AN ELEMENTARY TREATMENT

OF THE THEORY OF

SPINNING TOPS

AND

GYROSCOPIC MOTION

BY

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WITH ILLUSTRATIONS

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When the blunt peg is used this (practically) horizontal motion does not continue so long: and, in general, a top with a fine point or with a long leg will spin at a greater angle to the vertical than one with either a blunt point or short leg.

A loaded sphere when spun on a rough surface also presents a curious contradiction.

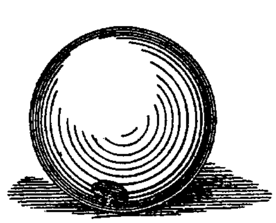

FIG. V. (a).

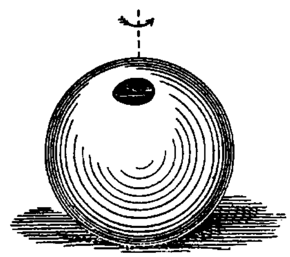

FIG. V. (b).

If, for example, a hole is made in the side of a croquet ball and filled up with lead, when placed on a table the ball will settle down to the position where the lead touches the table (Fig. V. *a*). But if a really good spin be given to the ball the loaded part will persistently rise, as indicated in Fig. V. (*b*), and under some conditions may get to the position where it is at the highest point of the sphere. (See Appendix IV.)

Many of the tops we have been accustomed to spin are hollow. Has it ever occurred to the reader to ask what would be the effect of filling a tin top with water, and making it water-tight? The answer that occurs to most minds at once is probably that it would spin much better; it is heavier, and

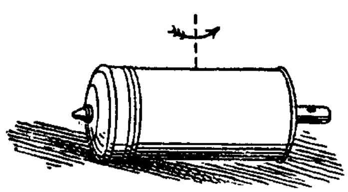

FIG. VI.—Top full.

FIG. VII.—Top empty.

there is more of it generally. Let the experiment be made. Figs. VI., VII. represent two tops which are in every particular the same, except that the left hand one is full of water hermetically sealed. The empty one, if spun in the ordinary way, will continue to spin in an upright position; the other one will lie down on its side at once, and spin violently lying at full length on the table. Some such tops are a little uncertain which to do. That in Fig. VIII. has been constructed so that the head can be unscrewed and water poured in. If empty it spins very well, whether a big or little spin be originally given to it. When it is full of water a little spin will only result in the top falling to the ground; a good spin will keep it upright in spite of the water. Such tops can be readily constructed out of small tins or similar receptacles capable of being soldered and made water-tight.

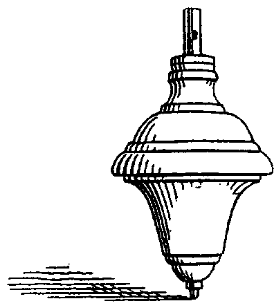

FIG. VIII.

Figs. IX. and X. represent two hollow china eggs, exactly similar, with the exception that one has been filled with water and the hole stopped up with sealing wax. If they are laid down on the table, and a spin is given to each of them about a vertical axis, they will behave in entirely different ways. The empty one jumps up briskly on its end and continues to spin in that position for a long time; the other will spin slowly on its side for a short time and continue this uninteresting motion till it stops: or if stopped prematurely by laying a finger on it, will begin to spin again on removing the finger.

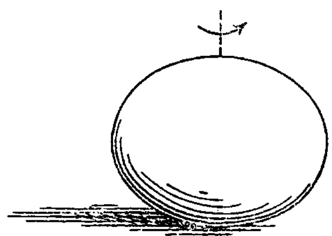

FIG. IX.—Egg full.

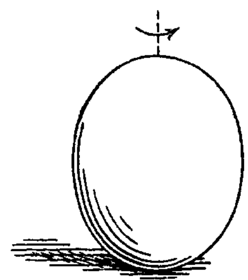

FIG. X.—Egg empty.

