A Particle Collision Model for Calculating the Energetic Cost of Legged Locomotion

Andy Ruina, Cornell TAM (&MAE)

with

Manoj Srinivasan, Cornell TAM & Princeton MAE
John Bertram, Calgary Medical School

Some support from N$F.

Read all about it:

J. Theor. Biol., **237**, Issue 2, Pages 170-192, Nov 2005

Nature, **439**, Pages 72-75, Jan 2006.

TAM, Cornell Feb 06
“Dynamic Walking”, May 06

General Goal: Understand coordination choices of animals (including people).

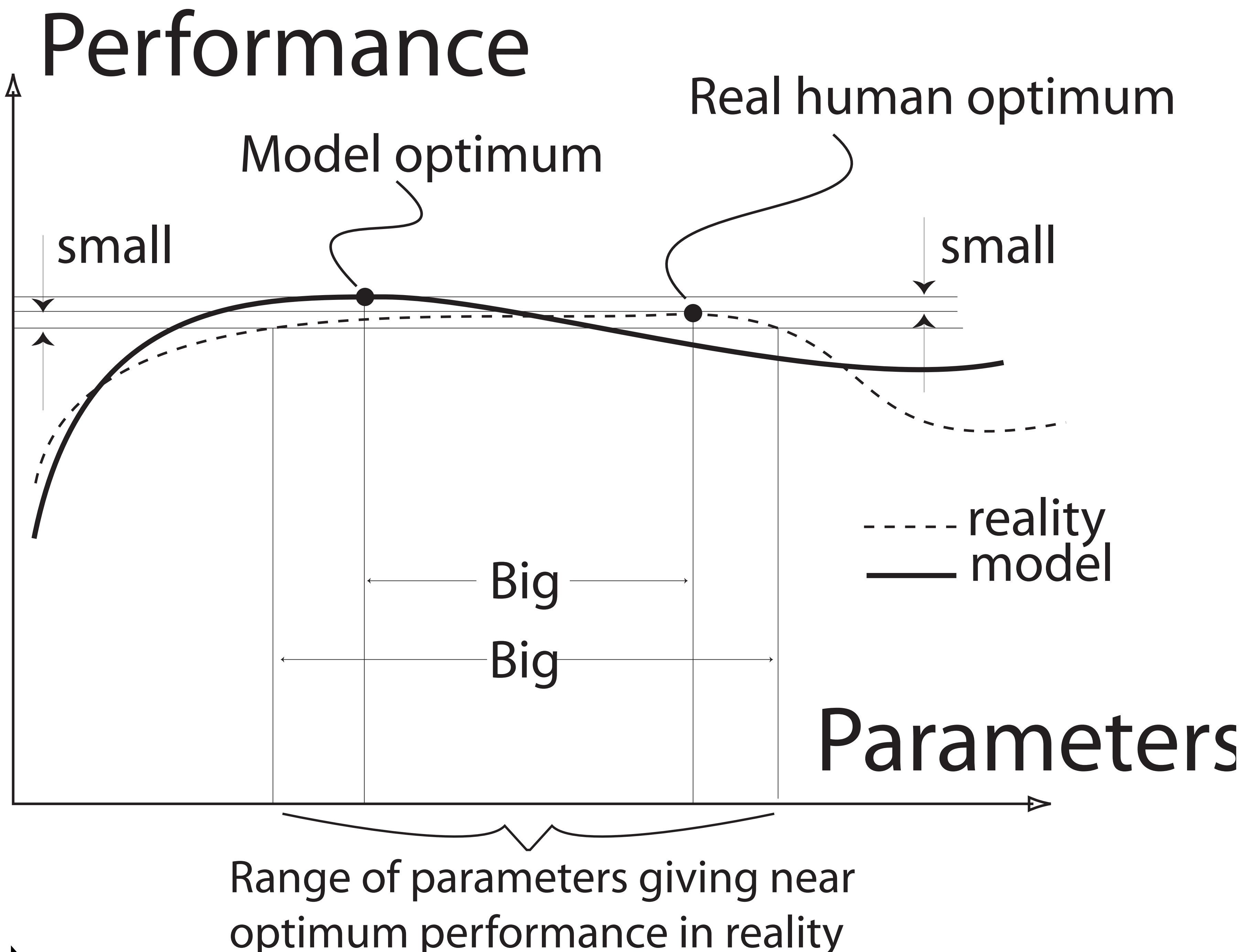
General Approach: Assume the principle of maximum laziness.

Animals move in such a way as to minimize their use of energy: Your great great . . . great . . .
. . . great grandmother's sister . . .
. . . died (young).

Some issues/caveats:

- Selection does not mean, exactly, optimization.
- There is selection for other things,
  - e.g., speed, weight bearing, IQ, sharp teeth,...
- Energy use per unit ________ ?
- How to calculate relation between motion and energy use (we use: cost = muscle work)
- Math 191, optimization is inherently inaccurate.

Optima tend to be insensitive to control parameters.

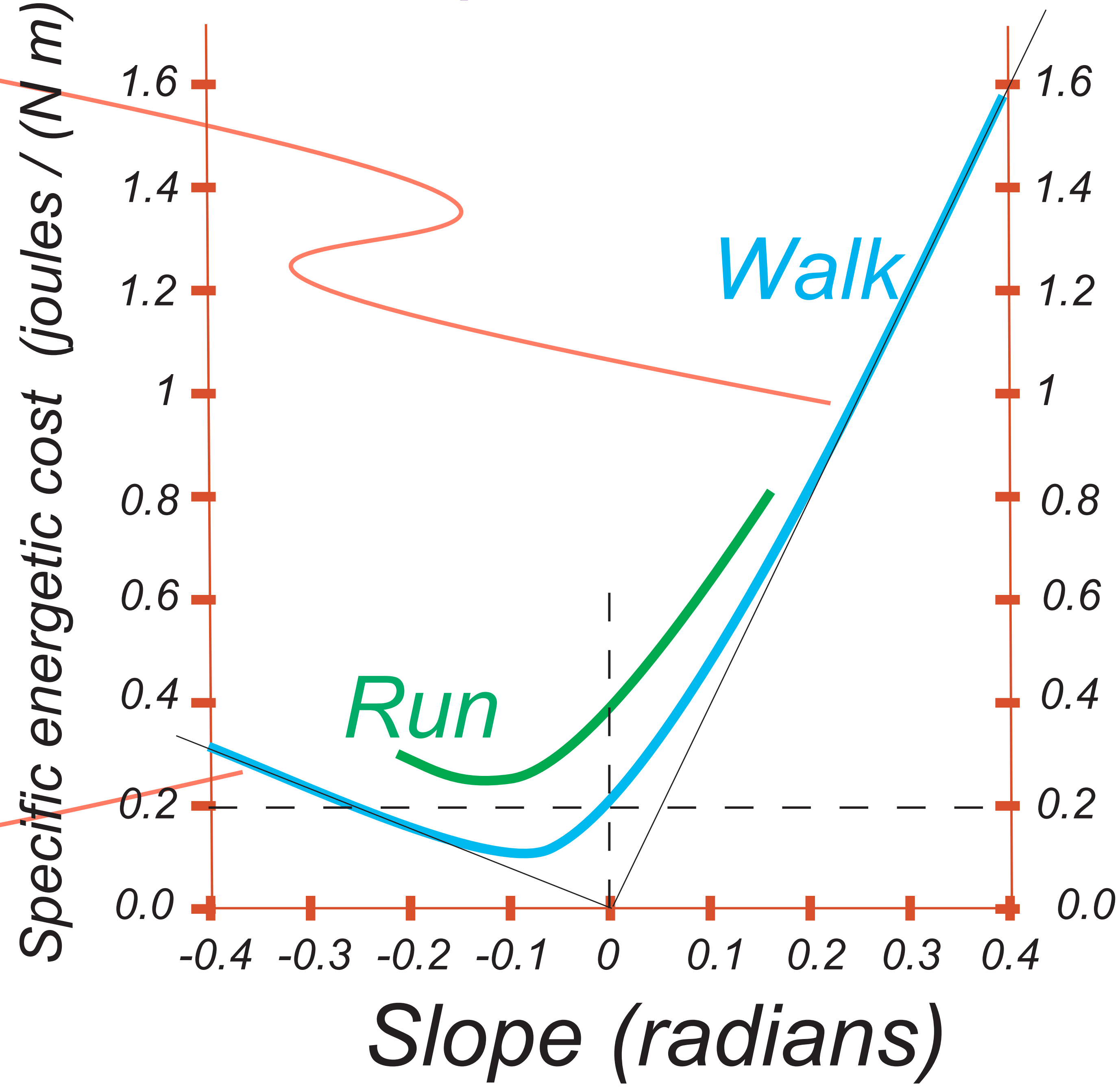


Optimization can’t predict parameters well.

Margaria, 1976

Chemical energy used to walk and run up and down hill

Chemical $\dot{E} = 4 \cdot (\text{Mechanical power})$

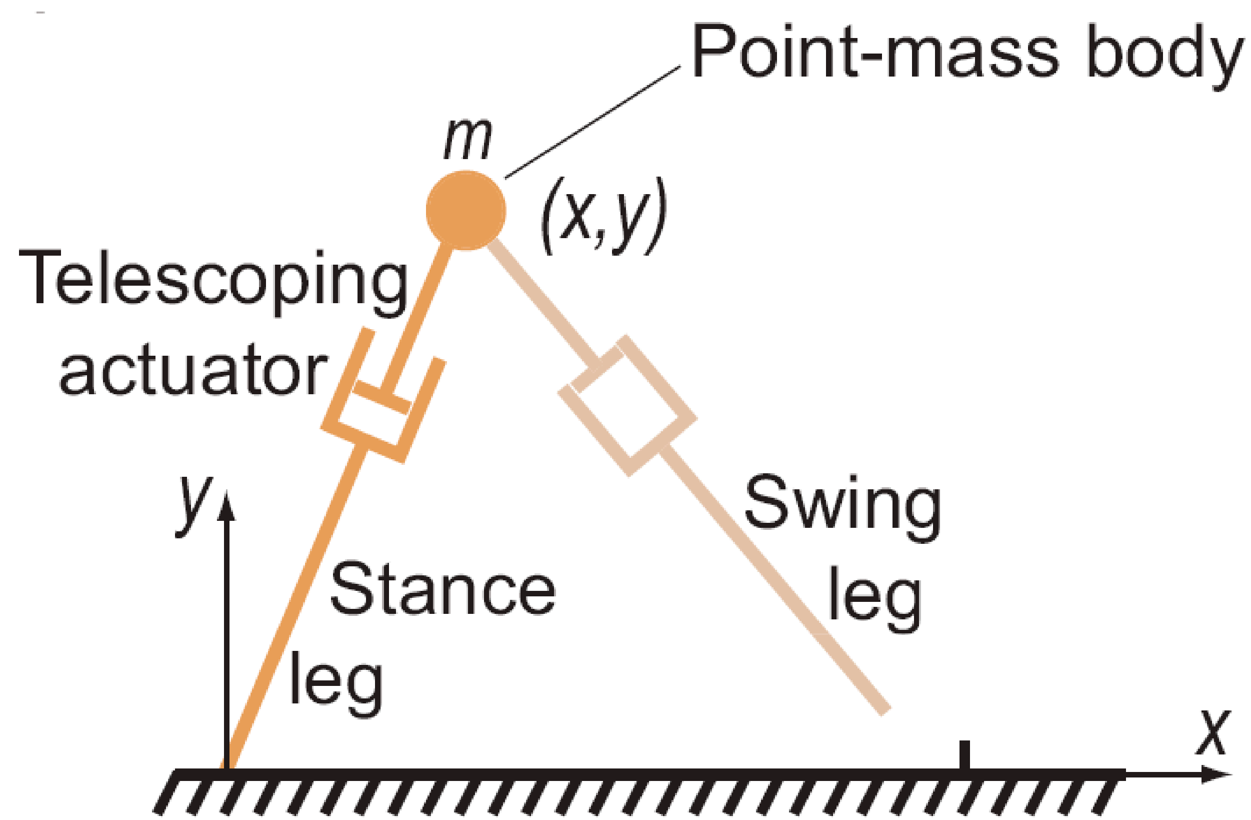


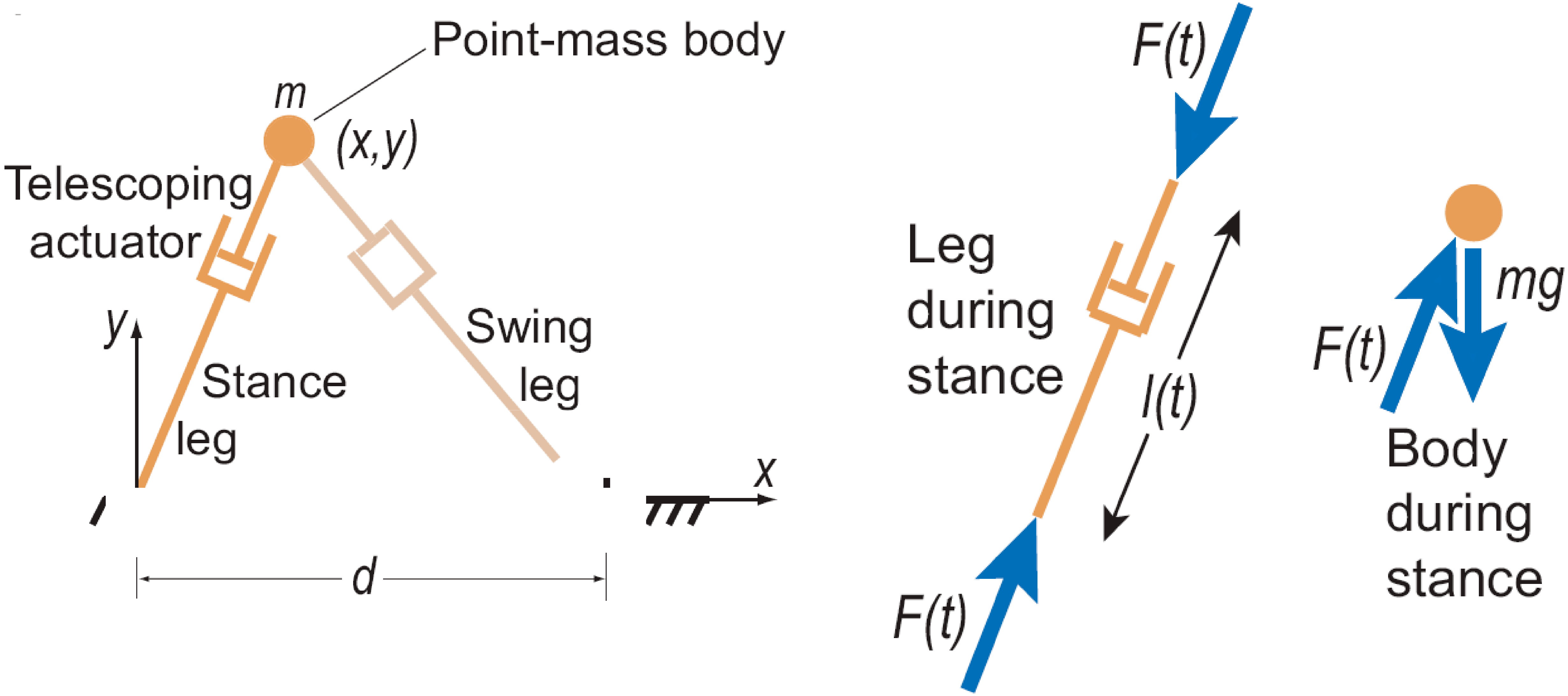
Chemical $\dot{E} = \frac{5}{6} \cdot \left|(\text{Mechanical power})\right|$

When work is substantial,
energy use is roughly proportional to work.

