Warning 2: Moment about a pt. in 3D is a 3D vector.

Ex. Replace this force system with a force & a couple:

\[ \Sigma \vec{F}_i = \vec{0} \Rightarrow \text{equiv force is} \]

\[ \vec{M}_{\text{c10}} = \Sigma \vec{F}_{\text{c10}} \times 3N \]

\[ \vec{M} = (6m \hat{i} + 2n \hat{k}) \times 3N \hat{j} = [-18Nm^2 + 6Nm^2] \]

The equiv. couple
FREE BODY DIAGRAMS (FBDs) read Chapter 4 in Rau/Mills

A picture of a collection of material with all external forces and couples showing.

FBD of arm

\[ \uparrow \downarrow mg \]

\[ \text{chainsaw} \]

Laws of Mechanics: Apply to any FBD of any system in static equilibrium:

\( \sum F_i = 0 \), \( \sum \tau = 0 \)

For every point:

Generally:

2D \( \Rightarrow \) 3 indep. scalar eqs., 3D \( \Rightarrow \) 6 indep. scalar eqs.

Ex: m = 10 kg

\[ \downarrow g = 10 \text{N/kg} \]

Assume no slip at B

Reactions at A & B

2D \( \Rightarrow \) 3 eqs.

3 unknowns: \( N_A, F_B, N_B \)

\[ F_B = \sum \tau_{B} = \frac{\tau_{B}}{10} \times (mg) \]

\[ + \frac{R_{B}}{10} \times N_A \]

\[ \sum F = \frac{3}{2} m \cdot 100 \text{N} \times \frac{3 \times 10^5 \text{N}}{A} \]

\( \Rightarrow \sum F = 75 N m - \frac{3 \times 10^5}{A} N_A m \)

\[ N_A = \frac{500}{0.03} N \]

(\( N_B \) = force comp., \( N = 1 \) Newton)

(cont'd next lecture)
TODAY: I/FBDs & Statics (cont'd), 2) Q8A, 3) Quiz 0 practice (due)

FBD: Pic. of system & all forces on it.

Mechanics: \( \Sigma F_x = 0, \ \Sigma F_y = 0 \)

Sum over all forces/couples on FBD and no others.

Note: "chainsaw" = External forces only

Continued from last lecture

\[
\begin{align*}
\sum F_x &= 0 \\
M &= 0
\end{align*}
\]

Continued from last class:

\( \Sigma F_x = 0 \)

Continued from last class:

\( \Sigma F_y = 0 \)

Continued from last class:

\( M = 0 \)

Continued from last class:

\( N_B = \frac{3}{4} \cdot 100 \text{ N} \)

Continued from last class:

\( F_B = -\frac{3}{4} \cdot 100 \text{ N} \)

\( Q&A \)

\( \text{Quiz 0} \)
Recall FBD = picture of isolated system and all external forces & couples that act on it.

Laws of Mechanics (Statics)

- For all forces/couples on a FBD:
  \[ \sum F_i = 0 \quad \text{and} \quad \sum \vec{M}_C = 0 \]
  - Generally 6 ind. scalar eqs in 3D

- Moment about an axis:
  - Tendency of forces (and couples) to cause rotation about an axis (given force system has different moment about different axes)

\[
\text{Moment about axis } AB = \frac{d \vec{F}}{r_{CI/A}} \quad \text{hard to evaluate from geometry (often)}
\]
\[
= (r_{CI/A} \times \vec{F}) \cdot \hat{\vec{r}}_{AB}
\]
\[
= (r_{CB} \times \vec{F}) \cdot \hat{\vec{r}}_{AB}
\]

Example 3.67 from Perian
Find unknown forces & reactions: T_{BE}, T_{BD}, T_{CA}, R_{OX}, R_{OY}, R_{OE} (6 unknowns)

\[ \sum F_i = 0, \quad \sum \tau_i = 0 \quad (i) \]

6 eqs.

Method 1: Pick any pt if you like.

1. Break into comps.
2. Solve 6 egs for 6 unknowns.
Method 2:

Try to get 1 eq. in 1 unknown.

\[ \sum \text{Max} a_{AB} = 0 \]

only \( R_{o} \) contributes

\[ \Rightarrow R_{o} = 0 \]

\[ \sum \text{Max} i_{OB} = 0 \]

only \( T_{ca} \) & \( \omega \) contribute

\[ (\sum \text{M} / B) \cdot \Delta_{0B} = 0 \]

\[ \Rightarrow \frac{o_{0B}}{l_{o0B}} \]

\[ \left( \frac{r_{B/B} \times W_{(\theta)}}{r_{A/B} \times T_{ca} \Delta_{AC}} \right) \cdot \Delta_{0B} = 0 \]

\[ \frac{r_{C/B} = -\frac{3}{4} \Delta_{N}}{r_{A/B} = -3.5 \Delta_{N}} \]

\[ \Delta = \frac{r_{ac}}{|r_{ac}|} = \frac{4}{\sqrt{6}} \]

\[ \frac{-2k_{-10}}{w_{1}} \]

\[ \Delta_{0B} = (1.5 \Delta_{N} + 2n_{K})V_{(15\Delta_{N})} \]

Plug in, crank \( T_{mc} = \ldots \)

\[ \sum \text{Max} i_{OA} \Rightarrow T_{BD} = \ldots \]
Feb. 5, 2007

TRUSSES:

Truss: A collection of "bars" connected at ends by hinges (2D), or ball & socket joints (3D).

All loads must be at "joints".

EX:

TRUSSES are GOOD.

a) useful
b) agreeable

Primary Goal: Given geometry, loads find tensions in bars & "bar forces" reactions.

Method:

FBDS of a) joints  b) part of truss

\[ \sum F_i = 0, \ \sum M_i = 0 \] 

Key idea: bars are "force members".
If only two forces act on a body in static equilibrium, they are equal in magnitude and opposite in direction along the line connecting the two points.

\[ \Sigma F_x = 0 \Rightarrow T_{AC} = 0 \]
\[ \Sigma F_y = 0 \Rightarrow T_{BC} = 0 \]
\[ \Sigma F_z = 0 \Rightarrow T_{BE} \sin \theta = 0 \]
\[ T_{BE} = 0 \]
\[ T_{BC} \sin \phi = 0 \]
\[ T_{BC} = 0 \]
Joint E

\[ \sum F_i = 0 \Rightarrow E \]
\[ \sum F_x = 0 \Rightarrow -T_{EK} - T_{EH} \cos \theta = 0 \]
\[ \sum F_y = 0 \Rightarrow -T_{CE} - T_{EH} \sin \theta = 0 \]

* is 2 eqs in 2 unknowns

T_{EH} = \frac{T_{CE} \sin \theta}{\cos \theta}

T_{EH} = -\frac{W}{\sin \theta}

"bar EH is in compression"
Today: Trusses (cont'd)

**Method** (most of time)

1. Draw FBD of whole structure (use to find reactions if possible)
2. Find 0-force bars by inspection
3. Draw FBDs of joints on sections as appropriate

**Method of Sections**

\[ \sum F_x = 0 \Rightarrow T_{AB} = ? \]
\[ \sum \Delta n_i = 0 \quad \checkmark \]

\[ \sum n_i = 0 \]

\[ -(T_{Ac} \cdot h) \hat{k} + (w \text{ terms}) \cdot \]

(horizonal distances) = 0

\[ h \ll \text{horiz. distances} \]

\[ T_{Ac} \gg \text{weight} \]

top in compression
both in tension

\[ \sum \Delta M = 0 \quad \Rightarrow \quad T_{DE} \]

Sometimes
a) luck
b) no luck
\Rightarrow multiple section
\Rightarrow set up sets
of eqns.

\[ (\sum \Delta i \times F_i) \cdot \Delta_{CD} = 0 \]
Feb 12, 2063
TODAY: 3D TRUSSES

Methods (just like 2D)

* joints

Each FBV \implies \Sigma F_i = 0

3 scalar eqs. \implies one joint at a time
work way around truss

\Sigma F_i = 0, \Sigma M = 0 or

\Sigma M = 6 different eqs

6 ind. scalar eqs.
$FBD$ joint + $D$

$\Sigma F_x = 0$

$T_{AD} \hat{A}DB + T_{DB} \hat{D}DB + T_{DC} \hat{D}DC + F = 0$

$\frac{F}{x} = \frac{m_1}{x} + \frac{m_2}{y}$

$BD \text{ in } xy \text{ plane}$

$T_{DB} = ?$
Side geometry: \( \frac{112}{123} \)

\[
\lambda_{DA} = \frac{\lambda_{OA}}{150a} = \frac{1.57 - 2.5 + 4k}{\sqrt{1.5^2 + 2^2 + 4^2}}
\]

\[
= \lambda_{DA} \hat{i} + \lambda_{DA} \hat{j} + \lambda_{DA} \hat{k}
\]

\[
\lambda_{OC} = \ldots
\]

\[
\lambda_{OD} = \ldots
\]

\[
\frac{503.7, 503.7, 503.7}{3 \text{ eqns in } 3 \text{ unknowns}}
\]

How to solve?