A real egg, if unboiled, and especially one which has had its yolk thoroughly shaken up into liquid form, will behave in precisely the same way; but a hard boiled egg will spin up on its end at once (especially if the table is rough) and continue spinning for a long time. A similar phenomenon can also be observed with acorns at a time when they are lying thick in the roads. If they are kicked, in any way whatever, they almost invariably skid along for a little way and then

[20.] Should we get precession? If so, how would the axle move when the motion is steady?

[21.] What is the condition that a top should *fall* to its steady position? What conduces to this condition?

[22.] Discuss the rising of that form of top where the "body" to which the spin is given revolves freely on a spindle carrying the toe; namely, where the toe and spinning body are not rigidly connected. Does it arrive at steady motion more quickly or more slowly than the other kind?

48. Tendency to spin about the greatest axis caused by gyroscopic resistance. The discussion given in Art. 47 explains the behaviour of the whip-top and loaded sphere* described in the Introductory Chapter, as also that of the acorns and hard-boiled eggs: for the friction at the point of contact tends to hurry precession, but, instead of doing so, owing to gyroscopic resistance, it turns the body in a direction which raises the centre of gravity, as in the case of the top. The phenomenon exhibited by the top of Fig. VI., and by the egg of Fig. IX., is caused by a tendency of the liquid to spin about its *shortest* axis, which overcomes the effect of friction tending to bring the top on to its longest axis with the centre of gravity raised. To understand this thoroughly let us consider under what conditions a body will spin about its shortest axis in preference to any other.

49. Tendency to spin about the least axis caused by centrifugal force. If a rigid body, with three perpendicular axes of symmetry, is free to turn in any direction about its centre of gravity G, and rotation be continuously given to the body about a fixed direction through G, for example the vertical, *but no other forces act on it*, the body will set itself so that its *least* axis through G becomes vertical, and it will spin stably in this position.

For every particle composing the body tends (owing to what is frequently called centrifugal force) to separate itself as far as possible from the vertical axis, and so increase the moment of inertia about that axis. Hence the body will only spin stably when the moment of inertia is as great as possible; namely when the *least axis* of the body through G is in a vertical position.

This can be easily illustrated by taking a stone or any other rigid body attached to a string and, starting with the string vertical, whirling the body round and round oneself as a vertical axis. The body will rise higher and higher until the string is horizontal; but having arrived at this position it will continue to maintain it, whatever additional spin may be communicated about the vertical axis.

Similarly, it will be seen that a liquid or viscous body which is being made to rotate about a vertical axis will tend to *change its shape*, in such a way that its vertical axis becomes smaller; and if there are no external forces tending to rotate the body, but rotation has already been communicated about some axis, this axis will tend to become smaller and smaller, as is the case with the Polar axis of the Earth.

* See Appendix IV.

50. We shall now be able to see the reason for the behaviour of the eggs and the tops which were full of liquid.

We know (Art. 47) that the shell, even though originally spun about its shortest axis, would, if empty, rise and spin about its longest axis, owing to the friction at the point of contact with the table. But between the liquid and the shell there is comparatively little friction, so that the liquid, once spun about its shortest axis, continues to spin about that axis; and being of much greater mass than the shell it overcomes the latter's tendency to rise on to its longest axis, with the result that the whole body, shell and liquid, spin about the shortest axis.

It is clear that the final behaviour of the egg, or top, must depend, amongst other things, on the relative masses of the liquid and shell, and this accounts for the top of Fig. VIII. spinning about its longest axis although full of water.

51. The gyroscopic top (Figs. XV. *a*, XV. *b*) is another, and perhaps more remarkable, instance of the precession of the axle.

The forces acting on the top are (Fig. 25):

(i) its weight;

(ii) the normal reaction S at the point of contact of the top with the spiral coil;

(iii) the tangential reaction F at that point;

(iv) the reaction at the point of support O—not lettered in the diagram.