Assume a spherical horse...

- That's too hard
- Make it a small sphere, a particle
- Massless legs

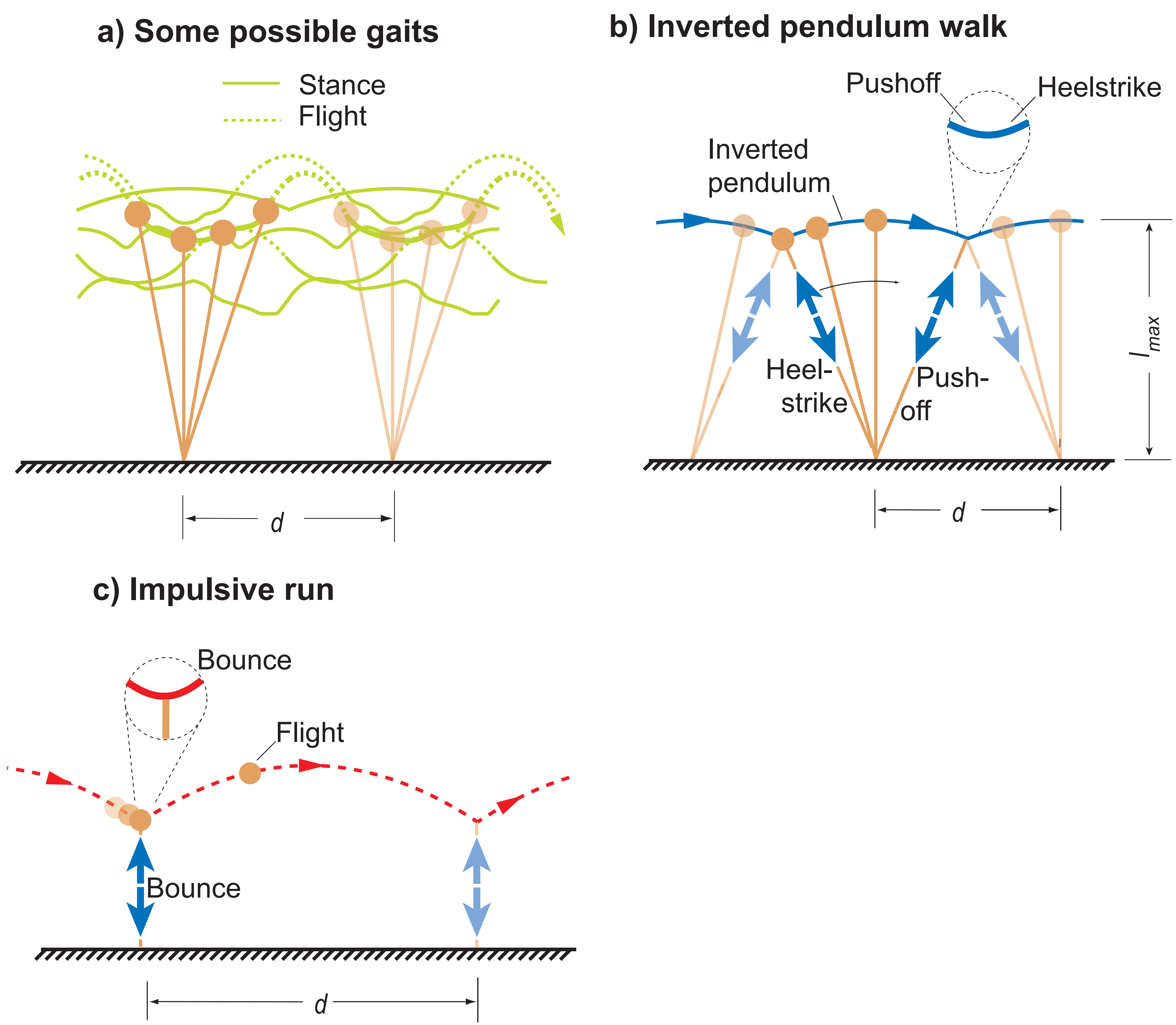




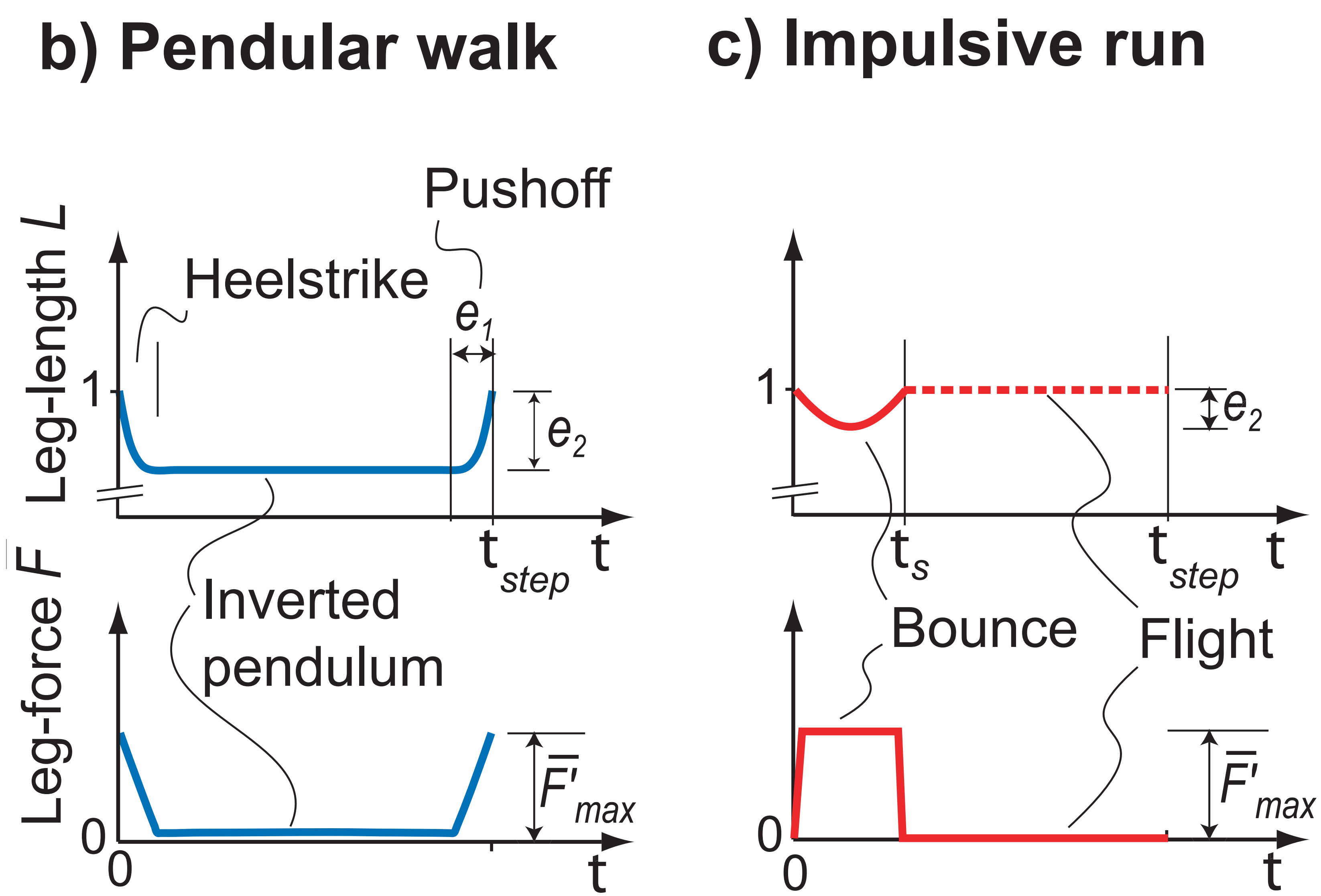
Leg Work W:

$$W = \int dW = \int P dt = \int \mathbf{F} \cdot \mathbf{v} dt = \int F \dot{\ell} dt = \int F d\ell$$

Minimizing work at fixed v and d finds solutions which spend most time with *Power* = 0: $\dot{l} = 0$ *or* $F=0$



Continuous solution gets close to an impulsive solution as numerical grid gets finer.

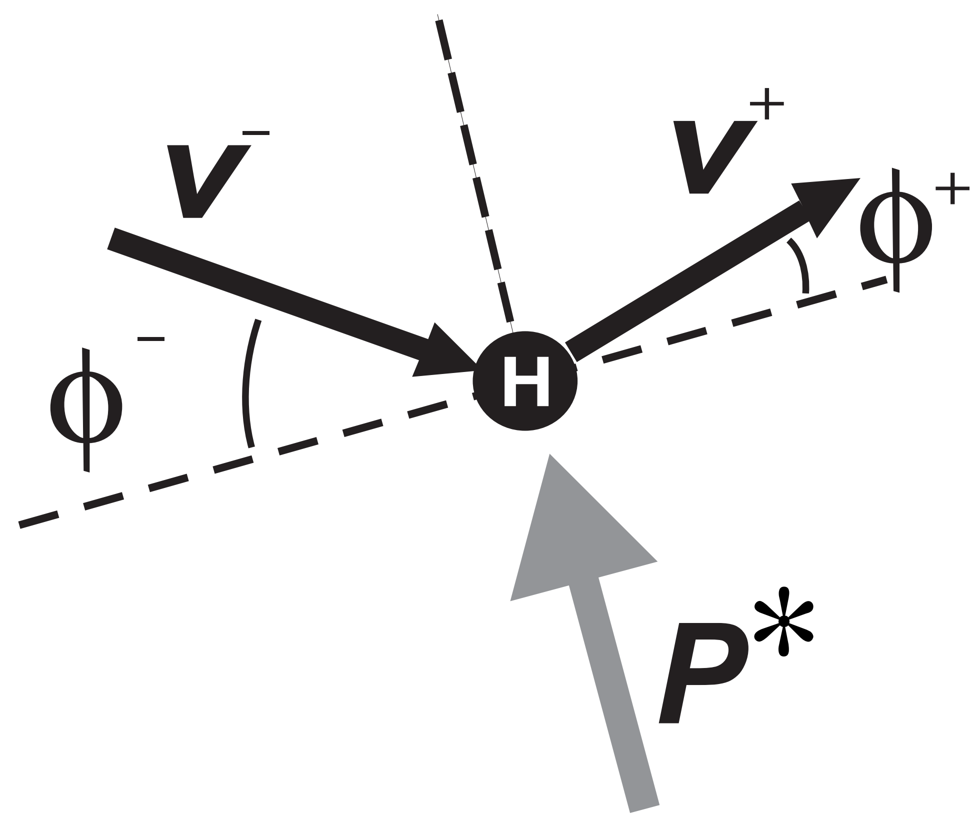


Save calculation effort with new assumption/approximation:

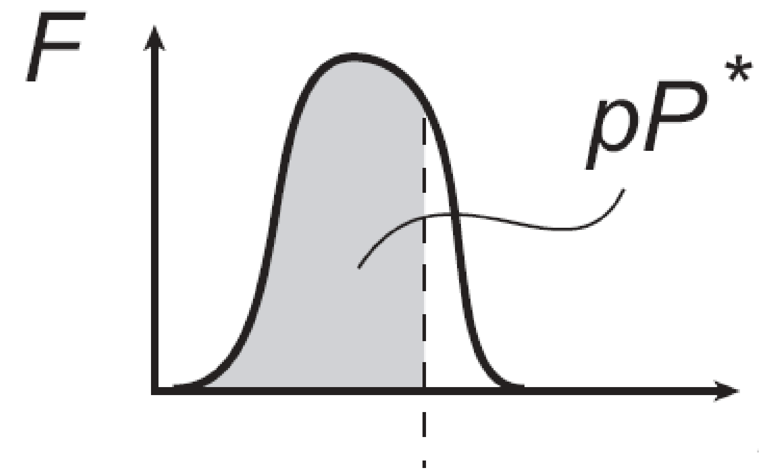
All work is done impulsively.

Work calculations ⇒ Collision calculations

$$m\mathbf{v}^+ = m\mathbf{v}^- + \mathbf{P}^*$$



Relate impulse and work (2 ways)



I. Net change:

The impulse is

$$\mathbf{P}^* = \int_{t_1}^{t_2} \mathbf{F} dt = \hat{\lambda} \int_{t_1}^{t_2} F dt$$

Impulse momentum

$$m\mathbf{v}^+ = m\mathbf{v}^- + \mathbf{P}^*$$

(net) Work Energy

$$\begin{aligned} W &= \Delta E = \frac{m}{2}\left(|\mathbf{v}^+|^2 - |\mathbf{v}^-|^2\right) \\ &= \mathbf{v}^- \cdot \mathbf{P}^* + |\mathbf{P}^*|^2/(2m). \end{aligned}$$

II. Integrate:

The partial impulse *P*, *0<p<1*, is

$$\mathbf{P}(t) \equiv \int_{t_1}^{t} \mathbf{F}(t')dt' = p\mathbf{P}^*$$

Calculate the (net) work

$$\begin{aligned} W &= \int dW = \int_{t_1}^{t_2} \mathbf{v} \cdot \mathbf{F} dt \\ &= \int_{\mathbf{0}}^{\mathbf{P}^*} \underbrace{(\mathbf{v}^- + \mathbf{P}/m)}_{\mathbf{v}} \cdot \underbrace{d\mathbf{P}}_{\mathbf{F}dt} \\ &= \int_0^1 (\mathbf{v}^- + p\mathbf{P}^*/m) \cdot \mathbf{P}^* dp \\ &= \mathbf{v}^- \cdot \mathbf{P}^* \underbrace{\int_0^1 dp}_{1} + \frac{\mathbf{P}^* \cdot \mathbf{P}^*}{m} \underbrace{\int_0^1 p dp}_{1/2} \\ &= \mathbf{v}^- \cdot \mathbf{P}^* + |\mathbf{P}^*|^2/(2m) \end{aligned}$$

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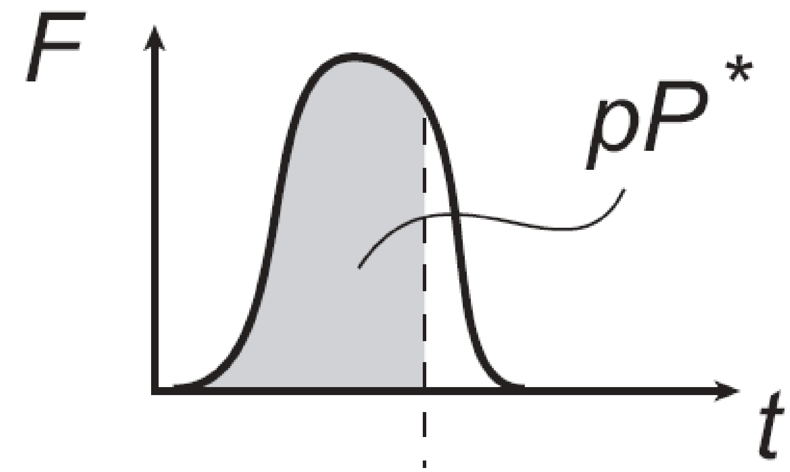