Write in matrix form

\[
\begin{bmatrix}
\lambda_{DA} \\ \lambda_{DA} \\ \lambda_{DA} \\
\end{bmatrix} 
\begin{bmatrix}
\tan \theta_{DA} + \lambda_{DA} \tan \theta_{DA} + \lambda_{DA} \tan \theta_{DA} \\
\tan \theta_{OD} + \lambda_{DA} \tan \theta_{OD} + \lambda_{DA} \tan \theta_{OD} \\
\end{bmatrix}
\]

\[
T_{CD} = -F_x
\]

\[
\lambda_{DA} \tan \theta_{DA} + \lambda_{DA} \tan \theta_{DA} + \lambda_{DA} \tan \theta_{DA} + \ldots
\]

\[
\lambda_{DA} \tan \theta_{DA} + \ldots
\]
\[
\begin{bmatrix}
\lambda_{0Ax} & \lambda_{0Bx} & \lambda_{0Cx} \\
\lambda_{0Ay} & \lambda_{0By} & \lambda_{0Cy} \\
\lambda_{0Az} & \lambda_{0Bz} & \lambda_{0Cz}
\end{bmatrix}
\begin{bmatrix}
T_{DA} \\
T_{DB} \\
T_{DC}
\end{bmatrix}
= 
\begin{bmatrix}
-F_x \\
-F_y \\
-F_z
\end{bmatrix}
\]

\[
\Rightarrow T_{BD} = \ldots
\]

ready for computer soln. calculator
MATLAB soln.

\[ r\text{DA} = [1.5 \ -2 \ -4]^T; \]
\[ r\text{DB} = [1.5 \ 0 \ -4]^T; \]
\[ r\text{DC} = [-1.5 \ 0 \ -4]^T; \]
\[ \text{lam}\text{DA} = r\text{DA}/\text{norm}(r\text{DA}); \]
\[ \text{lam}\text{DB} = r\text{DB}/\text{norm}(r\text{DB}); \]
\[ \text{lam}\text{DC} = r\text{DC}/\text{norm}(r\text{DC}); \]
\[ F = [100 \ 200 \ 300]^T; \]
\[ A = [\text{lam}\text{DA} \ \text{lam}\text{DB} \ \text{lam}\text{DC}]; \]
\[ \text{Tensions} = A \backslash (-F); \]
\[ \text{done} \]
\[ \text{backslash} \]

\[ 2/18 \text{ pg 5} \]
Shortcut

\( \{ 0 \} \cdot ( \Omega_0 x \Omega_0 ) \)

\( \Rightarrow \) i.e. for TDB
Feb 17, 2003

Recall:

Today: *3D forces (con't), review

Mechanics & Structures (first of four lectures)

In matrix form:

\[ \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \end{bmatrix} \]

solve by brute force (computer)

The columns are the camps of unit vectors of \( \mathbf{F} \).

\[ \mathbf{T}_1 = \mathbf{T}_2 \]

\[ \mathbf{T}_2 \times \mathbf{A}_1 = (\mathbf{T}_2 \times \mathbf{A}_1) = \mathbf{F} \times (\mathbf{A}_1 \times \mathbf{A}_3) \]

\[ \mathbf{T}_2 = \frac{\mathbf{F} \times \mathbf{A}_1}{\mathbf{A}_1 \times \mathbf{A}_3} \]
Math Aside 2/17 pg 2

$A \cdot B \times C$ is called mixed triple product (used also for moment about an axis: $M_a = J \cdot \tau \times F$)

Some Facts:

$A \cdot B \times C = A \times B \cdot C$

Why? a) Mult. out & check

b) Parallelipiped volume

Also: $A \cdot B \times C = \text{det} \begin{bmatrix} [A]' & [B]' & [C]' \end{bmatrix}$

$E \cdot \hat{a}_1 \times \hat{a}_3 = E \times \hat{a}_1 \cdot \hat{a}_3$

$= -\hat{a}_1 \times \vec{E} \cdot \hat{a}_3$

$\hat{a}_2 \cdot \hat{a}_1 \times \hat{a}_3 = -\hat{a}_1 \times \vec{E} \cdot \hat{a}_3$

$\Rightarrow T_2 = \frac{\text{det} \begin{bmatrix} [\hat{a}_3] & [\vec{E}] & [\hat{a}_3] \end{bmatrix}}{\text{det} \begin{bmatrix} [\hat{a}_1] & [\hat{a}_2] & [\hat{a}_3] \end{bmatrix}}$

Kramer's rule (we've derived using vectors)
Machines: 2/17 pg 3

(R Structures)

Collections of objects connected in simple ways.

Different from trusses in that some of objects are not 2-face members.

Consider the FBD of bolt-cutter & bolt. Assume top-bottom symmetric above.

Neglect Weight
\[ \Sigma M_c = 0 \text{ (upper handle)} \]
\[ \Rightarrow -l F_A + F_B w = 0 \]
\[ F_B = \frac{F_A w}{l} \quad \text{(1)} \]

\[ \Sigma M_D = 0 \text{ (cutter piece)} \]
\[ -h F_B + d F_E = 0 \]
\[ F_E = \frac{h}{d} F_B \quad \text{(2)} \]

1) \& 2) \Rightarrow \]
\[ F_E = \frac{h l}{w d} F_A \]

\[ \frac{8}{w} \approx 40, \quad \frac{b}{d} \approx 4 \]

\[ \Rightarrow F_E \approx 160 F_A ! \]

Note: "toggle mechanism"

\[ \theta \text{ is small} \]
\[ \Rightarrow F_2 >> F_1 \]

\[ \frac{F_1}{F_0} = \tan \theta \]

2/17

194
Feb 9, 2003
TODAY: Struct & Machine Control

Ex) (from final exam a couple years ago.)

Given \( w, l, h, W \), find \( F, T_{BC} \).
\[ \sum_{i=1}^{n} F_i = 0 \]

\[ -\omega \mathbf{W} \mathbf{K} + \frac{r_{c/E}}{r_{EC}} \times (T_{BC} \hat{e_{BC}}) = 0 \]

\[ r_{c/E} = r_{EC} = \omega \hat{i} + \mathbf{h} \hat{j} \]

\[ \hat{e_{BC}} = \frac{-\omega \hat{j} - \mathbf{d} \hat{k}}{\sqrt{\omega^2 \mathbf{h}^2 + d^2}} \]

\[ r_{BE} \times \hat{e_{CB}} = (\omega \hat{j} + \mathbf{h} \hat{i}) \times \left( \frac{-\omega \hat{j} - \mathbf{d} \hat{k}}{\sqrt{\omega^2 \mathbf{h}^2 + d^2}} \right) \]

\[ = \left( \omega \mathbf{d} + \frac{h \omega}{2} \right) \mathbf{K} \]

\[ \Rightarrow \]

\[ T_{BC} = W \frac{\mathbf{U} - \mathbf{d}}{(\frac{1}{4} - d)} \]

\[ T_{BC} > 0 \text{ if } d < \frac{1}{4} \]

\[ T_{BC} < 0 \text{ if } d > \frac{1}{4} \]

\[ T_{BC} \to \infty \text{ as } d \to \frac{1}{2} \]
\[ F = ? \quad \text{and} \quad \nabla^2 F \Rightarrow \nabla^2 T \quad \text{in} \quad \Omega \]

\[ \sum F = 0 \]

\[ \{ \sum T_{AB} \bar{\lambda}_{BA} + T_{AB} \bar{\lambda}_{AB} - F_j = 0 \} \]

Take x & y comps (\{ \bar{\lambda}_i, \bar{\lambda}_j \})

Symmetry on \( \sum F_x = 0 \)

\[ \Rightarrow \quad T_{AB} = T_{BA} \]

\[ \sum F_y = 0 \quad \Rightarrow \quad T_{BC} = \ldots \]

\[ \begin{cases} \sum m_A = 0 & \Rightarrow \text{F w/ out } T_{AB} \\ \sum \bar{\lambda}_{any pt \in \text{the } AB \text{ except } B} & = 0 \\ \sum \bar{\lambda}_{\text{any pt } \bar{\lambda}_{\perp + AB} } & = 0 \\ \sum \bar{\lambda} \times \bar{\lambda}_{AB} & = 0 \end{cases} \]

\[ \nabla \cdot (\nabla \times \bar{\lambda}_{AB}) = 0 \]
Feb 21, 2003

**TODAY**: Machines (cont'd)

a) last lecture example again.
b) force amplification
c) An old Prelim question

---

**last lecture**

---

Easiest & simplest FBDs

---

The ultimate values of $T_1$ & $T_2$ will not depend on whether $F$ is applied to the pin at $B$, or bar 1, or both bars.
How to get big forces from small forces?

Example: lever

\[ F_2 = F_1 \cdot \frac{d_1}{d_2} \]

Example: Toggle mechanism

\[ \sum \tau_c = 0 \]

\[ -F_1 l \cos \theta + F_2 \ell \sin \theta = 0 \]

\[ F_2 = F_1 \tan \theta \]

\( \theta \) small \( \Rightarrow F_2 \) big
ex) Wedge

\[ F_1 = F_D \sin \theta \quad (1) \]

\[ \mathbf{FBDII} : \Sigma F_x = 0 \]

\[ F_c = F_D \cos \theta \quad (2) \]

\[ \mathbf{FPDII} : \frac{F_c}{\tan \theta} \]

**Morals:** For frictionless passive machine

**Energy balance**

\[ |\delta_1 F_1| = |\delta_2 F_2| \]

\[ \delta_1 \text{ & } \delta_2 \text{ are disp. of } F_1 \text{ & } F_2 \]

**Force amplifiers are motion attenuators**
3) (30 pts) The proposed nutcracker design consists of two moving parts: a lever hinged at B and a punch hinged to the fixed base at A. All joints and slots are assumed to have negligible friction.

**Mechanism and geometry clarifications:** The vertical lever has a pin at C and a horizontal force $F$ applied at D. The punch has a slot in which the lever pin slides at C. The slot is parallel to the line AC. The spherical nut is cracked by being squeezed between the vertical surface of the punch at N and the vertical surface attached to the base. Point N at the left edge of the nut is level with the sliding pin at C. The horizontal distance from C to N does not enter the solution, but assume it is $c$ if you need it for an intermediate calculation.

**Quantities:** $F = 10$ lb, $a = 2$ in, $b = 10$ in.

---

a) (25 pts) Find the force acting on the nut at N. A number is desired (i.e., so many lb force).