In most models of this top, the centre of gravity is made to coincide with the point of support, though in some it can be adjusted so as to be either above or below as required.

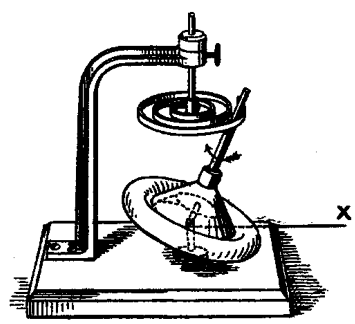

FIG. 25.

For the sake of simplicity we will consider that it is coincident with the point of support—though the following explanation would only require a little modification if this were not so.

Let us regard the axle of the top as in the plane of the paper, and let the spindle be rolling (left-handed as viewed from O, the point of support) along the inside of the coil, into the paper, and be approaching the end of the coil. The motion will be most easily explained by considering first the effect of the tangential reaction F at the point of contact P, and then the additional effect of the normal reaction S.

(1) *Tangential reaction.* It is evident that the friction acts into the plane of the paper and perpendicularly to it. This

If, however, we hurry the top when spinning with the larger precession Ω_1, we get motion (β): $\ddot{\theta}$ is positive and the top *falls.*

Top spinning on a blunt peg. We will now consider a top with a blunt peg spinning on an imperfectly rough surface, taking into account the forces already mentioned in Art. 47, and, in addition, *the resistance of the air.*

Fig. 84 shows the top with spin as marked, precessing *into* the plane of the paper, and if we take any elemental circular section, centre g, we see that forces due to air friction on this section are similar to those on a golf ball (Appendix II., p. 138)

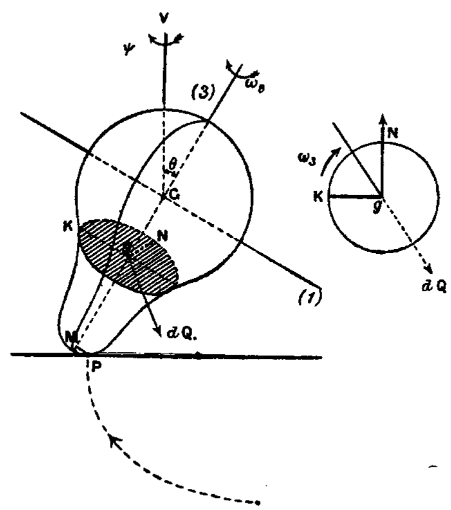

FIG. 84.

and reduce to a couple retarding the spin, and a force dQ in the quadrant KgN, in the direction marked, retarding precession.

Thus the resultant of the total air-friction on the top can be represented by a couple retarding the spin and some force Q acting so as to intersect GM.

We have, therefore, two possible results:

(1) If Q intersects GM below G, $\dot{\psi}$ is retarded.
(2) „ „ above G, „ hurried,

and it *depends on the shape of the top* which of the two occurs.

If the condition for the steady motion of the top be written down, taking into consideration the friction at P, it will be seen, as in the case of the top on a fine point, that for one value of α there are two possible values of $\dot{\psi}$, consistent with steady motion, which depend on the previous motion of the top; but since the blunt peg causes ω_3 to vary, the state of steady motion may only be momentary. Should the top reach this state, either the hurrying or retarding of precession may cause the top to rise (or fall): that is, either the friction at P or the force Q may contribute to its rise (or fall). Hence the question of whether a top can rise until its axis is vertical depends on the configuration of the top.