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Net Work in "Collision" is

positive (generated) work - negative (absorbed) work.

$$\Delta E = -E_a + E_g$$

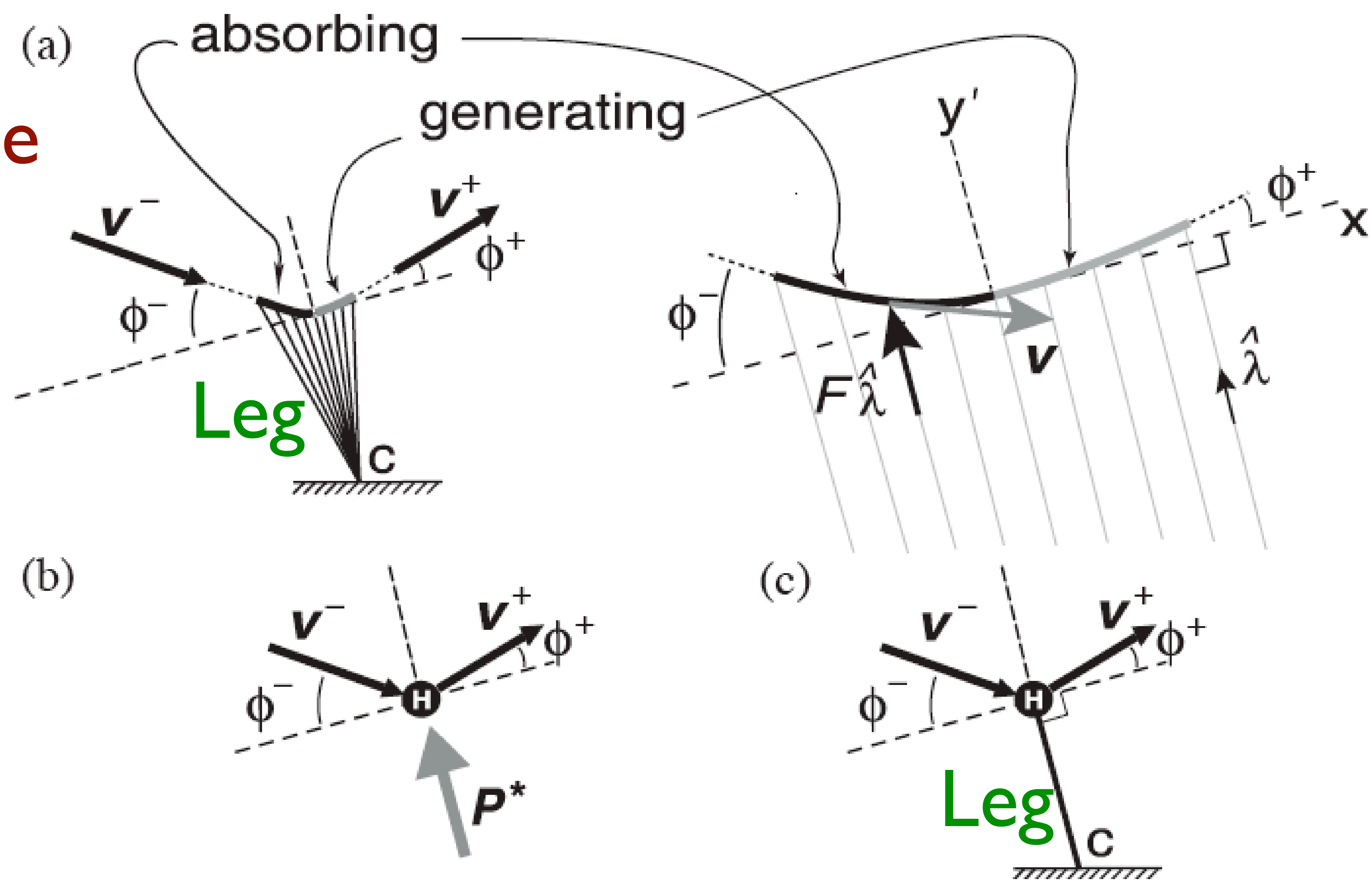
More assumptions:

- Leg is close to vertical
- Motion is close to horizontal
- Speed is close to constant

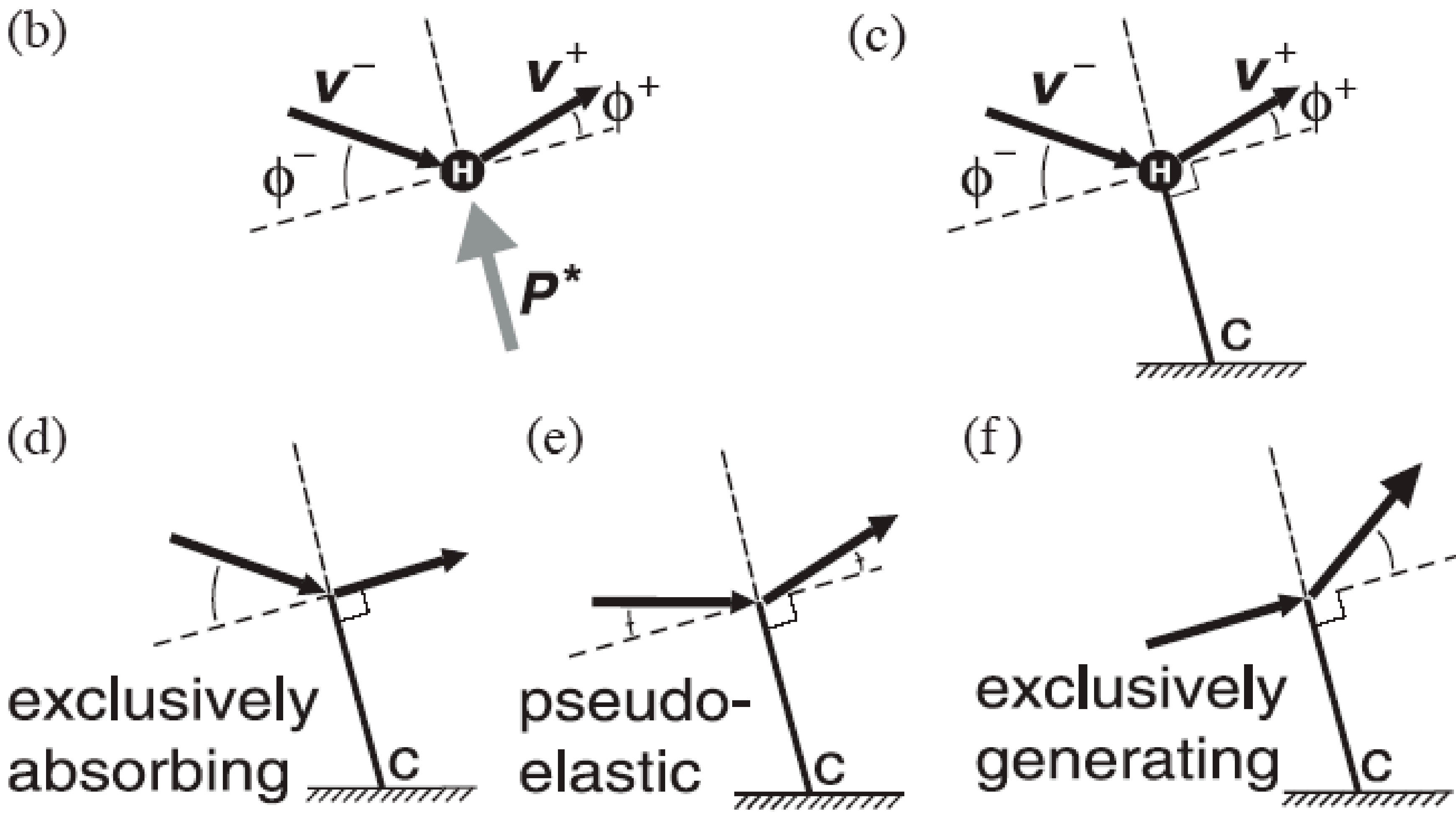
One shallow angle collision:

Spread in time

Impulse



Change in energy in collision: $\Delta E = -E_a + E_g$



where

$$
\begin{aligned}
E_a &= m(\phi^- v)^2/2 \\
E_g &= m(\phi^+ v)^2/2
\end{aligned}
$$

$$e_g = \frac{\phi^+ - \phi^-}{\phi}$$

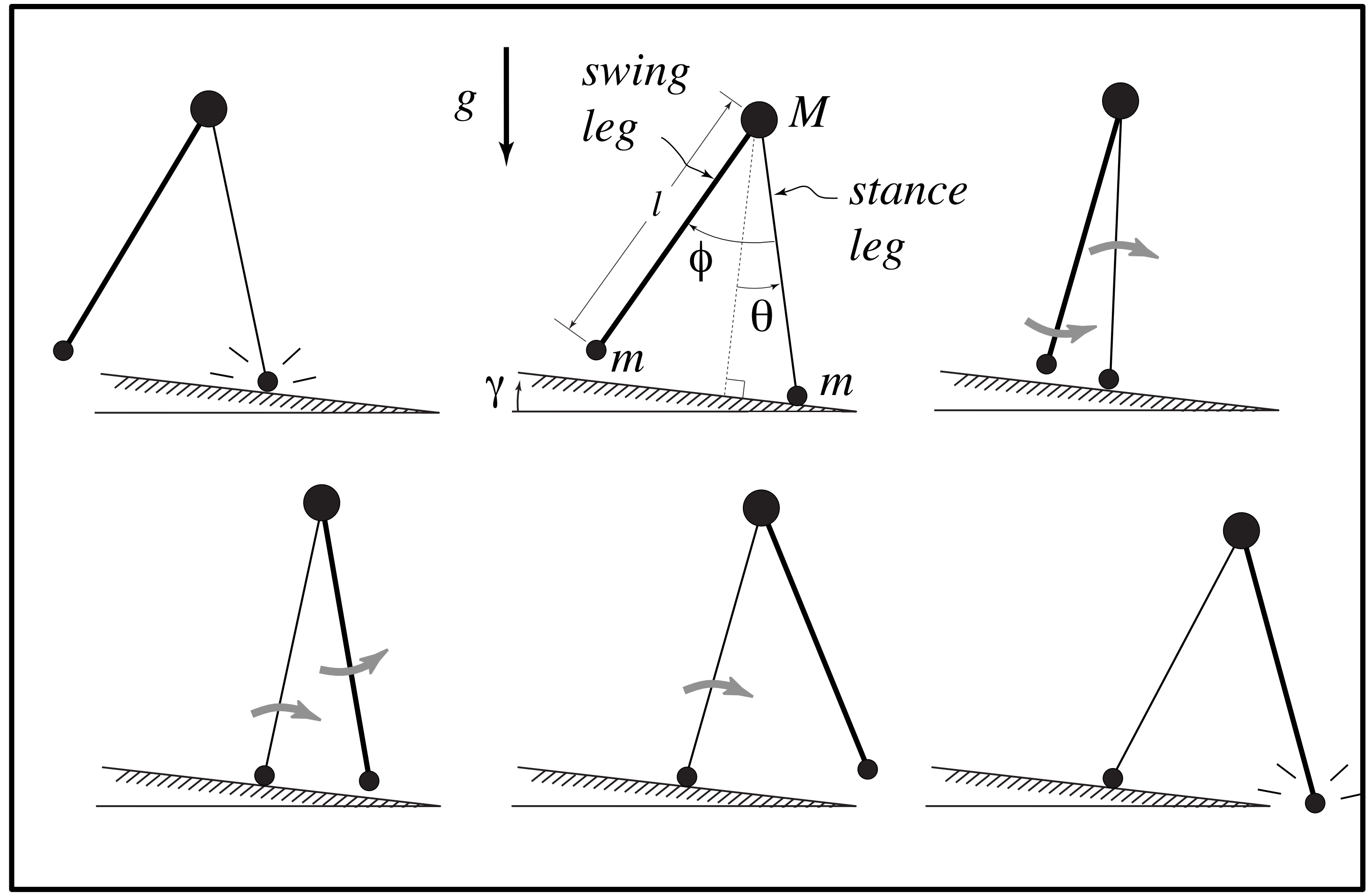
$e_r = 0$ | $e_r = 1$ | $e_r = \infty.$

$e_g = -1$ | $e_g = 0$ | $e_g = 1$

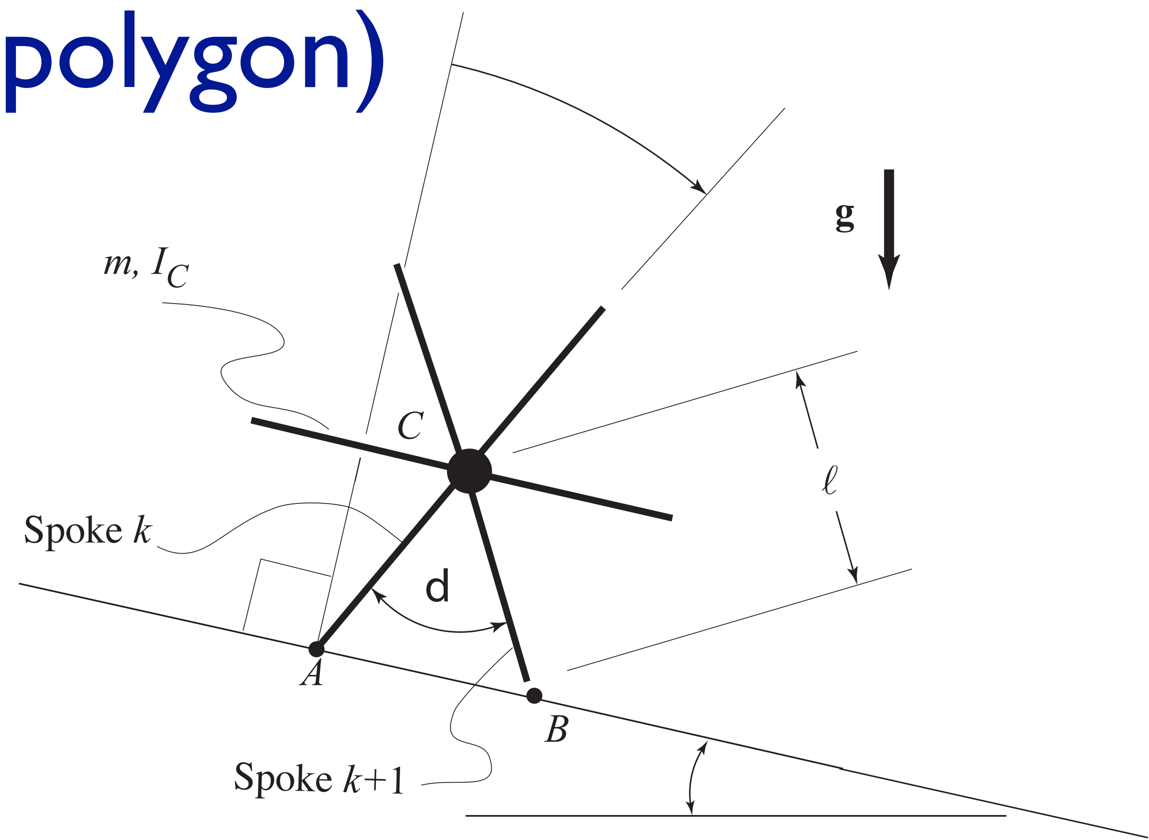
coefficient of generation

Passive Walking and rimless wheel (rolling polygon)