**Hint:** Only substitute in numbers when you have a formula for your answer in terms of $a$, $b$ and $F$.

b) (5 pts) The answer to (a) is conspicuous in its being either much smaller than $F$, very similar to $F$, or much bigger than $F$. Which is it? Explain, in words, why it is as it is. Part (b) will be graded independently of part (a). The best possible answer will generate an approximate formula for the force at N, using next-to-no equations.
This lecture picks up with the example (nut cracker) from last lecture.
CENTER of MASS, CENTROID
(center of gravity)

\[ \bar{r}_G = \text{average position of the mass} \]
\[ \text{position of C.O.M. w.r.t. } 0 \]

Analogy: test scores

\[ T_A = \frac{T_1 \cdot n_1 + T_2 \cdot n_2 + T_3 \cdot n_3}{n_1 + n_2 + n_3} \]
\[ \text{test score average} \]
\[ = \frac{65.3 + 75.5 + 95.2}{10} \]
\[ = \ldots \]

\[ \text{average} = \frac{(\text{value 1})(\text{amount of stuff})}{(\text{total amount of stuff})} \]
\[ = \frac{(\text{value 2})(\text{amount of stuff})}{(\text{total amount of stuff})} \]
\[ = \frac{(\text{value 3})(\text{amount of stuff})}{(\text{total amount of stuff})} \]

\[ \bar{r}_G = \frac{\sum r_i \cdot m_i}{\sum m_i} \]
\[ m_{\text{total}} = \sum m_i \]
\[ m_{\text{total}} \cdot \bar{r}_G = \sum r_i \cdot m_i \]
\[ m_{\text{total}} \cdot x_G = x_1 m_1 + x_2 m_2 + x_3 m_3 + \ldots + x_n m_n \]
Why be interested in \( r_G \)?

1) The gravity forces on a system (for near-earth gravity; \(-g\)) are constant downwards. They are "equivalent" to the total weight acting at \( r_G \).

2) Centroid is important for understanding beam cross sections (as you will see in a month or so).

\[
m_{\text{tot}} \cdot r_G = \sum r_i \cdot m_i \quad \text{discrete}\]

\[
\int r \cdot \text{dr} \quad \text{continuous}
\]

ex) Two equal pt. masses

\[\begin{align*}
\sum & m_i \\
& \Rightarrow \sum m_i r_i
\end{align*}\]

\[2m \cdot r_G = m \cdot r_1 + m \cdot r_2\]

\[r_G = \frac{r_1 + r_2}{2}\]

\[G = \text{Con is at midpoint}\]

[Note: "objective" location is independent of position or orientation of coord. syst. used for calculation]
Feb. 26, 2007

TODAY: 1) C.O.M. (cont'd) 2) Q & A

**Center of mass**

\[ m_{\text{tot}} r_{\text{cm}} = \sum m_i r_i \]

\[ \sum \int r_i dm = \int r dm \]

\[ dm = \rho dV = \rho dA = \rho ds \]

\[ g = \text{mass per unit} \]

- Volume
- Area
- Length as appropriate

**Ex:** Two equal masses (again)

\[ m_{\text{tot}} r_{\text{cm}} = \sum \int r_i dm \]

\[ g \rho A r_{\text{cm}} = \int (x^2 + y^2) \rho dxdy \]

\[ = \int (y^2 + y^2) \rho dxdy \]

C.O.M. is "objective" because it has a well-defined place w.r.t. object

You can use any coord. syst. you like to find C.O.M.

**Ex:** Uniform rectangular density

\[ m_{\text{tot}} r_{\text{cm}} = \sum \int r_i dm \]

\[ g \rho A r_{\text{cm}} = \int (x^2 + y^2) \rho dxdy \]

\[ = \int (y^2 + y^2) \rho dxdy \]
\[ \text{c.o.m.} \text{ on symmetry line} \]

\[ \text{ex) triangle} \]

\[ \text{ex) repeat for all 3 side bisectors} \]

Would have got same location w\/ this coord syst but a lot more calculation.

**Note:** C.O.M. respects symmetry of object.
Geometry fact: $C$ is $1/3$ of the way up from each base.

\[
\frac{h}{3}
\]

for\ triangle\ using\ calculus

\[
dA = g \left( \frac{b^2}{2} \right) \cdot dy
\]

\[
m_{5cm} = 55\text{dm}^3
\]

\[
m_{7cm} = 5\text{yd}m
\]

\[
\int_{0}^{b} y \cdot \frac{b-y}{b} \, dy
\]

\[
\left(\text{area}\, 42\right) y_{cm} = g \left( \frac{2}{3} \cdot \frac{b}{2} - \frac{2}{3} \cdot \frac{b}{2} \right)
\]

\[
y_{cm} = \frac{b}{3}
\]
\[ \text{ex)} \text{ quarter circle} \]

\[ \text{c.o.m. on thick line} \]

\[ \bar{x} = \bar{y} \]

\[ x_c = y_c \]

\[ ds = a \, d\theta \]

\[ dm = \frac{g a e \, d\theta}{2} \]

\[ m_{\text{tot}} x_{cm} = \int x \, dm \]

\[ = \int_{0}^{\pi/2} \left( \frac{2}{3} \cos \theta \right) a \left[ \frac{g a e}{a} \right] d\theta \]

\[ = \frac{1}{3} ga^2 \int_{0}^{\pi/2} \cos \theta \, d\theta \]

\[ \frac{\pi a^2}{4} x_e = \frac{1}{3} ga^3 \]

\[ x_{cm} = \frac{4}{3\pi} a \approx 0.4 a \]

\[ \bar{r}_{cm} = \frac{4}{3\pi} a \left( \hat{i} + \hat{j} \right) \]

\[ \text{ex)} \text{ composite object} \]

\[ m_{\text{tot}} \bar{r}_{cm} = m_1 \bar{r}_{cm} + m_2 \bar{r}_{cm} \]

\[ (m_1 - m_2) \bar{r}_{cm} = m_1 \bar{r}_{cm} - m_2 \bar{r}_{cm} \]
TODAY: Friction

Friction = force which resists or prevents slipping between contacting solids.

Some books: "smooth" = no friction  
"rough" = 00 friction  
(actually friction strength is not well correlated to surface roughness unless surface is lubricated = wet or water on oil)

Careful experiment: N=const, μ'static or stationary coeff. of friction
μ'dynamic or kinetic coeff. of friction
This is too complicated, so we use a simple model. $\tau = \mu \cdot N = K \cdot A$

Ammann's law for friction

Davinci law for friction

friction law called Coulomb's law

$F = \mu N$ for $v > 0$

$F = 0$ for $v \leq 0$

Try various cases

Next class

The constitutive law for dry friction which we will use

$N = \frac{F}{\mu} = \frac{F}{\tau}$

Curve is

$\mu$
\[ T_{AB} = ? \]

**Subex) No brakes on:**

\[
\sum F_x = 0 \Rightarrow T_{AB} - mg = 0
\]

\[
\sum F_y = 0 \Rightarrow N_c + N_0 + T_{AB} \sin \theta = 0
\]

\[
\sum \tau_{\theta} = 0 \Rightarrow \frac{F_0}{l} \times N_c = 0
\]

\[
T_{AB} = \frac{F_0}{\cos \theta + \sin \theta}
\]

\[
F_0 = \mu N_0
\]
Aside: Alt. description for friction: $\tan \phi = \frac{F_t}{N}$

$\phi \equiv \frac{k - k}{m} = 0$ for one eq. for $T_B$ and unknown $T_D$

Given $\theta$, $m$, and $b$, how do you calculate $F_B$?

Assume incipient slip to fall. Assume jet for given $\mu$.
3/5/03 pg.11

Today: 3 force bodies, friction, hydrostatics (last of 2).

3 force body

A body (in 2D or 3D) w/ 3 forces acting on it.

\[ \Sigma M_{AB} = 0, \text{ Force at } C \text{ only contributes to this.} \]

3) contrib. = 0 \Rightarrow force at C in plane ABC.

Likewise for \[ \Sigma M_{AC} = 0, \Sigma M_{BC} = 0 \]

\Rightarrow All forces co-planar in ABC plane.

\Rightarrow 2D problem in plane ABC.

\Rightarrow 3 force body all lines of action intersect at one pt. in plane ABC (that pt. can be at \( \infty \); all 3 lines \( \parallel \)).
Ex) repeat from end of last class

* ladder has no mass
* person $m$

What is biggest $d$ so ladder does not fall?

2 approaches

1) Assume no slip at $B$
   Find reaction at $B$
   Make sure $F_{\text{sd}} \leq \mu N_B$

Solve for $N_A$, $F_o$, $N_B$ as usual ($E F = 0$, $E L = 0$)
Answer in terms of $\theta$, $d$, $m$, $g$, $d_o$

\[
edfrac{F_{\text{sd}}}{N_B} \leq \mu
\]

Gives an inequality to find.
[Will get answer for all $d$]

\[\text{if } \theta < \phi\]
2) More tricky

3 force body

\[ \frac{d}{\cos \phi} = \mu \Rightarrow d = \mu d \cos \phi \]

\( H \) in friction cone associated w/ B,

Worst case \( H \) at edge of cone
How close can \( d \) get & still sat. equilib.

\[
\tan \phi = \mu
\]
March 3rd, 2003
Lecture given by Professor Alan Zehnder

Topic: Hydrostatics

Special case of distributed loads in which:
1) Force is perpendicular to surface it acts on
2) Pressure increases linearly with depth

\[ P = \rho gh \]

\[ \rho \rightarrow \text{density of fluid} \]
\[ h \rightarrow \text{depth} \]
\[ g \rightarrow \text{acceleration of gravity} \]

Let's just check units.

\[ P = \frac{\text{kg m}}{\text{m}^3 \cdot \text{s}^2} \cdot \text{m} \]
\[ = \frac{\text{kg m}}{\text{s}^2} \cdot \frac{1}{\text{m}^2} = \frac{\text{N}}{\text{m}^2} \checkmark \]

For reference,

1 atm = \(10^5 \frac{N}{m^2}\)

At the bottom of Cayuga Lake
\[ P \approx 10^3 \times 10 \text{ (100)} = 10^6 \text{ Pa} = 10 \text{ atm} \]

\(\rho, g, h\)

---

Start from Fundamentals
- Force & moment due to pressure on small area
- Integrate to get total \( F + M \)