Case of a top the toe of which is spherical. (Jellet's equation.) Let us suppose the toe of the top to be a portion of a sphere (Fig. 85) whose centre is O and radius r. Let $OG=h$ and the ratio $\frac{h}{r}$ be denoted by k. Draw PM perpendicular to CGO, and let F be the component of friction perpendicular to the azimuthal plane. Then, referring to Art. 128, we have by moments about GC:

$$C\dot{\omega}_3 - A\omega_1\dot{\theta} - A\omega_2\dot{\psi}\sin\theta = -F.r\sin\theta,$$

reducing to $$C\dot{\omega}_3 = -F.r\sin\theta.$$

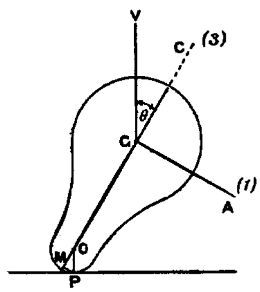

FIG. 85.

Again the angular momentum about the vertical GV is

$$A\dot{\psi}\sin^2\theta + C\omega_3\cos\theta;$$

therefore, by moments about GV,

$$\frac{d}{dt}(A\dot{\psi}\sin^2\theta + C\omega_3\cos\theta) = F.h\sin\theta.$$

Eliminating F, we obtain on integration,

$$A\dot{\psi}\sin^2\theta + C\omega_3\cos\theta = -C\omega_3 k + N,$$

$$\text{or} \quad -A\omega_1\sin\theta + C\omega_3(k+\cos\theta) = N, \quad \text{................(1)}$$

where N is a constant depending on initial conditions.

This integral is given by Professor Jellet in his *Theory of Friction*, Chap. VIII. (*Ed.* 1872), though obtained by a different process. It is clear that the equation is true whether the plane be smooth, imperfectly rough, or perfectly rough.

Angular momentum and velocity at any time of the top about GP and perpendicular axes. We have, by Jellett's relation,

$$-A\omega_1 \sin\theta + C\omega_3(k+\cos\theta) = N;$$

whence
$$-A\omega_1 \frac{r\sin\theta}{\rho} + \frac{C\omega_3 r(k+\cos\theta)}{\rho} = \frac{Nr}{\rho},$$

or if $\rho = GP$, and the angle $NGP = \phi$ (Fig. 86),

$$-A\omega_1 \sin\phi + C\omega_3 \cos\phi = \frac{Nr}{\rho},$$

i.e. the angular momentum of the top at any time about $GP = \frac{Nr}{\rho}$, *varies inversely as* GP, *and therefore decreases as the top rises.*

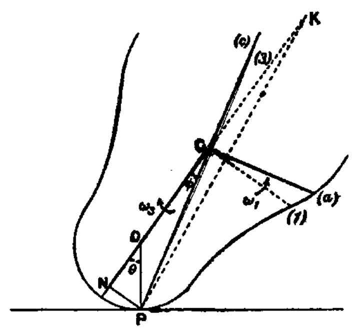

Fig. 86.

Again:

Moment of inertia about $GP = A\sin^2\phi + C\cos^2\phi$

$$= \frac{AX^2 + CZ^2}{\rho^2},$$

X, O, Z being coordinates of P, about axes (1), (2), (3) (Fig. 86).

Therefore the angular *velocity* about GP is given by

$$\omega = \frac{Nr \,.\, \rho}{AX^2 + CZ^2} = \frac{Nr \,.\, \rho}{\phi(X, Z)},$$

where $\phi(X, Z)$ stands for $AX^2 + CZ^2$.

If the motion were one of pure rolling about GP, with G at rest, we should have

$$\frac{-\omega_1}{r\sin\theta} = \frac{\omega_3}{h + r\cos\theta}.$$

Each fraction, by Jellett's relation,

$$= \frac{Nr}{r^2[A\sin^2\theta + C(k+\cos\theta)^2]}$$
$$= Nr/\phi(X, Z),$$

and the velocity of pure rolling about GP

$$=\frac{Nr \, . \, \rho}{\phi(X, Z)}.$$

Hence it follows that:

At any moment the velocity about (c) is always that necessary for pure rolling at angle θ with G at rest:
and if the velocities about (a) and (b) became zero either gradually or suddenly at any moment, the top would roll at the same angle, with the velocity about (c) which it had when the velocities about (a) and (b) vanished.