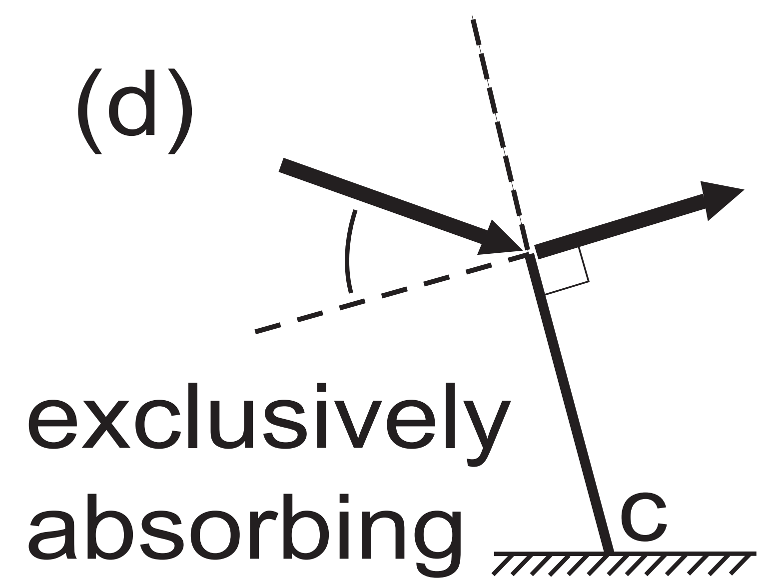
Simplest model of passive-dynamic walking



Rimless wheel (rolling polygon)



The collision model



Energetic cost of taking one step

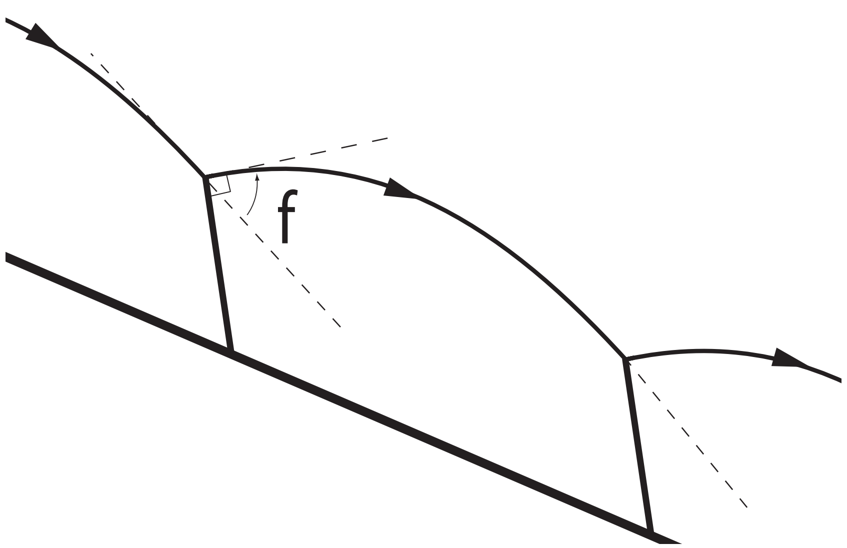
$$E_m = b\phi^2 v^2 m/2$$

inefficiency, about 4

(step length) / 2(leg length)

Similarly for running

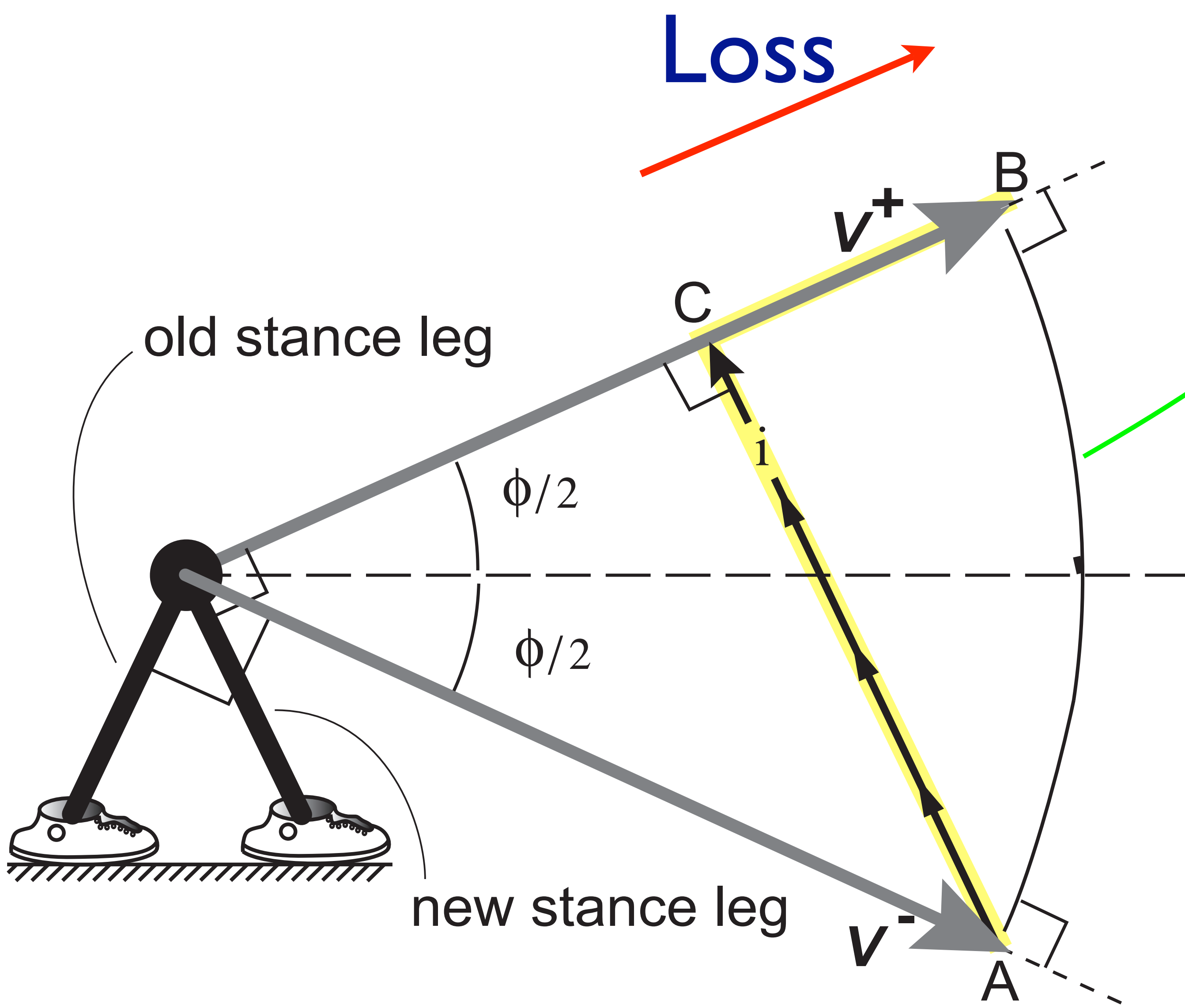
(b) Passive running downhill



$$E_m = b\phi^2 v^2 m/2$$

Balanced with gravitational energy supply.

Hodograph: trajectory of tip of velocity vector

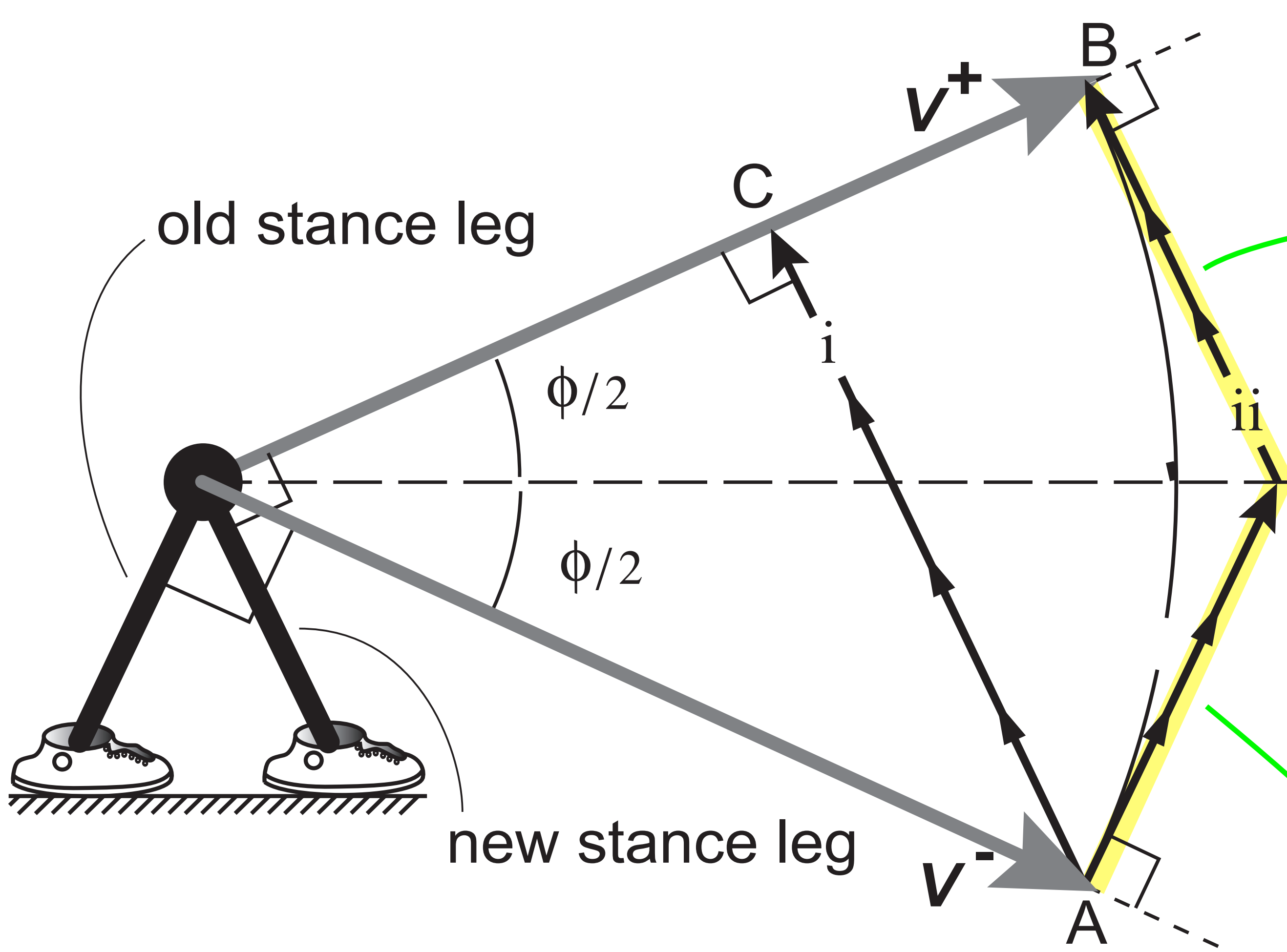


Constant energy curve,
Constant v circle

Loss scales with distance from constant *v* circle

Energy saving trick for walking:

then land on leading leg (e_g = -1)

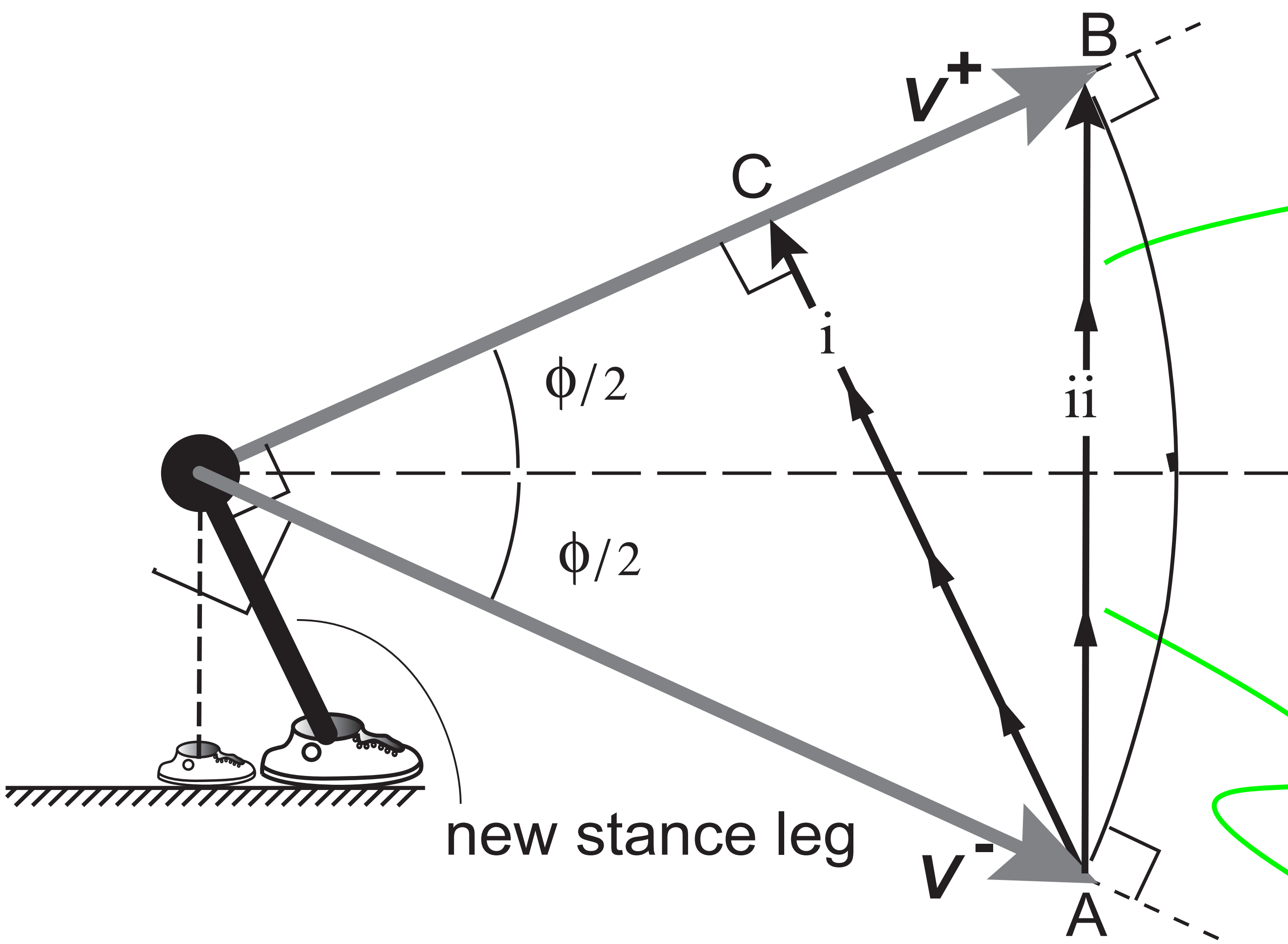


path	collision reduction factor $= J = E_m / (bmv^2\phi^2/2)$
i	1
ii	1/4

Pushoff with trailing leg (e_g = +1)