Build Shortcuts
- Replace distributed force by a single force acting at centroid of diagram of force distribution.
Example:

- Find force & moment due to water pushing on window
- Find reaction forces of wall pushing on windows

- Idealize problem as 2D
- Idealize as pinned on bottom supports, roller on top

Note: In this coordinate system:

\[ p = p(z) = \rho g (H_1 + H_2 - z) \]

FBD

Force due to \( p(z) \) acting on \( dA \)

\[ dF = p(z) \, dA \uparrow \]

\[ = p(z) \uparrow \, dx \, dy \]
Moment (about B) due to \( p(z) \) acting on \( dA \)

\[
dM_B = -p(z) z \, dy \, dz
\]

Integrate to get \( B + M_B \)

\[
B = \sum_{A} \int_{0}^{L} \int_{0}^{H_2} \rho g (H_1 + H_2 - z) \, dz \, dy = \rho g L \left( H_1 + H_2 + \frac{H_2^2}{2} \right)
\]

Total force of water pushing on window

\[
M_B = \sum_{A} \int_{0}^{L} \int_{0}^{H_2} -\rho g (H_1 + H_2 - z) z \, dz \, dy = -\rho g L \frac{H_2^2}{2} \left( H_1 + \frac{H_2}{3} \right)
\]

Determine Reaction Forces

\[
\sum F = 0 = \rho g L \left( H_1 + H_2 + \frac{H_2^2}{2} \right) \hat{i} + A_x \hat{i} + B_x \hat{k} + B_z \hat{k}
\]

\[
\sum M_B = 0 = -\rho g L \frac{H_2^2}{2} \left( H_1 + \frac{H_2}{3} \right) - A_x \cdot H_2
\]

Solving, we find that:

\[
B_z = 0
\]

\[
A_x = -\rho g L \frac{H_2^2}{2} \left( H_1 + \frac{H_2}{3} \right) = -\rho g L \frac{H_2}{2} \left( H_1 + \frac{H_2}{3} \right)
\]

\[
B_x = -\rho g L \left( \frac{H_1 H_2 + H_2^2}{2} \right) - A_x = -\rho g L H_2 \left( \frac{H_2}{2} + \frac{H_2}{3} \right)
\]
Let's try some numbers

Say, \( H_1 = 0.5 \text{ m} \)
\( H_2 = 1 \text{ m} \)
\( L = 5 \text{ m} \)

Approximate \( g = 10^4 \text{ m/s}^2 \)

\[ \Rightarrow \rho g = 10^4 \]

\[ A_x = -10^4 / (5)(1/2)(1 + 1/3) = - \frac{25}{12} \times 10^4 = -2.08 \times 10^4 \text{ N} \approx -4000 \text{ lbs} \]

\[ B_x = -2.92 \times 10^4 \text{ N} \]

Check if \( R + A_x + B_x = 0 \) \( \checkmark \)

Replace distributed force by single force at centroid of pressure diagram

\[ R = \sum p(y) \, dA = L \int p(y) \, dy \]

Note: \( y \) is centroid of area under \( p(y) \) curve

Moment around \( A \)

\[ M_x = \sum p(y) \, y \, dA = L \int y \, p(y) \, dy \]

\[ M_x = L \cdot \bar{y} \cdot R \]
Centroid of area under $p(y)$ curve

$$
\bar{y} = \frac{\int y \, p(y) \, dy}{\int p(y) \, dy}
$$

Previous Example:

$$
p = \frac{1}{2} \times 10^4 \text{N/m}^2
$$

$\rho = \frac{3}{2} \times 10^4 \text{N/m}^2$

\[ R = 1m \left( \frac{1}{2} \times 10^4 + \frac{1}{2} \left( 1 \times 10^4 \right) \right) \]

\[ = 5 \times 10^4 \text{N} = 50,000 \text{N} \]

\[ \bar{z} = \frac{1}{2} \left( \frac{1}{4} \times 10^4 \right) + \frac{1}{3} \left( 1 \times 10^4 \right) \]

Total Area

$$
\frac{5 \left( \frac{1}{4} + \frac{1}{6} \right) \times 10^4}{5 \times 10^4} = \frac{5}{12} \text{m}
$$
This week:

Mon: Hydrostatics (cont'd)
Wed: Quiz
Fri: Statics Capstone

Hydrostatics

\[ dF = -p \, dA \hat{n} \]

\( p = \) pressure

\[ \text{under water surface} \]

\[ \text{d}A = \text{bit of area} \]

\[ \hat{n} = \text{unit normal of surface} \]

\[ \text{pointing towards water.} \]

Of interest:

\[ F_{\text{net}} = \int dF = \int p(-\hat{n}) \, dA \]

\[ M_{\text{cmt}} = \int_{cmt} F \times dF \]

\[ dS = \int_{cmt} \frac{x}{(-\hat{n})} dA \]

\[ dS \] is like \( dA \)

\[ \text{bit of surface} \]

Key facts:

no shear \( \Rightarrow \) \( p \) is the same for all \( h \) at a given \( \rho \) in space

\[ \rho_1 = \rho_2 \]

\[ \frac{\rho_2 - \rho_1}{\rho_1} \]

Why?

\( \Sigma F_x = 0 \) \( \Rightarrow \) \( F_x = F_y \)

\[ \Sigma F_y = 0 \]

\[ \Rightarrow F_x = F_y = P \]
if $g =$ density of fluid $=$ const.

\[ \text{good model for water} \]

\[ \text{almost all the time} \]

\[ \text{bad model for air over length scales of kms.} \]

$\text{H}_2\text{O}$

$\downarrow h$

$\uparrow p = \rho g h$

\[ \text{"geese" pressure} \]

\[ \text{Real} = \text{Pressure} + \text{Atmospheric} \]

Beginner advice: ignore atmospheric pressure.

Archimedes Principle

"Aha!" Force from $H_2O$ pressure on submerged object.

Pressure at each pt. is same as it would have been if objects replaced w/ water.
Force on object from H_2O

= Force that would have acted on water w/ same shape.

FBD of water

\[ F = \rho g V \]

Because water is in equilibrium:

\[ \Sigma F = 0, \quad \Sigma M = 0 \]

\[ \nabla \cdot \mathbf{F} = \int \int \int \frac{\partial F}{\partial x} \, dx \, dy \, dz \]

\[ \int \int \int \frac{\partial F}{\partial x} \, dx \, dy \, dz = 0 \]

\[ \int \int \int \mathbf{F} \cdot d\mathbf{A} = \int \int \int \mathbf{F} \cdot \mathbf{n} \, ds \]
S = \int \rho(x, y, z) \, dV = \int \vec{F}_1 \cdot n_x \, dS + \int \vec{F}_2 \cdot n_y \, dS + \int \vec{F}_3 \cdot n_z \, dS

(\text{Case 1:} \text{ like the \textit{Coulomb}})

(\text{Case 2:} \text{ like the \textit{Coulomb}})

For hydrostatics:

\vec{F}_i = F_i
3/14/03 (pg. 1)

TODAY: (Statics review)

All of Statics

0. You can draw FBD of any system or subsystem.
   FBD = picture of system w/ all external forces showing (but no internal forces or "internal" forces).

Laws of Statics (for system in static equilibrium):

IA. \( \sum F_i = 0 \)
    = all ext.

IB. \( \sum M_i = 0 \)
    = any pt.
    (all pts)

\( M_{i/c} = F_{i/c} \times r_i \)

Observations:

Can, on one FBD replace a collection of forces w/ "equivalent" system

"equivalent" = same \( F_i \)
   same \( \sum M_i \),

\[ \sum F_i \rightarrow \sum M_i \rightarrow \sum \Delta \]

\[ \sum F_i \rightarrow \sum M_i \rightarrow \sum \Delta \]

(Couple) = a representation of a force system equiv. to a couple (no net force) by a vector showing net moment.
0. Principle of action & reaction.
   If \( A \) causes \( F \) on \( B \) then \( B \) will \( -F \) on \( A \) with same line of action.
   Models of interactions and of things.

If a motion is caused or prevented, a force or couple shows on FBD.

\[
\begin{align*}
\text{Friction} & \quad \mu = 0 \\
\text{String} & \quad T \leftarrow \text{load} \rightarrow T \\
\text{Fluid} & \quad T = k (2 - \phi_0) \\
\text{Slip Vel.} & \quad F/N \\
\text{Staticallly Determinate Problems} & \\
\text{You can solve using laws of statics.} \\
\text{Ex:} & \text{ Not statically determinate} \\
\text{Can't find } T_1 \text{ & } T_2 \text{ from statics.}
\end{align*}
\]
Need models of behavior (force/deformation relations) to solve statically indeterminate problems.

What problems are statically defi? No single precise answer. \( \begin{bmatrix} \text{Known coeff.} \end{bmatrix} [\begin{bmatrix} \end{bmatrix}] = [\begin{bmatrix} \end{bmatrix}] \)

eqs. are solvable

Rule of thumb:

\#eqs. = \#unknowns

ex) difficult stress

regular hexagon
no connection at H
given F find all bar tensions

FBD of P T1 T2 T3
All joint eqs

\[ 2 \times 6 \text{ eqs} = 12 \text{ eqs} \]

12 unknowns

9 tensions

3 react. forces \[= 12 \text{ unknowns} \]

Need to set up matrix to solve,

Yet the matrix is singular (not solvable)
3/24/03

Today
1. Friction binds
2. Stress & strain

Friction problems.

Given a slow hard motion, how does object move?
(Assume statics.)

\[
\tan(\phi) = \mu
\]
Assume elastic (linear) for most structural analysis.

Linear region

$$\Delta L = \frac{TL}{AE}$$

$E$ = elastic modulus

$$\frac{\Delta L}{L} = \frac{\sigma}{E}$$

III

$$E = \frac{1}{\frac{L}{\Delta L}}$$

$\varepsilon$ = strain, measure of lengthening def. = tension strain

$\sigma$ = stress, measure of force per unit area.
\[ \varepsilon = \text{"epsilon"} \] stress is not a synonym for strain in mechanics.