This state of steady motion would continue indefinitely if it were not for the resistance of the air and the frictional couple at the toe.

The state of general motion of this top may be built up from the above steady motion by considering the angular velocities of the top in the following manner:

We have

About (c) at any moment a velocity of the magnitude necessary for pure rolling about (c). In addition, there is in general:

About (a), an angular velocity ω_a, which, since the plane at P is rough, causes a translational velocity v perpendicular to the azimuthal plane.

Taking v as measured into the paper, we see,

$$\text{if} \quad \rho \, . \, \omega_a > v,$$

the point P will be moving out of the paper towards the reader: friction will hurry the precession, and the top will rise if it is spinning with the smaller of the two possible azimuthal velocities. (Similarly, if $\rho \, . \, \omega_a < v$, the top will fall.) This will cause,

About (b), an angular velocity $\omega_b = \dot{\theta}$. In this case, as before, P may move or tend to move *in* the azimuthal plane either to the left or right, so that the direction of friction in the azimuthal plane may be in either direction.

It is clear that, if $\quad \rho \, . \, \omega_a = v,$

we have the state of steady motion, *with G moving*, as described on pp. 48, 49, and the motion is one of pure rolling about some instantaneous axis PK. The above considerations show that, since friction is a passive force and only acts to *prevent* motion, the prime cause of the rise of a top is the angular velocity about axis (a). This velocity is the direct cause of G moving perpendicular to the azimuthal plane, and the indirect cause of G rising (or falling).

Provided the initial spin about GN is sufficiently large, the top will not fail to rise for want of velocity about axis (a), but it may fail to rise owing to the configuration of the top, as is shown by the following considerations.

To determine the least spin which will enable the axle of the top to become vertical. We will now proceed to a more detailed discussion of this rising of a spinning top on an imperfectly rough plane, employing the assumptions that the friction between the top and the ground may be represented by a single force at the point of contact, and that when slipping takes place the direction of the force of friction is opposite to the direction of sliding or, at any rate, acts so that kinetic energy is dissipated. It has already been pointed out (Arts. 47 and 66) that the friction we are here considering in no way contributes to the eventual fall of the top. Assuming that the top attains a state of steady motion, spinning in a vertical position with angular velocity n about its axle, having been started with initial velocity ω_0 about its axle, we can, from considerations of energy, obtain an inferior limit to n. For, putting $\theta=0$ in equation (1), we obtain

$$N=C(k+1)n.$$

Also, $N=C\omega_0(k+\cos\alpha)$ from initial considerations;

$$\therefore\ n=\frac{\omega_0(k+\cos\alpha)}{k+1}.$$

Now, since some kinetic energy has been dissipated, the total loss of kinetic energy is greater than the work done against gravity

Therefore $\frac{1}{2}C\omega_0^2-\frac{1}{2}\frac{C\omega_0^2(k+\cos\alpha)^2}{(k+1)^2}>gh(1-\cos\alpha)$;

whence, after dividing by $(1-\cos\alpha)$,

$$\omega_0^2>\frac{2(k+1)^2gh}{C(2k+1+\cos\alpha)}, \quad\text{.....................(2)}$$

the mass of the top being taken as unity.

Unless the initial spin satisfies this condition, it is impossible for the axle to become permanently vertical. It is obvious that if r is supposed to be small, so that k is large, the limiting value of n is large and tends to infinity as r diminishes to zero. Hence it is impossible for a top to rise on a perfectly fine peg. (See page 158, Case 1 (*c*).)

Conditions necessary in order that a top may rise to a permanently vertical position. The explanation given in Art. 47 on the rising of a spinning top is based on the assumption that friction acts in the same direction throughout the motion. Some tops, however, fail to rise to the vertical position *whatever the magnitude of the initial spin*, even if the direction of friction does not alter. The following investigation from considerations of energy alone has been adapted from an article* by Mr. E. G. Gallop, with the author's kind permission.