Energy saving trick for running:

then push off (*net e_g = 0*)



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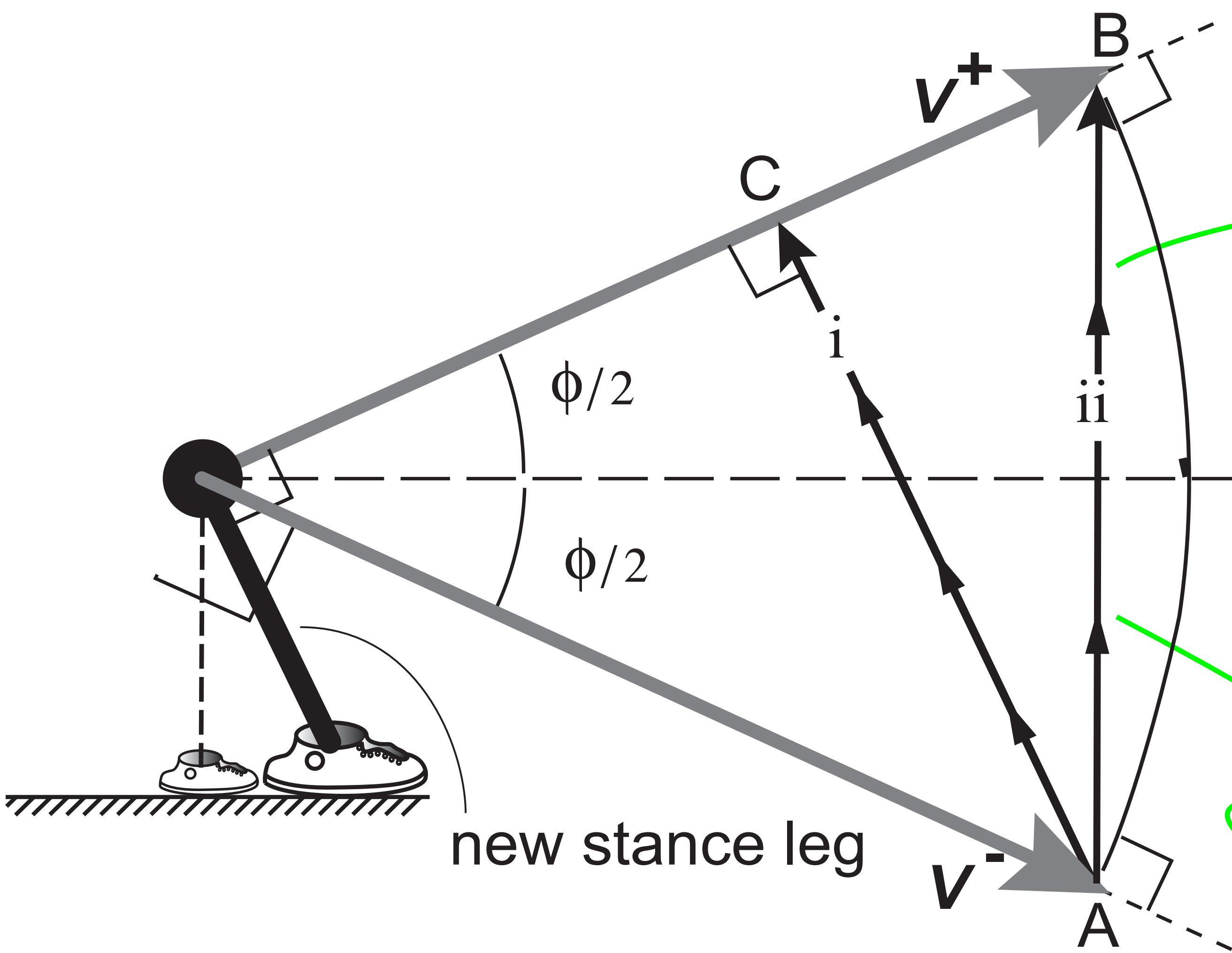
i	*1*
ii	*1/4*

Absorb first

Pseudo-elastic collision (no real elasticity).

Energy saving trick for running:

then push off (*net* $e_g = 0$)



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i	1
ii	1/4

Absorb first

Pseudo-elastic collision (no real elasticity).

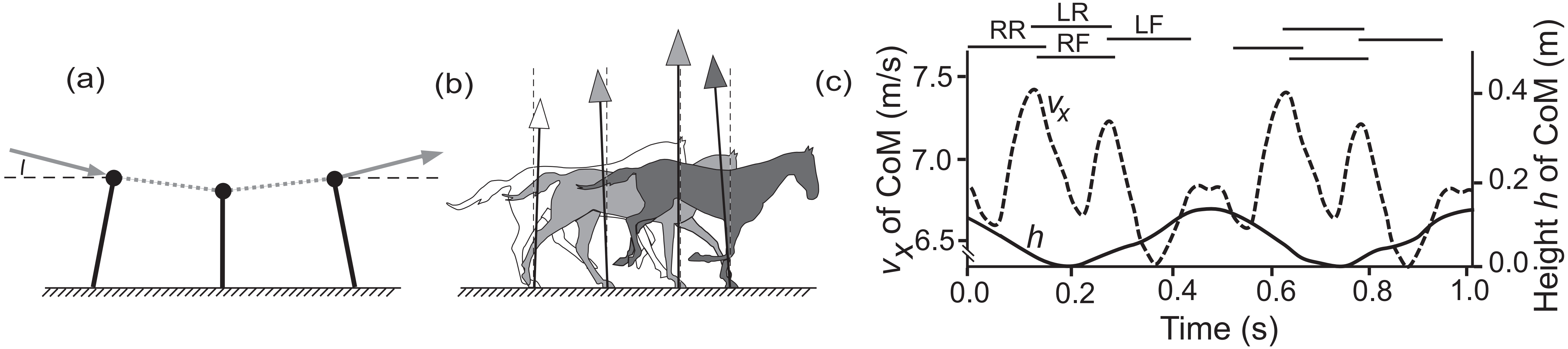
Two morals:

I) A sequence of collisions uses less energy than a single collision (for given deflection angle)

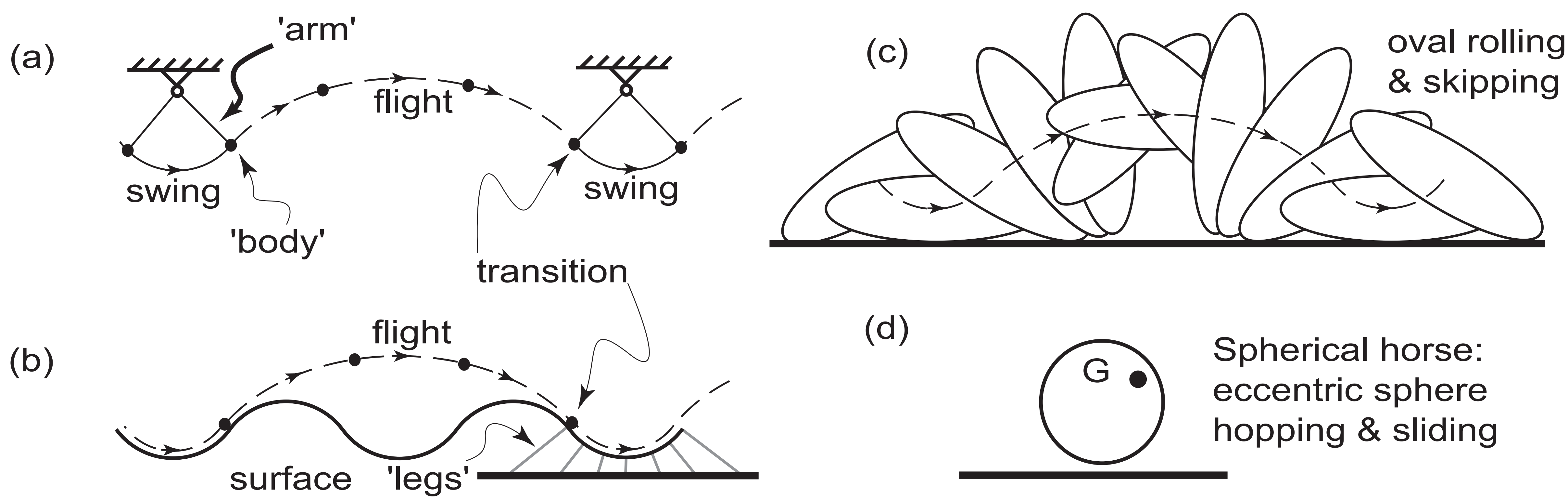
II) A pseudo-elastic collision uses less energy than a plastic collision.

Horse gallop: Seems to use both systems:

Ba-duh-dump ba-duh-dump



Relation between brachiation and galloping



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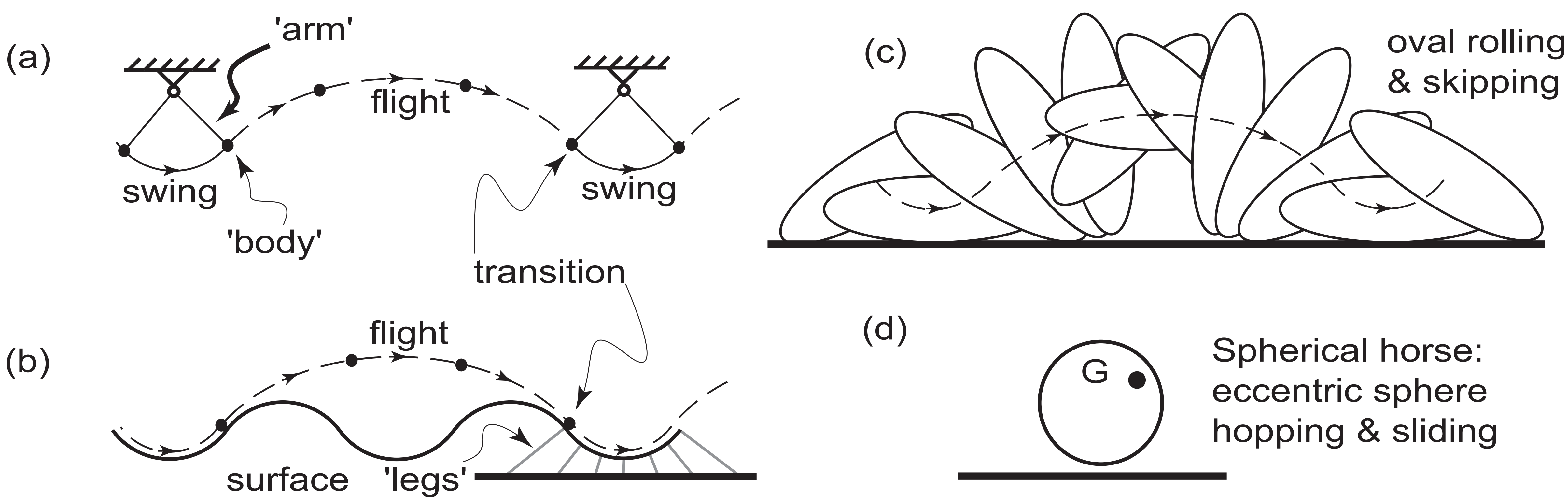