Redraw curve:

\[ \sigma \]

ultimite strength

prop. limit

yield pt.

\[ \varepsilon \]
typically an engineer wants \( \sigma < \sigma_y \) always.

\[ \varepsilon_t = \text{transverse strain} = \frac{AD}{D_0} \]

\[ \varepsilon_\varepsilon = -2\nu\varepsilon \]

\( \nu = -\varepsilon_\varepsilon / \varepsilon \)
Some numbers:

\[ E = \frac{\sigma}{\varepsilon} \text{ in linear prop. regime} \]

\[
= \begin{bmatrix}
200 \times 10^9 \text{ N/m}^2 \\
30 \times 10^6 \text{ lb/ft}^2 \\
70 \times 10^6 \text{ N/m}^2 \\
10 \times 10^6 \text{ lb/ft}^2
\end{bmatrix} \text{ steel}
\]

\[
= \begin{bmatrix}
30 \times 10^6 \text{ lb/lin}^2 \\
10 \times 10^6 \text{ lb/lin}^2
\end{bmatrix} \text{ Al}
\]

\[ \sigma_y = \begin{bmatrix}
200 \times 10^6 \text{ N/m}^2 \\
30 \times 10^5 \text{ lb/ft}^2 \\
200 \times 10^7 \text{ N/m}^2 \\
30 \times 10^4 \text{ lb/ft}^2
\end{bmatrix} \text{ steel}
\]

\[
\begin{array}{c}
0 \leq \nu \leq 0.5 \\
-1 \leq \nu \leq 0.5
\end{array}
\]

Note: \( E \) at yield is typically \( 1 \times 10^6 \text{ to } 1 \times 10^7 \text{ ksi} \) for most materials.

\( \sigma_y \) usually not much bigger than \( \sigma_y \) (usually 10 - 20% bet. up to 3x)
Stress - force per unit area

\[ \sigma = \frac{T}{A} \]

Normal Stress

\( \sigma > 0 \) - tension

\( \sigma < 0 \) - compression

\( \Delta L \) - extension

\( \Delta L = \frac{AL}{E} \)

\( \sigma = E \varepsilon \)

Shear Stress

\[ \tau = \frac{V}{A} \]

* But tangential forces also exist.

- When forces \( V \) are tangential over an area \( A \)
Ex. plates held together by rivets.

Top half

Fr = equilibrium \( v \) must = \( T \)

Distributed tangential force - which can vary along surface.

\[
T = \frac{V}{A}
\]

Ex.

Oblique cut

area \( A \)

\( T \)

because of this cut + surface it has both tangential + normal stress.

Resolve \( T \) into components of

Shear + normal to surface.

\[
T = \frac{I}{A} \quad T_{\alpha} = T_{e}
\]

On this cut

Normal stress \( \sigma = \frac{T_{n}}{A'} = \frac{1/2}{A'z} = \frac{T}{2A} \)

Shear stress \( \tau = \frac{T_{e}}{A'} = \frac{T}{2A} \)

State of stress on a material surface depends on forces and orientation of surface.
General State of Stress (3D)

shear stress
what is relation btw $T_1 + T_2$?

if No Normal Stress

relationship btw $T_1 + T_2$?

Moment Balance about o

FBD

$T_{Ax}, T_{Ay}, T_{Az}$ = forces = stresses + areas

$T_1 (bxc)$


$2H_0 = -2T_{Ax} (bxc) + \delta (bxc)(T_2)(bxc) = 0$

$(-T_1 + T_2) abc = 0$

$T_1 = T_2$

* shear stress acting on faces must be same

not a scalar or vector, stress turns out to be a matrix.

response to shear stress is a shape change

Hooke's Law for Shear

$T = G \delta$
Shear Modulus

$G = \frac{E}{3(1+\nu)}$
Poisson's Ratio, Young's Mod, Shear Mod - not independent of each other.
TODAY: More stress & strain

Stress: Force per unit area on a surface exposed by a FBD cut. Usually cut plane inside some solid (same idea works for fluids)

\[ \sigma = \frac{F}{A} \]

stress

- a scalar: \( \sigma = \frac{F}{A} \)

- in this context \( \sigma \) is a scalar like "tension"

- is a scalar

the stress vector:

\[ \sigma = \frac{\Delta F}{\Delta A} \]

- for small \( \Delta A \)

- "\( \Delta A \to 0 \)"

- \( \sigma \) will depend on where in solid & \( \theta \)
A component of stress vector: $\sigma = \frac{\Delta F}{\Delta A}$

The stress matrix: $3 \times 3$ matrix

The normal stress $\sigma = \text{comp of } \sigma$ in $\hat{n}$ dir.

Shear stress $\tau$
cut out a small cube

\[ \overrightarrow{h}_x = \hat{j} \]
\[ \overrightarrow{h}_y = \hat{k} \]
\[ \overrightarrow{h}_z = \hat{i} \]

\[ \sigma = \sigma_{xx} \hat{i} + \sigma_{xy} \hat{j} + \sigma_{xz} \hat{k} \]
like wise for other 2 surfaces

\[ [\sigma] \text{ matrix } = \begin{bmatrix}
\sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{bmatrix} \]

\[ \overrightarrow{\sigma} \text{ vector for } \hat{n} = \hat{i} \]

stress tensor:

**Explanation 1**: Huh?

**Expl. 2**: Forget about it you'll learn it in courses like TAM 610, 663 etc.

**Expl. 3**: Analogy
\[ [\sigma] \text{ tensor is to } [e] \text{ like } \]
At some point in space, a solid (protrusion)

Aside on continuum mechanics:

\[ \sigma = \frac{\Delta F}{\Delta \text{area}} \]

Small \( \Delta x \rightarrow 0 \)

\[ \text{atomic dim} \longrightarrow c \text{ main scale} \]

\( \log \sigma (\text{in mat}) \)

"Small dim" \\
"Side of thing"

\[ \frac{\partial F}{\partial \sigma} \]

\[ \frac{\partial F}{\partial x} \]

A function that has as output \([A] \text{ input}\]

\[ \frac{\partial}{\partial x} \frac{1}{x} = \frac{1}{x^2} \]

\[ \exp(x) \]

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March 31/03  R  A. Poisson's ratio

Today: B. Collection of beam in tension

A. Poisson's ratio

Ex1 Metals $\nu \approx 0.3$
Rubber $\nu \approx 0.5$
Jello $\nu \approx 0.49$

$0.5 \leq \nu$ corresponds to constant volume

Recall $\nu = -\frac{E_I}{E_g}$

Given: $h_1, h_2, E_1, E_2, A_1, A_2, F, L, a$
Find deflection at pt. B.
Treat C & D as hinges.
FBD of {eq}CDB \ ( \in \ \mathbb{R}^3 \ ) \ \Rightarrow \ \text{IC} \ \text{in} \ \text{B}

\text{Sign Convention: Tension is positive}

\begin{align*}
\Delta l &= 0, \ \Delta F_i = 0 \Rightarrow T_D = \frac{-2}{l} F \\
T_C &= \frac{2(l-a)}{l} F
\end{align*}

Mech. Properties, Cauchy Laws

\begin{align*}
\text{strain} &= \frac{\text{stress}}{E} \\
\varepsilon_0 &= \frac{\sigma_0}{E}
\end{align*}

\[ s = \frac{T x}{AE} \]

\[ \sigma_0 = \frac{F}{A_2} \frac{1}{E_1} \]

\[ \delta_0 = \frac{-\alpha h_2}{l A_2 E_2} F \]

\[ \delta_0 = \frac{2a}{l} \frac{F}{A_2} \]

Mechanics, Geometry

Assume small slopes \Rightarrow neglect non-linear displacements

Increase in length is positive
\[ \delta_c = \frac{-Q - a}{2A_1E_1} \cdot F \]

(3) apply to (2)

\[ \delta_P = \text{mess...} \]

involving things we know

---

Mechanically:

\[ \frac{T_1}{2} \uparrow \quad \uparrow_{T_2} \quad \uparrow_{T_1/2} \]

\[ \downarrow_{F} \]

\[ \delta F_i = 0, \quad \delta M = 0 \Rightarrow \]

\[ T_1 + T_2 = F \]

can't find \( T_1 \) and \( T_2 \) from statics alone \( \Rightarrow \)

statically indeterminate

---

Given \( F, A_1, E_1, A_2, E_2, l \)

Find \( \delta_c \)

Symmetry \( \Rightarrow \) no rotation
Kinematics / Geometry

\[ \delta_1 = \delta_2 \]  \hspace{1cm} (2)

Material Properties

\[ T_1 = \frac{E_1 A_1}{l} \delta_1 \] \hspace{1cm} (3)
\[ T_2 = \frac{E_2 A_2}{l} \delta_2 \]

1, 2, 3 to get answer.