* Part III., Vol. XIX., of the *Transactions of the Cambridge Philosophical Society*, to which the reader is referred for a fuller investigation.

To find the minimum value of the energy, for a given value of θ, subject to Jellett's equation. Let v be the velocity of G at any time; then T, the kinetic energy, is given by

$$2T = v^2 + A\omega_1^2 + A\omega_2^2 + C\omega_3^2.$$

If the potential energy is reckoned zero in the initial position, the total energy E at any time is given by

$$E = T + gh(\cos\theta - \cos\alpha).$$

The minimum value ever possible for E at any given inclination θ to the vertical will occur when $v=0$ and $\omega_2=0$ (for v and ω_2 are independent of θ and can be zero), in which case the top will be for the instant in a state of steady motion with G at rest, the axis describing a fixed cone of which G is the vertex. In this case of minimum energy we have, θ being regarded as constant,

$$A\omega_1 d\omega_1 + C\omega_3 d\omega_3 = 0,$$

while, for variations of $d\omega_1$, $d\omega_3$, subject to Jellett's condition,

$$-A\sin\theta\, d\omega_1 + C(k+\cos\theta)d\omega_3 = 0.$$

Hence
$$\frac{-\omega_1}{\sin\theta} = \frac{\omega_3}{k+\cos\theta} \quad \text{.................................(3)}$$

$$= \frac{N}{A\sin^2\theta + C(k+\cos\theta)^2}, \quad \text{from (1),}$$

and
$$\frac{1}{2}A\omega_1^2 + \frac{1}{2}C\omega_3^2 = \frac{1}{2}\frac{N^2}{A\sin^2\theta + C(k+\cos\theta)^2}.$$

It follows that the minimum value of energy possible at the inclination θ to the vertical is

$$E = gh(\cos\theta - \cos\alpha) + \frac{1}{2}\frac{N^2}{A\sin^2\theta + C(k+\cos\theta)^2},$$

or writing $\cos\theta = x$,

$$E = gh(x - \cos\alpha) + \frac{1}{2}\frac{N^2}{A(1-x^2) + C(k+x)^2},$$

while from relation (3) we have

$$\frac{-\omega_1}{r\sin\theta} = \frac{\omega_3}{h + r\cos\theta},$$

$$\textit{i.e.} \quad \frac{-\omega_1}{PM} = \frac{\omega_3}{GM},$$

and the motion is one of pure rotation about PG; so that there is no sliding friction at the point of contact, and the top is spinning steadily, as already pointed out above.

The minimum energy equation may be written

$$E = gh(x - \cos\alpha) + \frac{1}{2}\frac{N^2}{f(x)} = F(x) \text{ say.}$$

It will be noticed that the last term represents the minimum *rotational* energy.

Considering then the minimum energy curve $E=F(x)$, we see that it forms a kind of barrier between the actual value of the energy of the top and the zero value. It is clear that if it is of the form (I.) in Fig. 87, the top may reach a position E_1 of minimum energy for the particular value of θ corresponding to x_1, and on subsequent dissipation of energy it must spin at a larger value of θ: further, it will subsequently reach the position

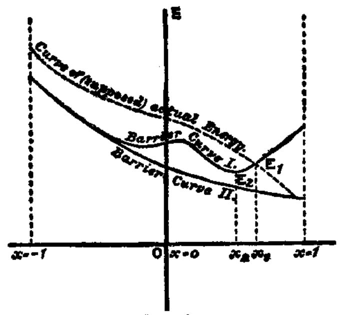

FIG. 87.

x_2, from which it must fall to the ground, since the energy at x_2 is the minimum possible for spinning at all. If, however, the curve is of the form (II.), continued dissipation of energy insures eventual rising to the vertical position.