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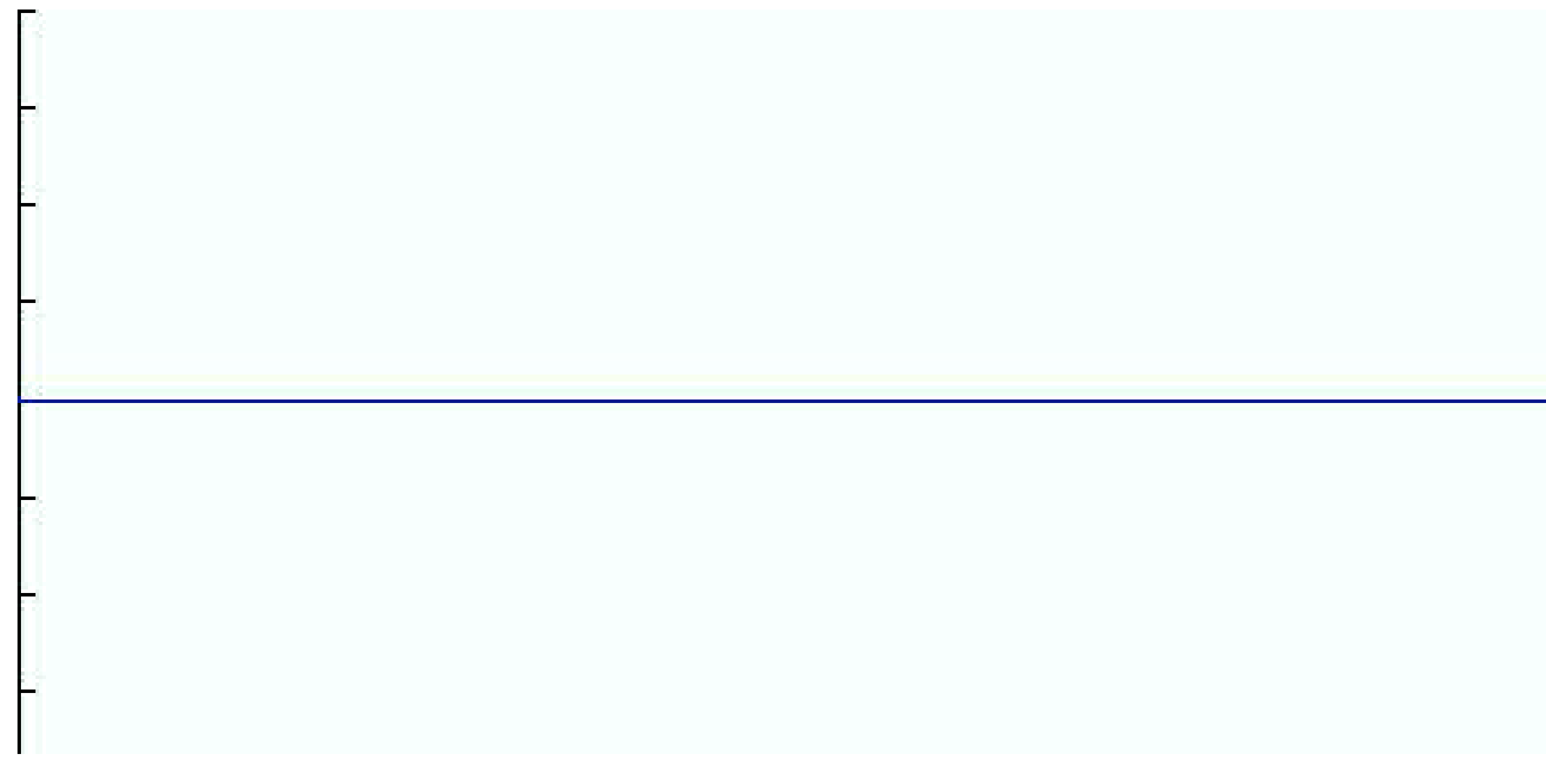
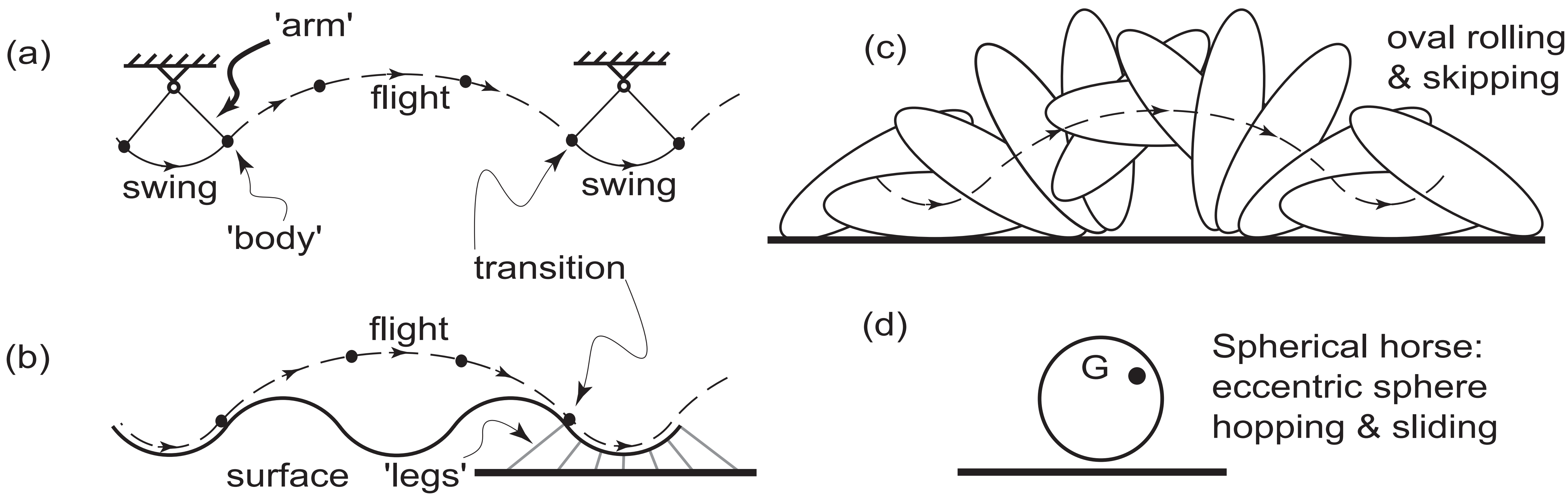
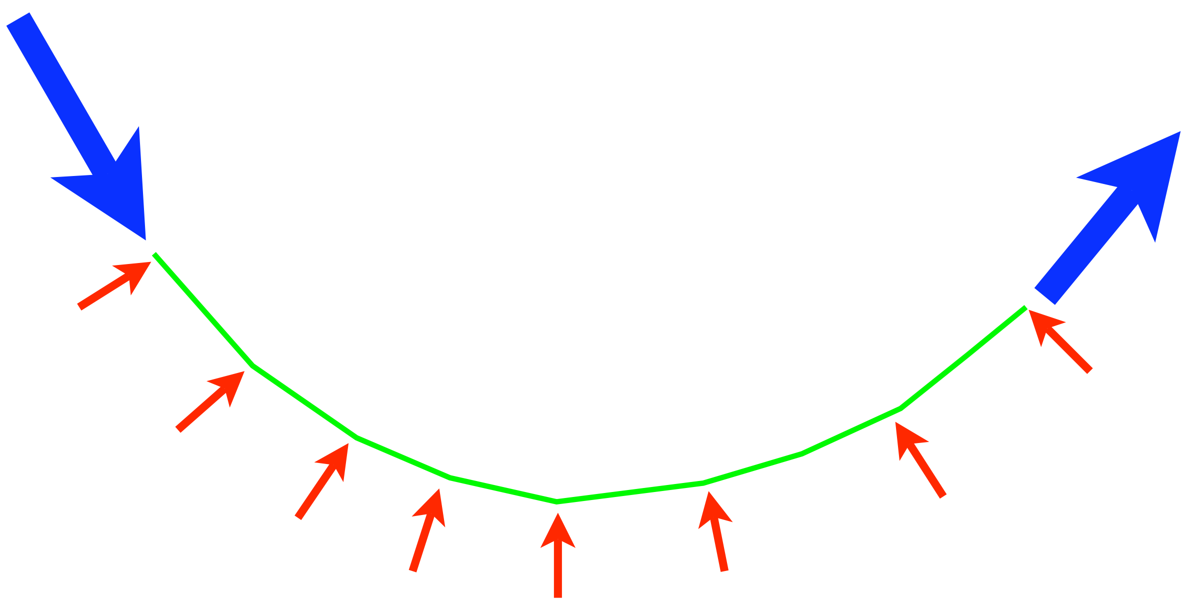
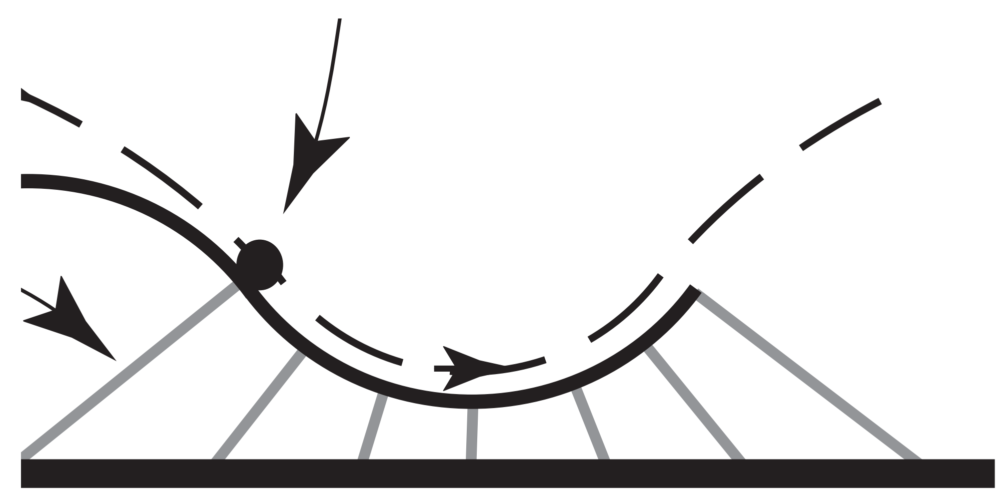




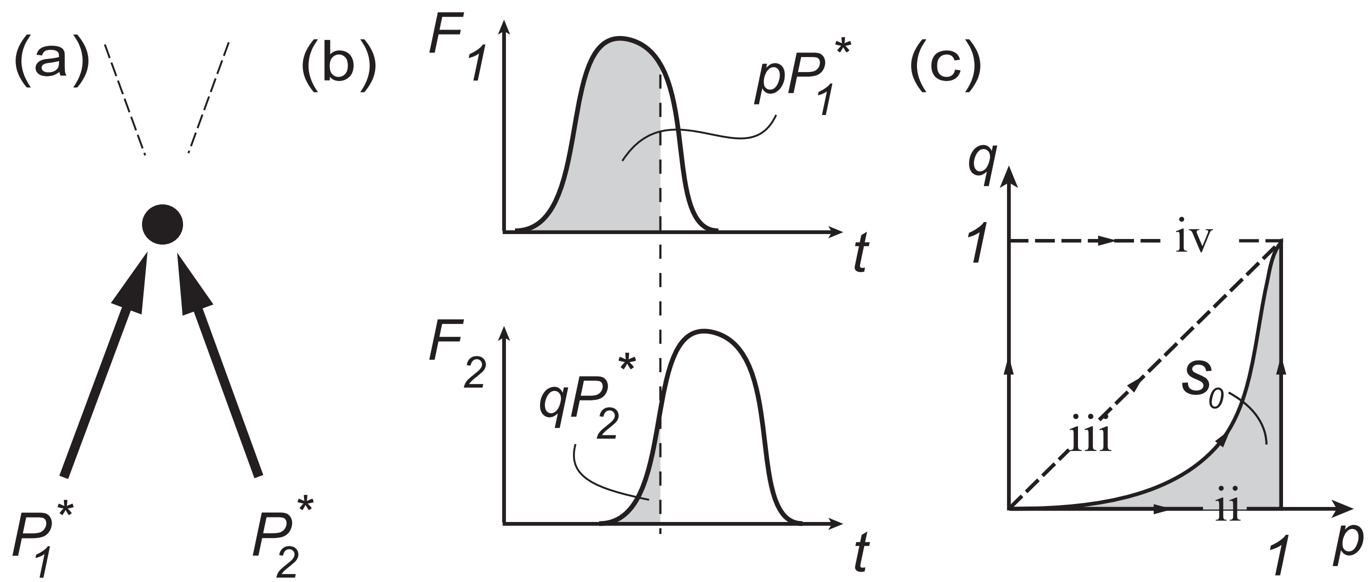
Somewhat odd result:

An infinite number of infinitely small collisions,
each "orthogonal" to the path,
tends to **perfectly elastic**,
no matter the nature of the individual collisions
(plastic or generative or in-between).





"Simultaneous" collisions



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\begin{aligned}
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W &= \Delta E = \frac{m}{2}\left((v^+)^2 - (v^-)^2\right)
\end{aligned}
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\begin{aligned}
W_1 \geq 0 \quad &: \quad \mathbf{v}^- \cdot \hat{\lambda}_\mathbf{1} \geq 0 \quad \text{and} \quad \mathbf{v}^+ \cdot \hat{\lambda}_\mathbf{1} \geq 0 \\
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W_1 &= \int dW_1 = \int_{t_1}^{t_2} \mathbf{v} \cdot \mathbf{F_1} dt \qquad (4 \\
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&= \int_0^1 \left(\mathbf{v}^- + (p\mathbf{P_1^*} + q\mathbf{P_2^*})/m\right) \cdot \mathbf{P_1^*} dp \\
&= \mathbf{v}^- \cdot \mathbf{P_1^*} \int_0^1 dp + \frac{\mathbf{P_1^*} \cdot \mathbf{P_1^*}}{m} \int_0^1 pdp \\
&\qquad + \frac{\mathbf{P_1^*} \cdot \mathbf{P_2^*}}{m} \underbrace{\int_0^1 qdp}_{s_o} \\
&= \mathbf{v}^- \cdot \mathbf{P_1^*} + |\mathbf{P_1^*}|^2/(2m) + (\mathbf{P_1^*} \cdot \mathbf{P_2^*}) s_o/m
\end{aligned}
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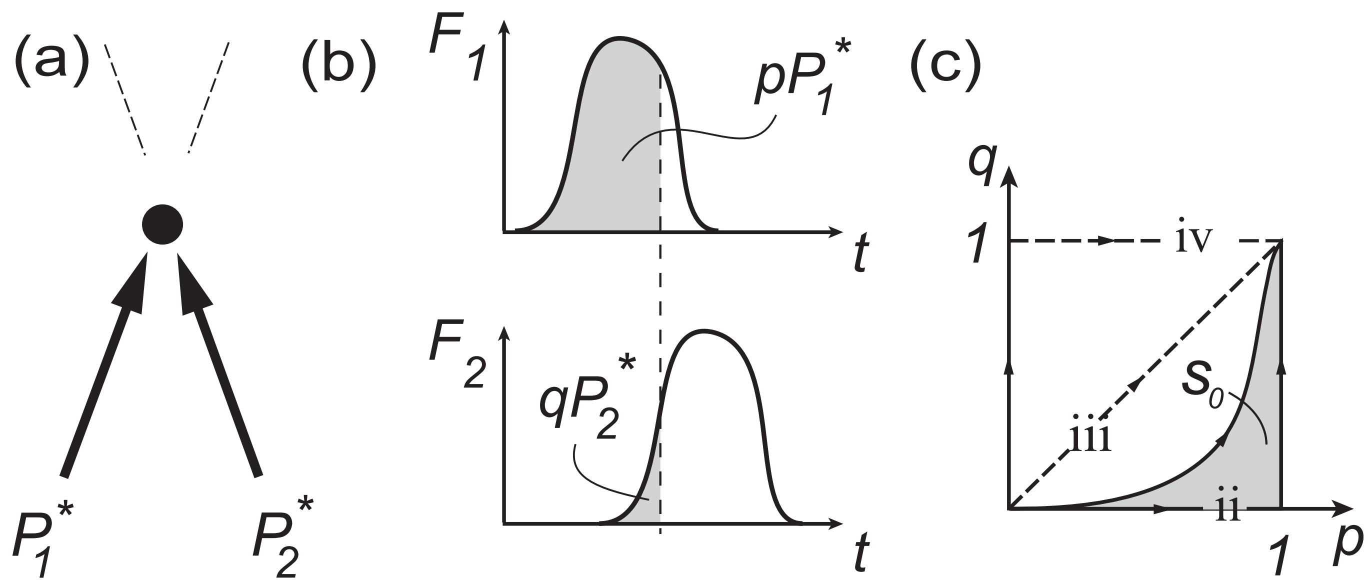
$$
W_2 = \mathbf{v}^- \cdot \mathbf{P_2^*} + |\mathbf{P_2^*}|^2/(2m) + \mathbf{P_1^*} \cdot \mathbf{P_2^*}(1 - s_o)/m
$$

“Simultaneous” collisions



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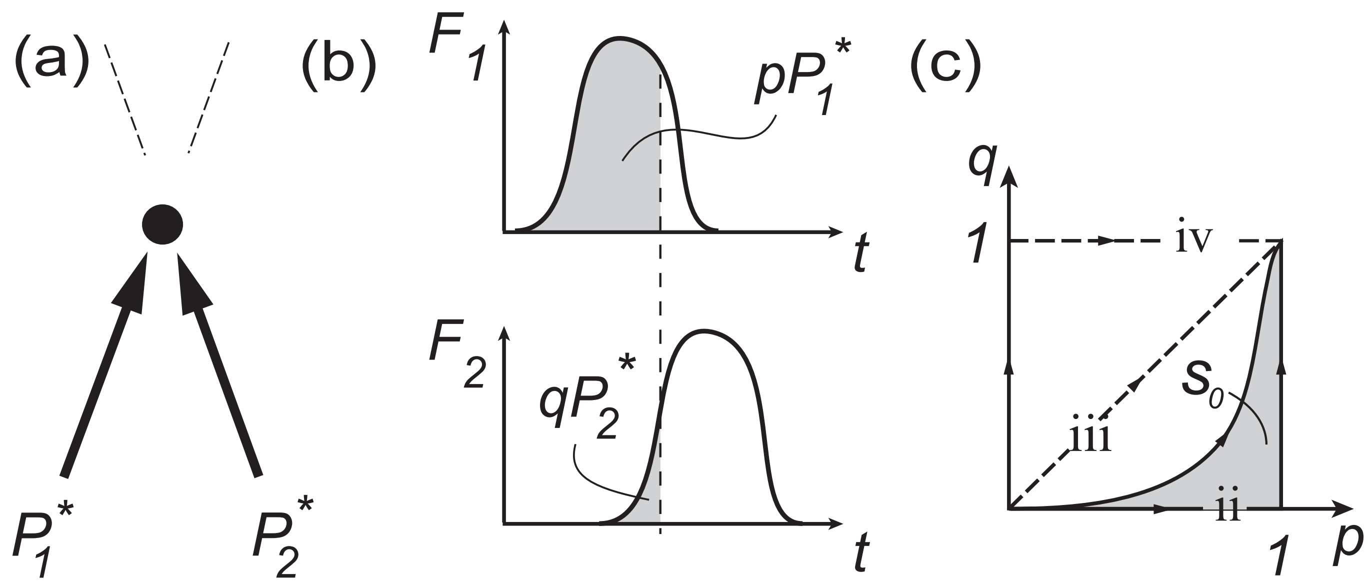
"Simultaneous" collisions