Graphical derivation

\[ \sigma = \frac{F}{A} \]

\[ F = T_1 + T_2 \]
\[ \frac{E_2 A_2}{l} \delta \]
\[ F = \left[ \frac{E_1 A_1}{l} + \frac{E_2 A_2}{l} \right] \delta \]
TODAY: More tension

\[ \varepsilon = \frac{\Delta L}{L} \] elongation strain

\[ \Delta L = \text{increase} \]

\[ \Delta x = \text{change in length} \]

\[ \sigma = \frac{F}{A} \] Linear elastic constitutive law

"Young"

\[ T = \frac{\sigma A}{A} \]

\[ T = k \sigma \]

\[ \varepsilon = \frac{1}{E} \sigma \]

\[ \Delta \sigma = \frac{F}{A} \]

Think of rod as a spring

\[ k = \frac{F}{\Delta L} \]

\[ \Delta L = \frac{F}{k} \]

\[ \Delta L = \frac{F}{E} \]

\[ \Delta L = \frac{F}{E} \]

\[ \varepsilon_v = \frac{v \Delta L}{L} \]

Poisson's ratio
Ex. 1)  
\[ E A_1 \rightarrow T \rightarrow \delta \]

Ex. 2)  
\[ F \rightarrow E A_2 \]

Ex. 1)  
\[ l_1 + l_2 + a_1 + a_2 \rightarrow 1 \]

Ex. 2)  
\[ \delta \]

FBDS

\[ T \leftarrow \square \rightarrow T \]

Geometry

\[ \delta = \delta_1 + \delta_2 \]

\[ \delta = \left[ \frac{A_1}{E_1} + \frac{A_2}{E_2} \right] T \]

\[ T = \frac{\delta}{\frac{1}{E_1} + \frac{1}{E_2}} \]

\[ K = \frac{1}{E_1} + \frac{1}{E_2} \]
No prestress

Assume: \( T_1 = T_2 = 0 \)

\[ 5_p = -5 \]

\[ \gamma \]

\[ \sigma = \frac{T}{A} \]

\[ \varepsilon \]

\[ \delta \]

\[ \alpha \]

\[ \beta \]

\[ \gamma \]

\[ \delta \]

\[ \alpha \]

\[ \beta \]

\[ \gamma \]

\[ \delta \]
4/7/03  (Ruia not Hai)

TODAY

1) Trig. add. formulas
2) Stress on inclined sections

TRIG. ASIDE

Consider 2 unit vectors \( \mathbf{a} \) and \( \mathbf{b} \):

\[
\begin{align*}
\mathbf{a} &= \cos \theta \hat{i} + \sin \theta \hat{j} \\
\mathbf{b} &= \cos \phi \hat{i} + \sin \phi \hat{j}
\end{align*}
\]

1) \( \mathbf{a} \cdot \mathbf{b} = a \cdot b \)

\[
\begin{align*}
|\mathbf{a}| = |\mathbf{b}| = 1 \\
\cos \theta = \cos \phi \cos \theta + \sin \phi \sin \theta \\
\theta = \phi - \phi
\end{align*}
\]

2) \( \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \)

\[
\begin{align*}
\mathbf{a} = \left(\cos \theta \hat{i} + \sin \theta \hat{j}\right) \\
\mathbf{b} = \left(\cos \phi \hat{i} + \sin \phi \hat{j}\right)
\end{align*}
\]

\[
\begin{align*}
\mathbf{a} \times \mathbf{b} &= \left| \mathbf{a} \right| \left| \mathbf{b} \right| \sin \theta \hat{k} \\
&= \left(\cos \theta \hat{i} + \sin \theta \hat{j}\right) \left(\cos \phi \hat{i} + \sin \phi \hat{j}\right) \sin \theta \hat{k}
\end{align*}
\]

\[
\begin{align*}
\mathbf{a} \times \mathbf{b} &= \left(\cos \theta \hat{i} + \sin \theta \hat{j}\right) \left(\cos \phi \hat{i} + \sin \phi \hat{j}\right) \\
&= \left(\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} + \sin \theta \cos \phi \hat{k}\right) \\
&= \left(\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \cos \phi \hat{k}\right)
\end{align*}
\]

\[
\begin{align*}
\cos(\theta - \phi) &= \cos \theta \cos \phi + \sin \theta \sin \phi \\
\cos(\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi
\end{align*}
\]
Stresses on inclined sections

Ex) 2.6-1

\[ \sigma = \frac{P}{A} \leq \sigma_{ult} \]

To avoid failure,
\[ \sigma_{ult} \Rightarrow P \leq \sigma_{ult} A \]

\[ P = 16000 \left( \frac{2}{3} \right) (4 \text{ in}) \]

\[ P \leq 64 \times 10^3 \text{ lb} \]

Hey! no shear on surface w/ normal in \( \hat{t} \) direction. What about crooked surfaces?

\[ \hat{n} = \cos \theta \hat{i} + \sin \theta \hat{j} \neq \hat{i} \]

Area of surface:

\[ A = \frac{A_o \cdot \cos \theta}{\hat{n}} \]

\[ A = \int_{0}^{\theta_0} \frac{A_o \cdot \cos \theta}{\hat{n}} \ d\theta \]

\[ \theta = 0 \quad \theta = \pi/4 \]
Stress vector \( \sigma = \frac{P}{A} \hat{i} \)

force on cut surface = \( \frac{P \cos \theta}{A_0} \hat{i} \)

Want to look at normal & shear comp's.

\[
\sigma \hat{n} = \sigma_n \hat{n} + \gamma \hat{t} \\
\text{a unit vector tangent to surface}
\]

(see text for FBD approach)

\[
\sigma = \sigma_n \\
\sigma \cdot \hat{n} = \sigma_n \cdot \hat{n} \\
(\sigma_n \hat{n} + \gamma \hat{t}) \cdot \hat{n} = \left( \frac{P \cos \theta}{A_0} \right) \cdot \hat{n}
\]

\[
\sigma_n = \frac{F}{A} \cos^2 \theta
\]

\[
\sigma_n = \frac{F}{A} \cos^2 \theta
\]

Note \( \sigma_n \) is max for \( \theta = 0 \Rightarrow \) answer used before for tension stress

\[
\sigma \cdot \hat{t} = \sigma \cdot \hat{t} \\
(\sigma_n \hat{n} + \gamma \hat{t}) \cdot \hat{t} = \left( \frac{F \cos \theta}{A_0} \right) \cdot \hat{t}
\]

\[
\gamma = \frac{F \cos \theta}{A_0 \sin \theta}
\]

\[
\gamma = \frac{F \cos \theta}{A_0 \sin \theta}
\]
Can use our trig formulas
\[ \sin \theta \cos \theta = \frac{1}{2} \left[ \sin 2 \theta + \sin \theta + \cos \theta \right] \]
\[ = \frac{\sin 2 \theta}{2} \]
\[ \gamma = \frac{P}{A_o} \cos \theta \sin \theta = \frac{P}{A_o} \frac{\sin 2 \theta}{2} \]

\[ \sigma = \ldots \ldots \left( \frac{1}{2} + \frac{\cos 2 \theta}{2} \right) \frac{P}{A_o} \]
\[ \text{Take smallest} \]
Two answers
1) \[ P \leq 64000 \text{ lbf} \]
2) \[ P \leq 72000 \text{ lbf} \]

Shear stress is biggest when \( \frac{\sin 2 \theta}{2} \) biggest
\[ \Rightarrow \theta = \pi/4 \quad (\sin 2 \theta = 1) \]
Shear biggest on surface at 45° from tension dir.

\[ \cos^2 \theta = \frac{\cos \theta + \sin \theta}{2} \]
\[ + \sin^2 \theta = \frac{\sin \theta + 1}{2} \]
\[ = \frac{(\cos^2 \theta + \sin^2 \theta)}{2} \]
\[ + \cos^2 \theta - \sin^2 \theta \]
4/9/03

Today: 1) Tension  2) Torsion

TENSION (Cont'd)

Recall:

\[ \sigma_0 = \frac{F}{A_0} \]

\[ \sigma = \frac{F}{A} \]

\[ \tau = \frac{\tau}{A} \]

\[ \varepsilon = \frac{\Delta L}{L_0} \]

\[ \delta = \frac{\Delta x}{x_0} \]

Why interesting?

A. Might want to know surface w/ biggest \( \sigma \) (\( \theta = 0 \)) or biggest \( \tau \) (\( \theta = \frac{\pi}{4} \))

B. For a weld or glue joint, \( \sigma \) & \( \tau \) may be specified for a given \( \theta \)

\[ \tau = \sigma_0 \sin \theta \cos \theta = \frac{\sigma_0}{\sin 2\theta} \]
TORSION

WARNING!! For round bars only!! If you care about not-round bars it is incorrect to use the formulas below by just recalculating J!!!

TORSION OF ROUND BARS (possibly hollow)

\[ T = \text{torque} \]

\[ \phi = \text{net rotation of one end rel. to other} \]

\[ \phi = \frac{T R}{G (some measure of modulus)} \]

\[ \sigma = \frac{\varepsilon}{E} \]

\[ G = \text{the shear modulus} \]

\[ T = \text{torque} \]

\[ t_{\text{max}} = \frac{G (\pi T)}{2} \]

What about stress?

\[ \tau = \frac{T (where i=6)}{(t \times g) (\text{torque})} \]
4/11/03 1

TOPIC: TORSION

TORSION

ex/ paperclip

⇒
plastic bend

⇒
Torsion specimen

perspective

hold in one hand

Assume deformation
1) like a bunch of stacked pennies each rotated sliding compared to neighbor

question:

Given d, d, material w. Find F to twist given θ.

Recall from last lecture

\[ T = \frac{\pi d^3}{4} \theta \]

Torsion Theory

(Geometry, Mat. props, Mechanic)

Geometry

Assume deformation
1) like a bunch of stacked pennies each rotated sliding compared to neighbor.
4/11/03

---

Material Properties:

Linear elastic:

\[ \tau = G \gamma \]

Shear modulus:

\[ \gamma = \frac{\tau}{\tau} \]

---

Rectangular drawn on outside

\[ s = \text{radius from center line} \]

\[ \gamma = \frac{(d\phi)g}{dl} = \frac{(d\phi)g}{dl} \]

"gamma"

---

Mechanics

recall "Tension = \sigma A"

we want something like this for torsion.
Look at an end cap (internal cut surface)

\[ T = \int \sigma T \, dA \]

All linear elastic torsion formulas come from \( \theta, \theta, \theta \).