The condition then that the curve of minimum energy shall be of form (II.), shown in the figure, is a sufficient (though not necessary) condition that the top *can* rise to the vertical position, provided the initial spin is sufficiently large.

We require, then, that the possible minimum energy shall decrease as x increases, and consequently, since $gh(x-\cos a)$ increases with x, we must have two conditions:

(a) $\frac{1}{2}\frac{N^2}{f(x)}$ decreases as x increases,

(b) " " faster than $gh(x-\cos a)$ increases, for there is a loss of energy on the whole.

Condition (a) gives

$$f(x)=A(1-x^2)+C(k+x)^2 \text{ is increasing;}$$

$$\therefore f'(x)=2(C-A)x+2Ck>0.$$

If $C>A$ this condition is always satisfied since k is positive.
If $C<A$ it is satisfied when

$$C-A+Ck>0,$$

$$i.e. \quad C(k+1)>A.$$

for writing $x=1$ gives the greatest value to the negative part of the expression.

Condition (b) gives:

$$\text{Decrease in } \frac{1}{2}\frac{N^2}{f(x)} > \text{increase in } gh(x-\cos a),$$

$$\text{or} \quad \frac{1}{2}\frac{N^2 f'(x)}{[f(x)]^2} > gh$$

for all values of x between -1 and 1.

If we find the condition that $\frac{f'(x)}{[f(x)]^2}$ continually *diminishes*, it will be sufficient to ensure, in addition, that

$$\frac{1}{2}\frac{N^2 f'(1)}{[f(1)]^2} > gh.$$

Now, with our present conditions, $f(x)$ continually increases from -1 to 1; hence its graph is as drawn (Fig. 88), according as

$$C \text{ is} \lesseqgtr A.$$

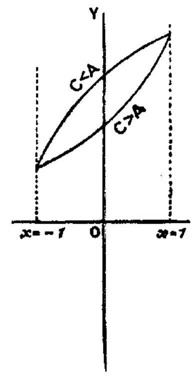

FIG. 88.

If C is $< A$, we see that $f'(x)$ decreases and $f(x)$ increases continually;

$$\therefore \frac{f'(x)}{[f(x)]^2} \text{ continually decreases.}$$

If C is $> A$, both $f'(x)$ and $f(x)$ increase, but it can be shown* that $\frac{f'(x)}{[f(x)]^2}$ continues to diminish if $f(x)$ cuts the axis in real points, namely, if

$$4C^2k^2 > 4(C-A)(Ck^2+A),$$

which reduces to $\qquad k^2 > 1 - \frac{A}{C}.$

* Considering the parabola $\qquad y = Kx^2 - L,$

we have $\qquad \frac{dy}{dx} = 2Kx.$

Now $\qquad y\Big/\frac{dy}{dx} = \frac{Kx^2-L}{2Kx}$

$$= \frac{1}{2}\left(x - \frac{L}{Kx}\right).$$

The differential coefficient of this $\qquad = \frac{1}{2}\left(1 + \frac{L}{Kx^2}\right),$

which is always positive if L is positive, *i.e.* if the parabola cuts the axis of x in real points.

$\therefore\ y\Big/\frac{dy}{dx}\left(\text{and much more so } y^2\Big/\frac{dy}{dx}\right)$ increases under these conditions, and $\frac{f'(x)}{[f(x)]^2}$ decreases if $f(x)$ cuts the axis of x.

Hence the conditions required are:

If $C < A$, $\qquad C(k+1) > A$; ....(4)

$C > A$, $\qquad k^2 > 1 - \frac{A}{C}$, ...(5)

and in both cases $\frac{1}{2}\frac{N^2 f'(1)}{[f(1)]^2} > gh$,

$$\text{or} \quad N^2 > \frac{ghC^2(k+1)^4}{C(k+1)-A}. \qquad (6)$$

If the initial motion consists of a spin ω_0 about the axis of figure, $N = C\omega_0(k+\cos\alpha)$, and the last condition reduces to

$$\omega_0^2 > \frac{(k+1)^4 gh}{\{C(k+1)-A\}(k+\cos\alpha)^2}. \qquad (7)$$

It follows, therefore, that (4) and (7), or (5) and (7), each represent a complete set of conditions which insure the top rising to the vertical position. It will be noticed that (4) and (5) refer to the construction of the top only, while (7) refers to the initial motion also.