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$$= \mathbf{v}^-\cdot\mathbf{P_1^*}\int_0^1 dp + \frac{\mathbf{P_1^*}\cdot\mathbf{P_1^*}}{m}\int_0^1 pdp + \frac{\mathbf{P_1^*}\cdot\mathbf{P_2^*}}{m}\underbrace{\int_0^1 qdp}_{s_o}$$

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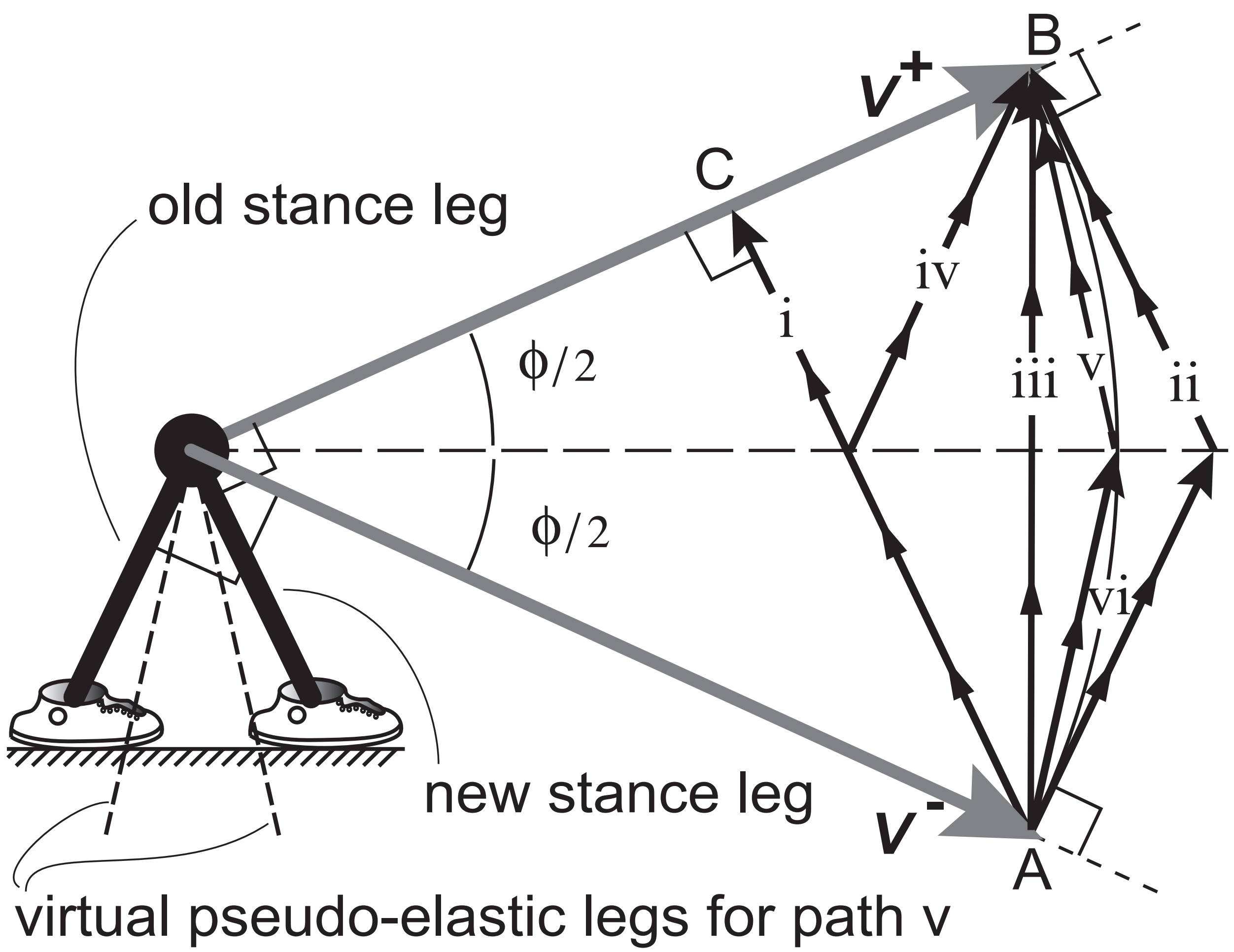
The work depends on the order parameter s_0.

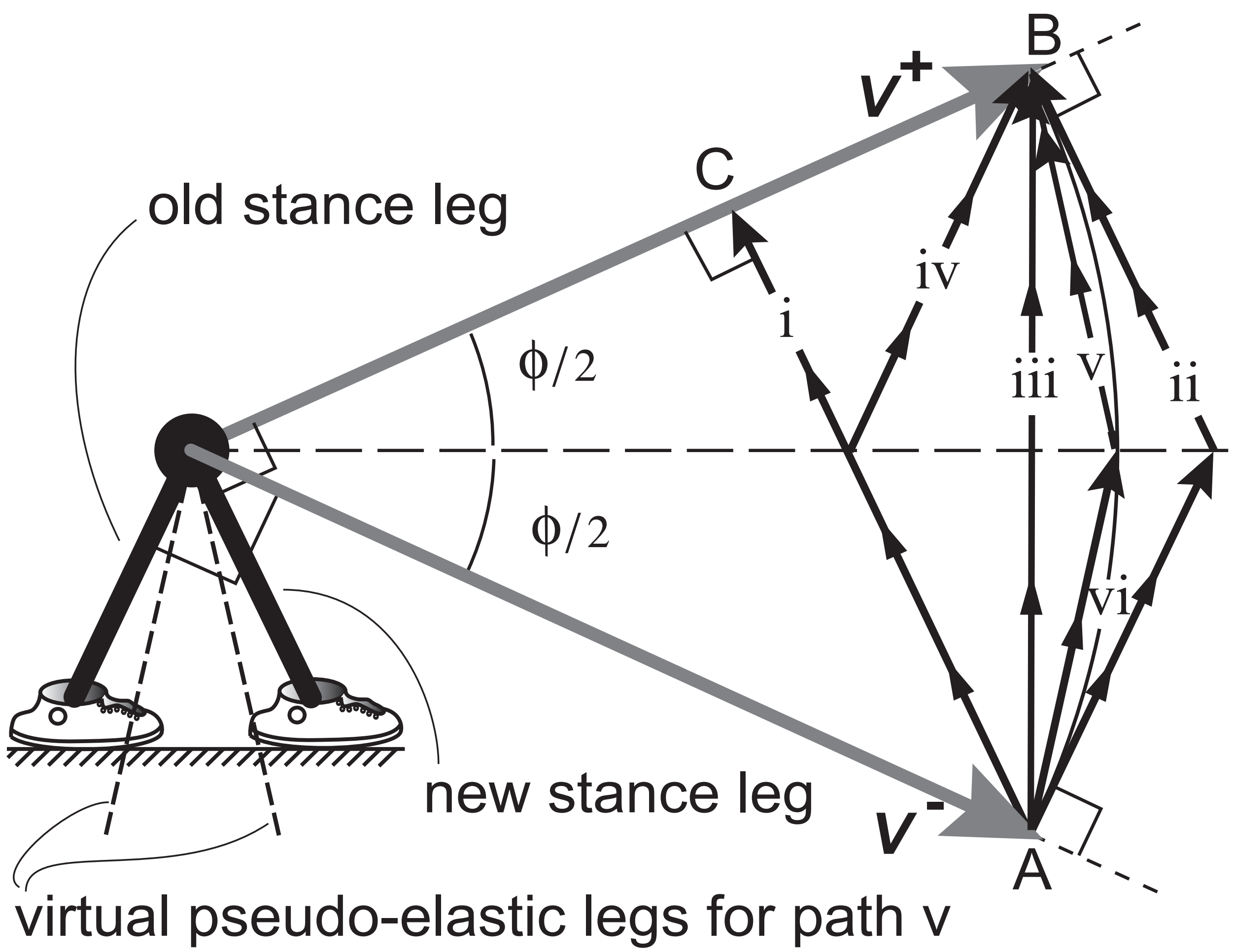
Another distinguished limit.

Back to walking



path	collision reduction factor $= J$ $= E_m / (bmv^2\phi^2/2)$
i	1
iv	3/4
iii	1/2
vi	1/3
ii	1/4
v	1/8

Back to walking



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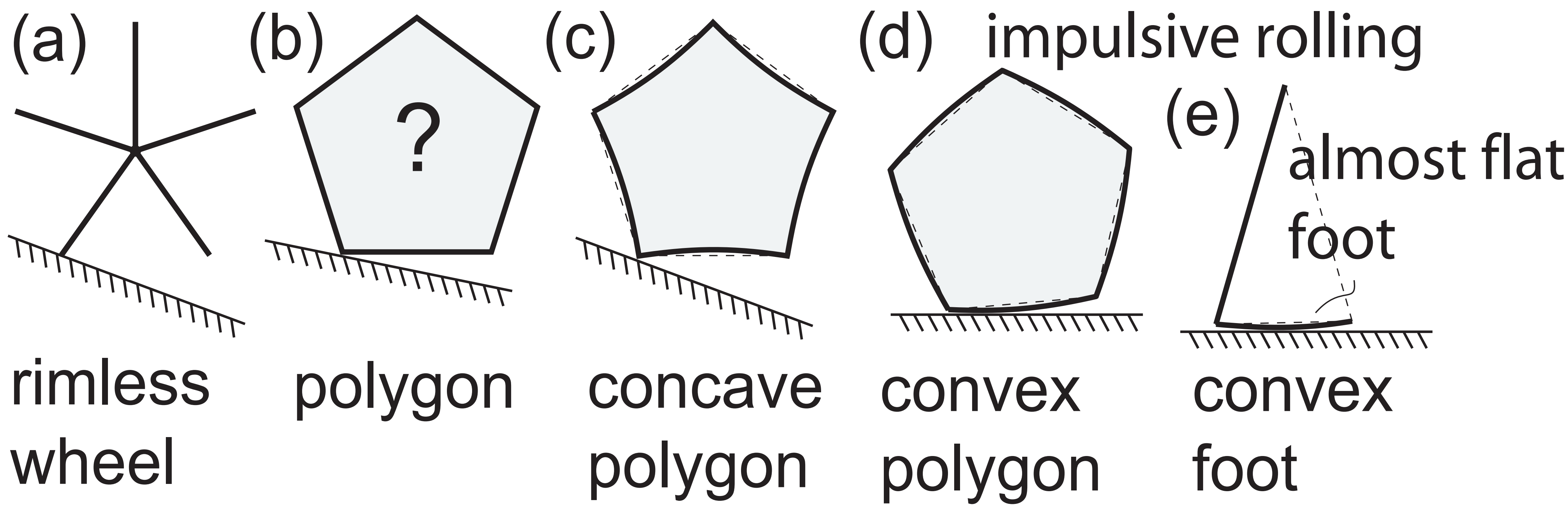
Even within collisional/rolling model, energetics is sensitive to details.