Algebra & Calculus:

\[ T = \int \sigma T \, dA \]

\[ = \int \frac{G \theta}{2} \, dA \]

\[ = \frac{G I}{2} \int \frac{G \theta}{2} \, dA \]

\[ = \frac{G I}{2} \int \frac{G \theta}{2} \, dA \]

\[ T = \frac{G I}{2} \int \frac{G \theta}{2} \, dA \]

= polar area moment of inertia
Theory only holds for round bars (penny stack is geometry is otherwise wrong.)

\[ J \text{ for round bar} \]

\[
\int a^2 \, da = \int_0^{2\pi} \int_0^a a^2 \, da \, dz = \pi r^4 \]

\[ J = \frac{\pi r^4}{2} \]

twice the radius =

16 times the torsion stiffness

\[ \phi = \frac{\pi}{10} \]

\[ \gamma = \frac{T\phi}{GJ} \]

Back to paper clip

\[ T = \frac{6\phi J}{a} \]

\[ wF = \frac{6\phi J}{a} \]

w = .5", \quad \phi = 1"

\[ \phi = 12 \times 10^{-6} \text{ in/in} \]

\[ J = \frac{\pi}{4} \left( \frac{1}{a} \right)^4 \text{ in}^4 \]

\[ \gamma = \frac{T\phi}{J} \]
4/14/03

Torsion (w/ Tension Review)

**Tension**

**Mechanics**

\[ P = \sigma A \]

\[ \left( \sigma = E \varepsilon \right) \]

\[ \varepsilon = \frac{\delta}{L} \]

\[ \sigma = E \varepsilon \]

\[ \left[ \varepsilon = \frac{\delta}{E} + \alpha \Delta T \right] \]

**Torsion**

\[ T = S \gamma \tau dA \]

\[ \gamma = \frac{\tau dA}{J} \]

\[ J = S \rho dA \rho^2 \sin \alpha \]

\[ \gamma = \frac{M I}{J G} \]

\[ \tau = \frac{T A}{J G} \]
<table>
<thead>
<tr>
<th>Bar in Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1 = p_2$, $\delta = \delta_1 + \delta_2$</td>
</tr>
<tr>
<td>$K_{\text{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$</td>
</tr>
<tr>
<td>$k_1 = \frac{E_1}{l_1}$</td>
</tr>
<tr>
<td>$k_2 = \frac{E_2}{l_2}$</td>
</tr>
<tr>
<td>$C = C_1 + C_2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bar in Parallel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1 = T_2$, $\phi = \phi_1 + \phi_2$</td>
</tr>
<tr>
<td>$K_{\text{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$</td>
</tr>
<tr>
<td>$k_1 = \frac{E_1}{l_1}$, $k_2 = \frac{E_2}{l_2}$</td>
</tr>
<tr>
<td>$C = C_1 + C_2$, $C = \frac{1}{\nu_1 + \nu_2}$</td>
</tr>
</tbody>
</table>

$\phi_1 = \phi_2$

$T = T_1 + T_2$

$K_{\text{eff}} = k_1 + k_2$

$C_{\text{eff}} = \frac{1}{\nu_1 + \nu_2}$
Statically ind. problems require mixing up different parts of the theory (hint: be especially careful w/ signs).

Given all geometry & properties, find $\sigma_i$.

Geometry temp. rise

\[ \delta_1, \delta_2 \text{ are increase in length} \]
\[ \text{we expect that if } \frac{A_i}{E_i}, \alpha_i \text{ big & } AT \text{ big then } \delta_i < 0 \]

1. $\delta_1 + \delta_2 = \delta_3$ [if $\sigma_f$ exists]

2. $\delta_i = \left[ \frac{\alpha_i}{E_i} + (\alpha T) \frac{A_i}{A_f} \right] l_i$

3. $\frac{p}{\sigma_f} = \frac{p}{\sigma_{f_2}}$

\[ \sigma_f, A_f = \sigma_{f_2} A_2 \]
Apply 3 to 2, 2 to 1

⇒ 1 eqn. for 1 unknown \( \sigma_1 \)

\[ \delta_1 + \delta_2 = \delta_3 \]

⇒ \[ \left[ \frac{\sigma_1}{E_1} + \Delta T \alpha_3 \right] \delta_1 + \left[ \frac{\sigma_1 A_1}{A_2 E_2} + \Delta T \alpha_2 \right] \delta_2 = \delta_3 \]

1 eqn. in 1 unknown \( \sigma_1 \)

\( \sigma_1 = \text{---} \)

if answer is \( \sigma_1 > 0 \)
replace \( \sigma_1 \) with \( \sigma_1 = 0 \)

if answer is \( \sigma_1 < 0 \)
leave alone \( \sigma_1 = \)

Torsion indeterminate problem

Rigid bar AB is welded to end of shaft DC. When gap DC is covered \( (\phi_{DC} = \phi_3) \) then contact made with outer tube.

Given \( F \& \) all gecm 8 props. Find \( \gamma \) in tube (2) outer wall.
Today
1) Torsion ex cont'd
2) More about tension & torsion
3) Long narrow thin
4) Intro to bending.

end view:

Given all geometry & properties & F find max stress in bar 2. Assume F big enough to close gap:

\[
\frac{(Fw)l}{r_1 G_1} > \phi_g
\]
Geometry

\[ \phi_2 = \phi_1 - \phi_5 \]

Mechanics

FBP of rigid bar & shafts

\[ T_1 = \frac{J_1 \phi_1}{l_1 \sigma_1}, \quad T_2 = \frac{J_2 \phi_2}{l_2 \sigma_2} \]

\[ l_1 = l_2 \]

\[ J_1 = \pi r_1^4 l_1 \]

\[ J_2 = \frac{\pi}{2} (r_3^4 - r_2^4) \]

\[ = \frac{\pi}{2} (r_3^4 - r_2^4) r_3^2 \approx \frac{\pi}{2} r_3^2 \quad \text{if} \quad r_3 - r_2 \to 0 \]

\[ r = r_2 \to 0 \]

Note: We neglect fumy effects at ends

\[ T_2 \text{ comes from } 2 \text{ nubs} \]

\[ -T_1 + -T_2 = F w \]

\[ \gamma_{\text{max}} = \left| \frac{T_2 \gamma_{\text{max}}}{J_2} \right| \]

Note: cannot solve for \( T_2 \)
Tension $P$, Torsion $T$ can vary along length, as can properties & geometry.

Example:

Given $E$, $A(y)$, $L$, $P$, what is change in length?

$\sigma(y) = \frac{P + w(y)}{A(y)}$

$\epsilon(y) = \frac{\sigma(y)}{E}$

$\delta = \int_{y_1}^{y_2} \sqrt{\frac{A(y)}{E}} \, ds$
Likewise for tension

\[ \phi_{\text{tot}} = \int \frac{T}{J} \, dl' \]

HW hint:
1) "Rate of twist" means it is average rate of twist = \( \phi_{\text{tot}} / l \)
2) Exercise your integration skills.

Long narrow things

Some structure/machine

We are dealing with these one at a time.
Now, new topic, bending:
Most important of all:

ex)

\[ \text{We start to find } U \& M \text{ at cut.} \]

\[ \text{important.} \]
4/21/03

TODAY: 1) Q&A, 2) V, M diagrams

Q. Shear stress

\[ \sigma \sigma_0 = \frac{P}{A} \]

\[ \gamma = \gamma_0 \cos \theta \sin \theta \]

\[ \sin \frac{B}{2} = \cos \frac{B}{2} = \frac{\sqrt{2}}{2} \]

Q. Thermal Expansion

\[ \delta = \alpha (\Delta T) L + \frac{A_f}{E} \]

\[ \triangledown \text{ Mat Prog.} \]
"Sign Convention" issues:
Need to have pre-defined dots of up \( \uparrow \) & clockwise \( \downarrow \).
(A drawing defines this.)

**Sign Convention**

\[
N = N_1, \quad V = V
\]
(but vectors \( \neq \))

- Smiling beam
- Distributed load \( \mathbf{F} \) & \( \mathbf{F} \) down

- Moment is \( + \) if counterclockwise
  - Shear is \( + \) if counterclockwise rotation: (down on right, up on left)

\[
\frac{A_0}{A} = \cos \theta
\]
\[
\frac{A_0}{A} = \sin \theta
\]

- \( \nabla \) M diagrams
A common issue: How do V & M depend on psi in beam?

\[ V(x) = ?,\quad M(x) = ? \]

Example:

\[ L = 10',\quad F = 100\text{ lb} \]

"weld", "built in"

called "clamped-free" beam, or "cantilever" beam

For:

\[ \sum F_y = 0 \quad V(x) = F = 100\text{ lb} \quad \text{const.} \]

\[ \sum M(x) = 0 \quad -MA - F(x-L) = 0 \]

\[ M(x) = 100\text{ lb}(10-x) \]

\[ V = 100\text{ lb} \]

\[ M = 100\text{ lb}\cdot\text{ft} \]
Alternative Soln.  

FBD of whole beam  

\[ F = 0 \]  

\[ V_0 = V(0) \]  

\[ \sum N_0 = 0, \quad \sum F_y = 0 \Rightarrow V_0 = 100 \text{ lb} \]  

Now make cut at \( x_1 \) but look at left piece  

\[ V_0 = 100 \text{ lb} \]  

\[ N_0 = -1000 \text{ ft lb} \]  

\[ V(x_1) \]  

\[ \frac{dF_1}{dx} \]  

\[ \sum F_y = 0 \Rightarrow V(x) = 100 \text{ lb} \cos x \]  

\[ \sum M_{10} = 0 \Rightarrow \]  

\[ -(-1000 \text{ ft lb}) - V(x_1) \cdot x \]  

\[ \Rightarrow 0 \]  

\[ M(x) = -1000 \text{ ft lb} \]  

\[ + (100 \text{ lb}) \cdot x \]  

Same as before  

Novel: Same answer for left FBD or right FBD.
Recall smiling beam

\[ \text{L} \downarrow \text{L} \uparrow \text{F} \to x \]

Ex) 2

"Simply supported" beam

FBD of whole beam

\[ \text{cut at } x \]

First \[ x < L/2 \]

\[ \text{max} \]

\[ \text{max} \]
\[ \sum F_x = 0 \Rightarrow 0 = 0 \]

\[ \sum F_y = 0 \Rightarrow \frac{32E}{8} - 2x - V(x) = 0 \]

\[ V(x) = \frac{32E}{8} - 2x \quad x \leq \frac{1}{2} \]

\[ \sum M_x = 0 \Rightarrow \Pi(x) - \frac{32E}{8} \cdot x + \frac{E}{2} \cdot x = 0 \]

\[ 
\begin{align*}
\Pi(x) &= \frac{32E}{8} x - 2x \frac{1}{2} \\
x &\leq \frac{1}{2}
\end{align*}
\]

Next look at \( x > \frac{1}{2} \)

\[ \pi(x) \]

\[ V(x) \uparrow \frac{E}{8} \]

\[ x > \frac{1}{2} \]

\[ \sum F_y = 0 \Rightarrow V(x) = -24E \quad x \geq \frac{1}{2} \]

\[ \sum M_x = 0 \Rightarrow -\Pi(x) + \frac{4}{3} \left( 2 - x \right) = 0 \]

\[ \Pi(x) = \frac{32}{3} \left( 2 - x \right) \quad x \geq \frac{1}{2} \]

Graphs:

[Graphs showing different sections with arrows and values indicated.]
Fact: Always \( \frac{dM}{dx} = V(x) \)

\[ \frac{dV(x)}{dx} = -2 \sigma(x) \]

\[ V(x) = V_0 + \int_0^x (-\sigma(x')) \, dx' \]

\[ M(x) = M_0 + \int_0^x V(x') \, dx' \]

Need to worry about concentrated loads & couples.

\[ F \]

\[ L \]

\[ \Delta V = F \]

discontinuity in slope
Note: \[ \frac{dv}{dx} = -\varepsilon \]
\[ \frac{d\eta}{dx} = \eta \]

is a shortcut.