It can be easily verified that the value of ω_0 obtained in (7) is greater than that obtained in (2). This result is to be expected, since in (7) is involved also the condition that throughout the motion rotational energy is lost faster than potential energy is gained.

It will be noticed in the above investigation, that

(1) The *translational* velocity of the top has not come under consideration.

(2) The expression $f(x)$, which is found to be increasing, can be written

$$f(x) = \frac{1}{r^2}\{Ar^2\sin^2\theta + C(h + r\cos\theta)^2\}$$

$$= \frac{1}{r^2}(AX^2 + CZ^2),$$

where X, Z are the coordinates of the point P of the top.

Hence, at any given inclination θ, the point P lies on some ellipse whose equation is

$$AX^2 + CZ^2 = K, \text{ some constant},$$

and the condition that $f(x)$ is increasing is the condition that the next position of P is *outside* the ellipse in question. It is clear that this ellipse is one of the principal traces of the ellipsoid of inertia, drawn to such a scale that it passes through P.

This representation is somewhat analogous to Poinsot's representation of the motion of a body under no forces, if we imagine the body to be subsequently disturbed.

(3) Jellett's relation,

$$-A\omega_1 \sin\theta + C\omega_3 (k + \cos\theta) = N,$$

can be written $$-A\omega_1 X + C\omega_3 Z = Nr,$$

showing that the point P of the top always lies on the polar of the point $(-\omega_1, \omega_3)$ with respect to the ellipse

$$AX^2 + CZ^2 = Nr$$

which is an ellipse similar and similarly situated to the one mentioned previously.

Explanation of the rising egg. The above investigation also shows the limiting conditions under which a hard-boiled egg or an acorn will rise on to its longest axis (see Chapter I. and Art. 48). In this case C is $< A$, and the end of the egg or acorn is in general approximately spherical in shape. The remaining condition that $C(k+1) > A$ is also in general satisfied.

Further condition in the case of the loaded sphere. In the case, however, of the loaded sphere described on page 5, the diameter of the sphere which passes through the centre of the load will not become vertical for every load, however great the initial spin may be, since the condition $C(k+1) > A$ will not be satisfied for all positions of the centre of the load. Let O be the centre of the original sphere, R its radius, and M the mass of the complete sphere before being hollowed out. Let o be the centre of the loaded portion, r its radius, and let m be the excess of the mass of the load above the mass removed. Let $Oo = c$.

Then $$OG = h = \frac{mc}{M+m}, \quad Go = \frac{Mc}{M+m},$$

$$C = \frac{2}{5}MR^2 + \frac{2}{5}mr^2, \quad A = \frac{2}{5}MR^2 + \frac{2}{5}mr^2 + \frac{Mmc^2}{M+m}.$$

Hence $C < A$.

The condition $$C(k+1) > A$$

becomes $$Ck > A - C > \frac{Mmc^2}{M+m};$$

that is, $$C\frac{m}{M+m} \cdot \frac{c}{R} > \frac{Mmc^2}{M+m},$$

$$\text{or} \quad \frac{c}{R} > \frac{Mc^2}{\frac{2}{5}MR^2 + \frac{2}{5}mr^2} > \frac{\frac{Mc^2}{MR^2}}{\frac{2}{5}\left(1 + \frac{mr^2}{MR^2}\right)},$$

$$\text{or} \quad \frac{c}{R} < \frac{2}{5}\left(1 + \frac{mr^2}{MR^2}\right).$$