Can always use FBD instead.
TODAY: \( \sigma, \tau \) in beams

Procedure: Use statics to find \( N, V \)

\[ V, M \text{ at pt. } x \text{ of interest} \]

Goal: to find stresses.

Theory of stresses in beams

Geometric assumption

plane-normal sections remain plane & normal
Similarly, we have:

\[
\frac{\Delta l}{l_0} = \frac{S-y}{S} \Rightarrow \epsilon = \frac{\Delta l}{l_0}
\]

\[
\Delta l = \frac{S-y}{S} l_0
\]

\[
\epsilon = \frac{\Delta l}{l_0} = \frac{S-y}{S} \frac{l_0}{l_0} = \frac{S-y}{S} - 1
\]

\[
\epsilon = \frac{S-y}{S} - 1 = \frac{S-y}{S} - \frac{S-y}{S} + y
\]

\[
\epsilon = \frac{-y}{S}
\]

\[
\epsilon = \frac{-y K}{S}
\]

\[
M = -\int y \sigma \, dA
\]

\[
= -\int y E (y K) \, dA
\]

\[= KE \int y^2 \, dA = I_y K
\]

\[
\kappa = \frac{1}{S}
\]

\[
M = EI \kappa
\]
Geometry: $E = -y/\rho \Rightarrow p = \rho \cdot \rho = E \cdot u'' \frac{d^2 x}{dx^2}$

Mat. Props.: $\sigma = E \cdot \varepsilon$; Mechanics: $M = E \cdot I = E \cdot I \cdot u''$

$I = \text{area moment of inertia} = \int y^2 \, dA$

Location of neutral axis: $\text{Net tension} \Rightarrow \sigma = \int_0^x dA = \int y \, dx = 0 \Rightarrow \text{neutral axis at centroid}$

of interest is stress: $\sigma = E \cdot \varepsilon = E \cdot (-y \cdot \kappa)$, $\sigma = \frac{-My}{I}$

$\sigma_{\text{max}} = \frac{M y_{\text{max}}}{I} = \frac{M}{I}$

$S$ = geometric quantity; $S = \text{section modulus} = \frac{I}{y_{\text{max}}}$

Example: What is the yield stress of Andy’s ruler? Bend ruler to radius of 2 in (50 in thick) to get permanent $\Delta$.

$r = 2'' \Rightarrow y = \frac{1}{50} \text{ in}$, $y_{\text{yield strain}} = \frac{\sigma_{\text{max}}}{E} = \frac{1}{50} \frac{1}{2} \text{ in}$

$\sigma_y = \text{yield stress} = \frac{E \cdot y_{\text{yield strain}}}{2''}$

$\sigma_y = 10^9 \text{ Pa} \text{ or } 15 \times 10^6 \text{ lb/in}^2$

$E = 200 \times 10^9 \text{ Pa}$

$\kappa = 50,000 \text{ lb/in}^2$ hard steel

Tedious Skill: Calculating $I$ for various cross sections
4/28/03

Today: EI deflection

Max M at left end = 100.16

Vectors:
V = 100 lb.
M = -1000 lb-ft

Need to find I, y value
(Find G = N/M)

Area | I
---|---
1 | 1/12 lb-ft²
2 | 3 lb-ft²

Area = area of region
Z_i = vert. loc. of centroid of area
I_i = area moment of inertia about centroid

Question:
σ (tension) max = ?
σ (bending) max = ?

First find M (EI)
Where is $G$? Z value of centroid of composite.

$$Z_G = \frac{\sum A_i z_i}{\sum A_i} = \frac{3\frac{1}{4} [3.5 \text{ in} + 1.5 \text{ in}]}{(3 + 3) \text{ in}^2}$$

$$= \frac{15 \text{ in}}{6} = 2.5 \text{ in}.$$

$$I = \sum (I_i + d_i^2 A_i)$$

Parallel axis' thm,

$$= \sum \left[ \left(4 + 1^2 \cdot 3 \right) + \left(\frac{4}{4} + 1^2 \cdot 3 \right) \right] \text{ in}^4$$

$$= \frac{17}{2} \text{ in}^4$$

For tension $|Y_{max}| = 1.5 \text{ in}$

For comp. $|Y_{max}| = 2.5 \text{ in}$

$$\sigma_{max} = \left| \frac{P_{max}}{I} \right|$$

$$= \text{arith. of } \sigma_{max}$$

$$\left| \sigma_{max} \right| = \left| \frac{P_{max}}{I} \right|$$

(because bottom surface is further from $G$ than top surface)
What is shear stress in beam?

\[
\bar{T} = \text{ave. shear stress} = \frac{V}{A} = \text{sum} \left[ \begin{array}{c}
\end{array} \right]
\]

not constant in cross section

varies according to

\[
\approx \frac{V_0}{I}
\]

which is covered in lab but not in the lecture

\[\text{This formula not on final exam.}\]

\[
\text{Deflection in beams}
\]

\[
\sqrt{\pi} = \frac{M}{EI}
\]

\[
K = \frac{M}{EI}
\]

\[
\frac{d^2u}{dx^2} = \frac{M}{EI}
\]

\[
u'' = \frac{M}{EI}
\]

\[
K = \frac{\pi}{2}
\]

\[
I = \frac{b^4}{12}
\]
TODAY:
1) Deflection of beams
2) Something you've always wanted.

Deflection at end = ?
(Know all nodes)

\[ M = \frac{F}{2} \]

\[ \sum F_y = 0 \Rightarrow V = F \]

\[ \sum M_x = 0 \Rightarrow -F(x-x) - M = 0 \]

\[ M = -F(x-x) \]

\[ u(x) = \int u'(x') dx' + C(x) \]

\[ u'' = \frac{1}{EI} (F(x - x)) \]

\[ u' = \int^x u''(x') dx' + C(x) \]

\[ u = \frac{1}{EI} \left( \frac{Fx^2}{2} - Fx + C(x) \right) \]

B.C.:
Value of \( u \) at left edge = 0
Deflection at end = \( u(l) \).

\[ u(l) = \frac{F}{EI} \left( \frac{l^3}{6} - \frac{l^3}{2} \right) \]

\[ u(l) = -\frac{F \cdot l^3}{3EI} \]

\[ \begin{align*}
\text{(ex)} \quad l &= 8', \quad b = 2'' \quad h = 4'' \\
F &= 100 \text{ lb} \\
E &= 10^6 \text{ psi} \\
I &= \frac{1}{12}bh^3 = \frac{1}{12} \cdot 2 \cdot 4^3 \text{ in}^4 \\
I &= \frac{32}{3} \text{ in}^4
\end{align*} \]

\[ u(l) = -\frac{(100 \text{ lb}) \cdot (8+4)^3}{3 \cdot (10 \text{ in}) \cdot \left( \frac{32}{3} \text{ in}^4 \right)} \]

\[ = -\frac{100 \cdot 8^3}{3 \cdot 10^3 \cdot 32/3} \cdot \frac{4^3}{1 \text{ in}^4} \]

\[ = -\frac{100 \cdot 2^9}{3 \cdot 2^5} \cdot \frac{4^3}{1 \text{ in}^4} \]

\[ = -10^{-4} \cdot 2^{10} \cdot 3^3 \text{ in} \]

\[ \approx 2 \times 10^{-3} \text{ in} \]

\[ 2^9 \approx 10^3 \]

\[ 3^2 \approx 10 \]

\[ u = -10^{-4} \cdot 10^3 \cdot 10^{-3} \text{ in} \]

\[ = -3 \text{ in} \]

\[ \text{(ex)} \text{ same 2 x 4 sideway,} \]

\[ \text{Key variable is I more flexible by form of beam/fold.} \]

\[ \frac{0.8^2/12}{2.4^2/12} \]

\[ = \frac{2}{27} \]

\[ = \frac{4}{27} \]

\[ \approx 4 \text{ times stiffer.} \]
How much bigger is deflection of case B than case A.

Aside: nominal 2x4

Trick: symmetric
(cantilever) \( u(x) = \frac{F x^3}{3EI} \)

simply supported, load in middle \( u \left( \frac{L}{2} \right) = \frac{(F/2)(L/2)^3}{3EI} = \frac{FL^3}{48EI} \)

& 6A

bend formula

\[ K = \frac{d^2u/dx^2}{\sqrt{1 + (du/dx)^2}} \]

smalls hape

\( ds = dx \)

\( \theta = \tan \theta = u' \)

circle:

\[ 9d\theta = ds \]

\[ \frac{d\theta}{ds} = \frac{du}{dx} (\frac{dx}{du})^{-1} = u'' \]