Collisions compared to rolling



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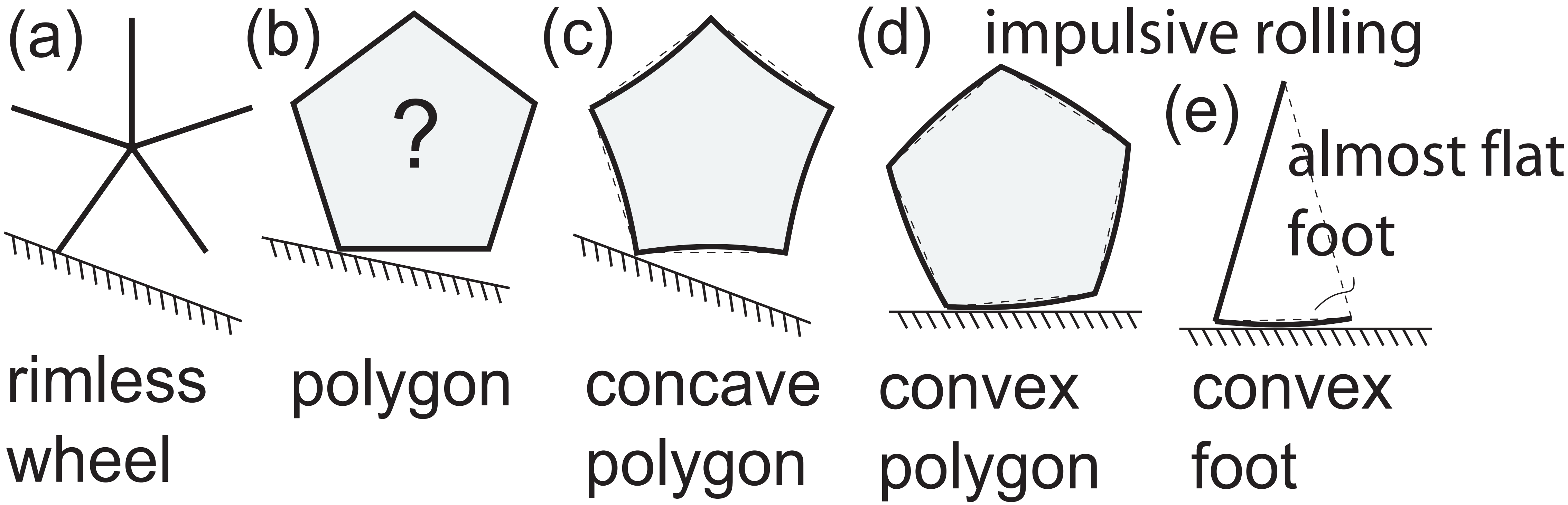
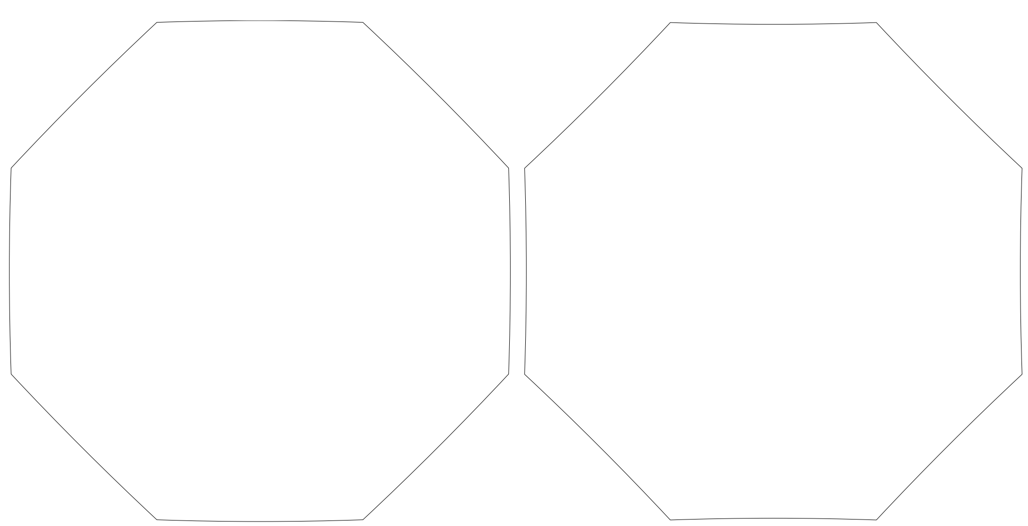
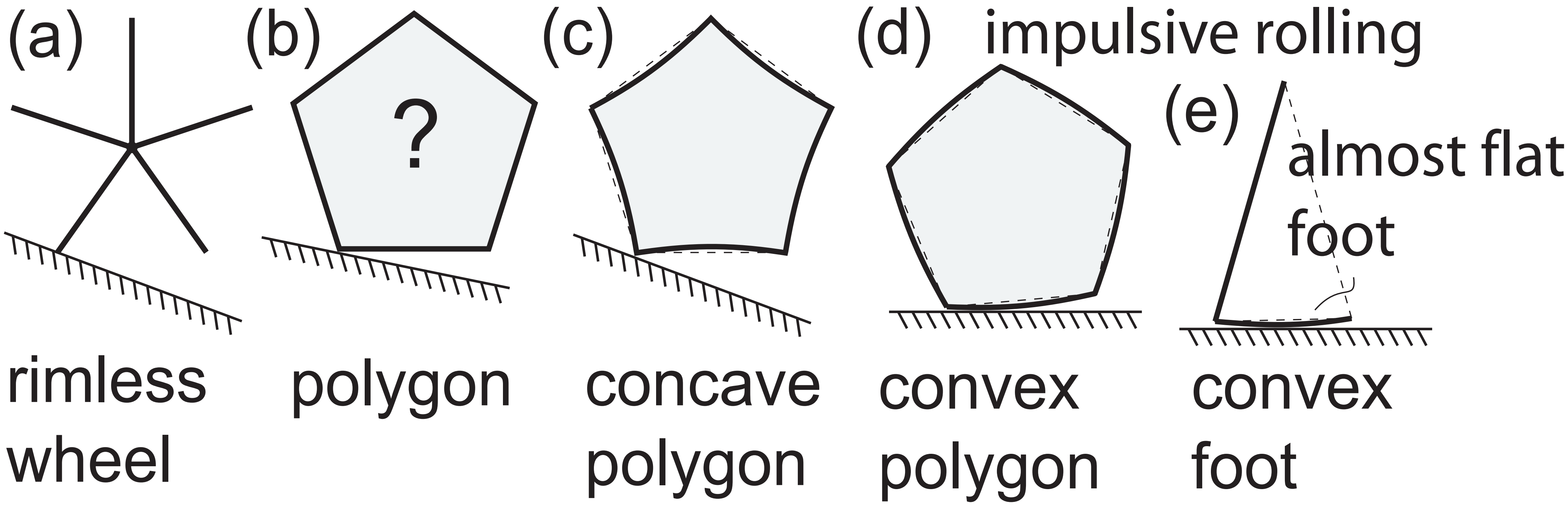
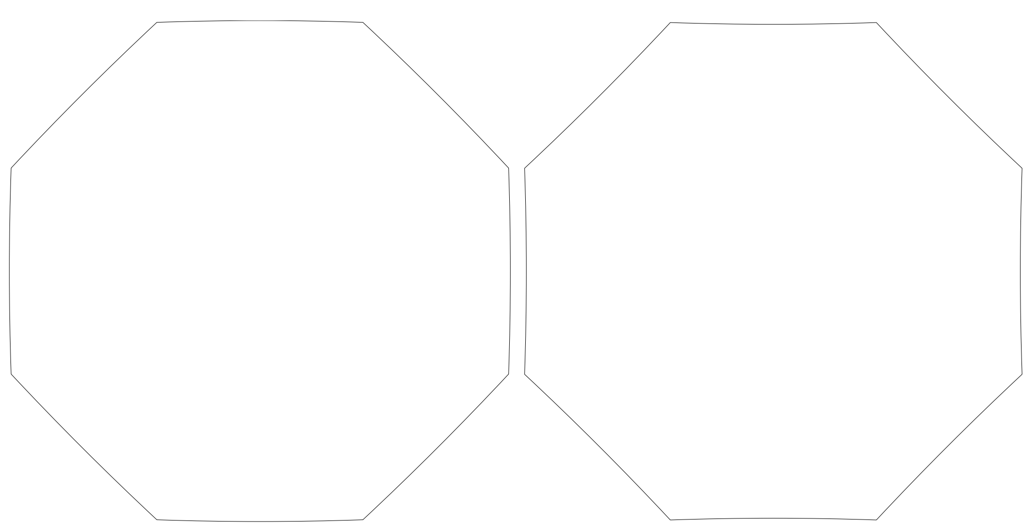




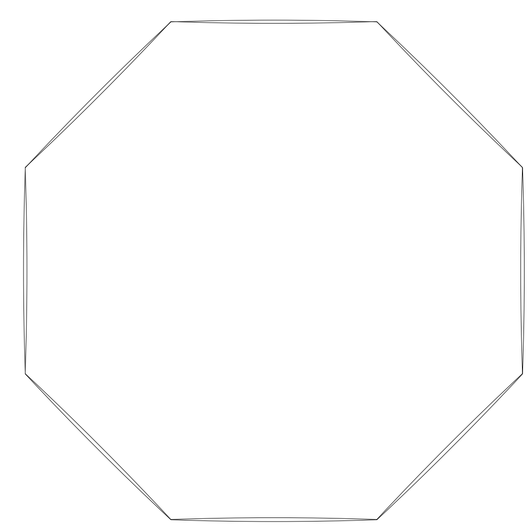
CAD drawings

Collisions compared to rolling



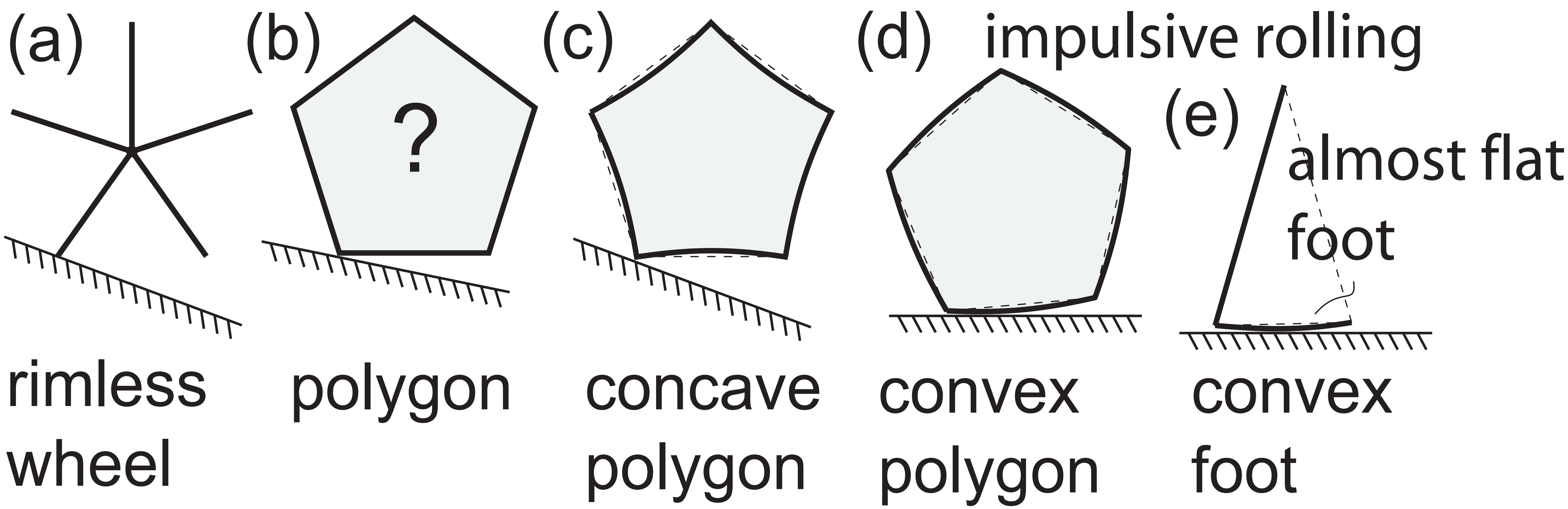


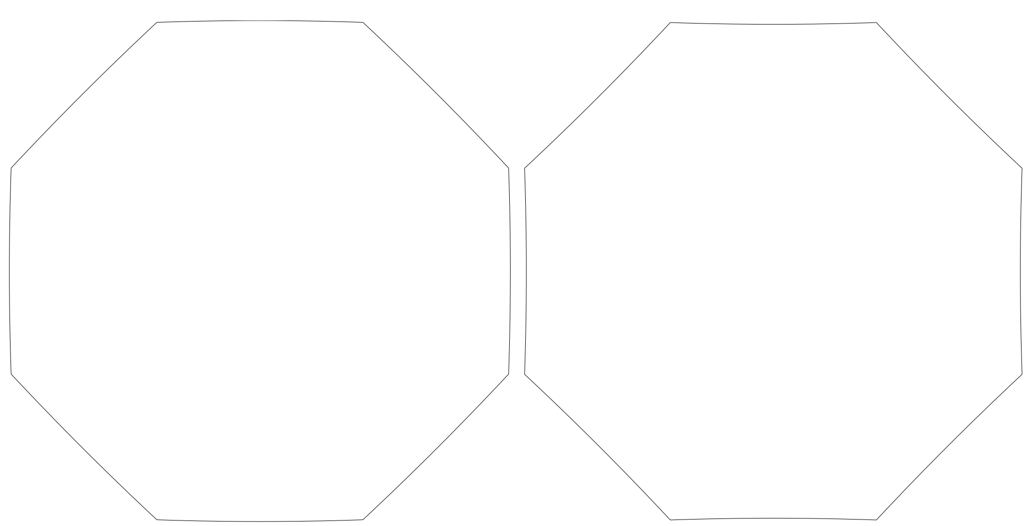
CAD drawings



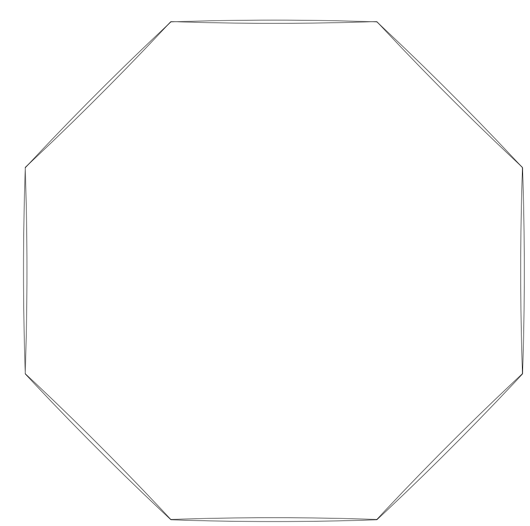
superposed

Collisions compared to rolling





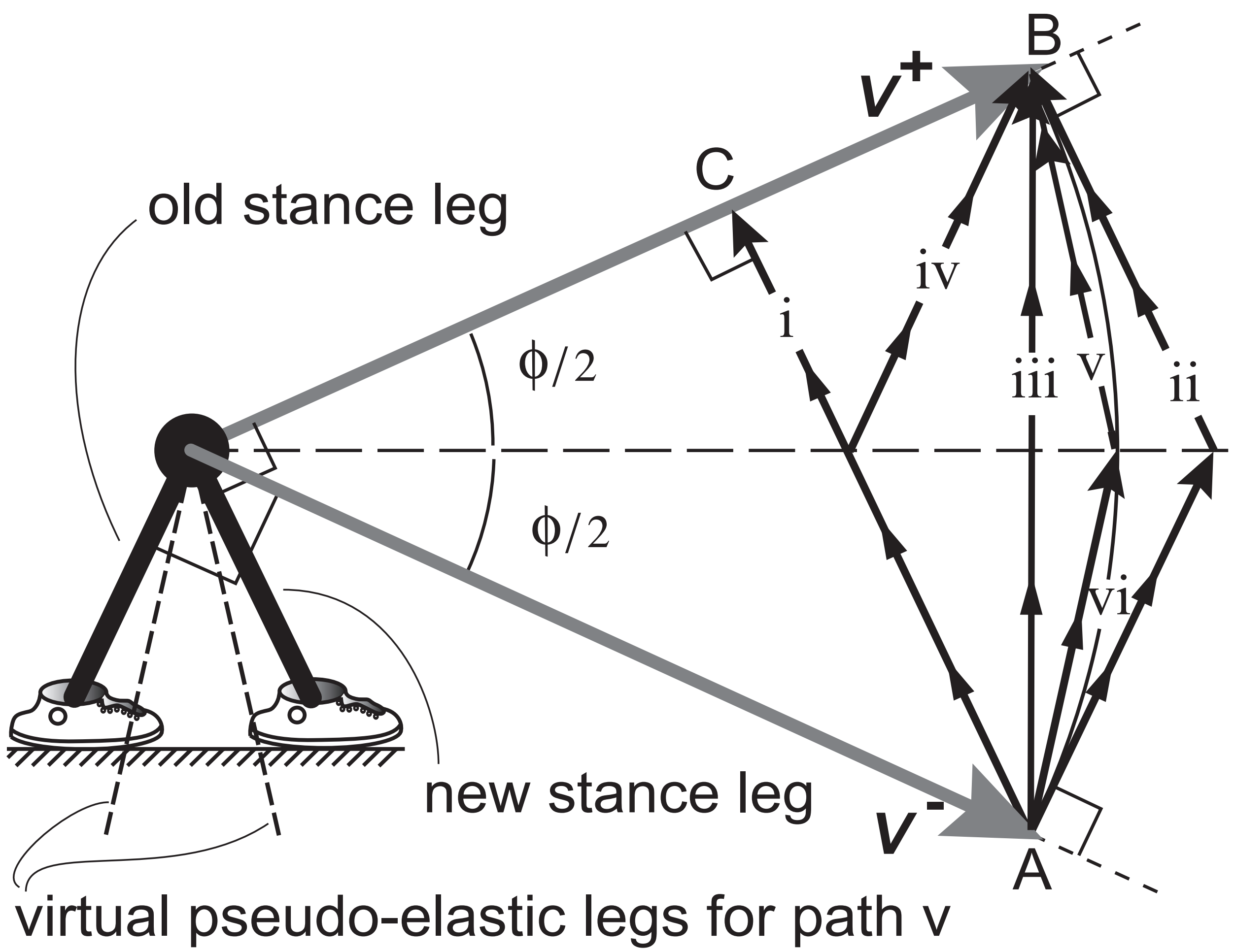
CAD drawings



superposed

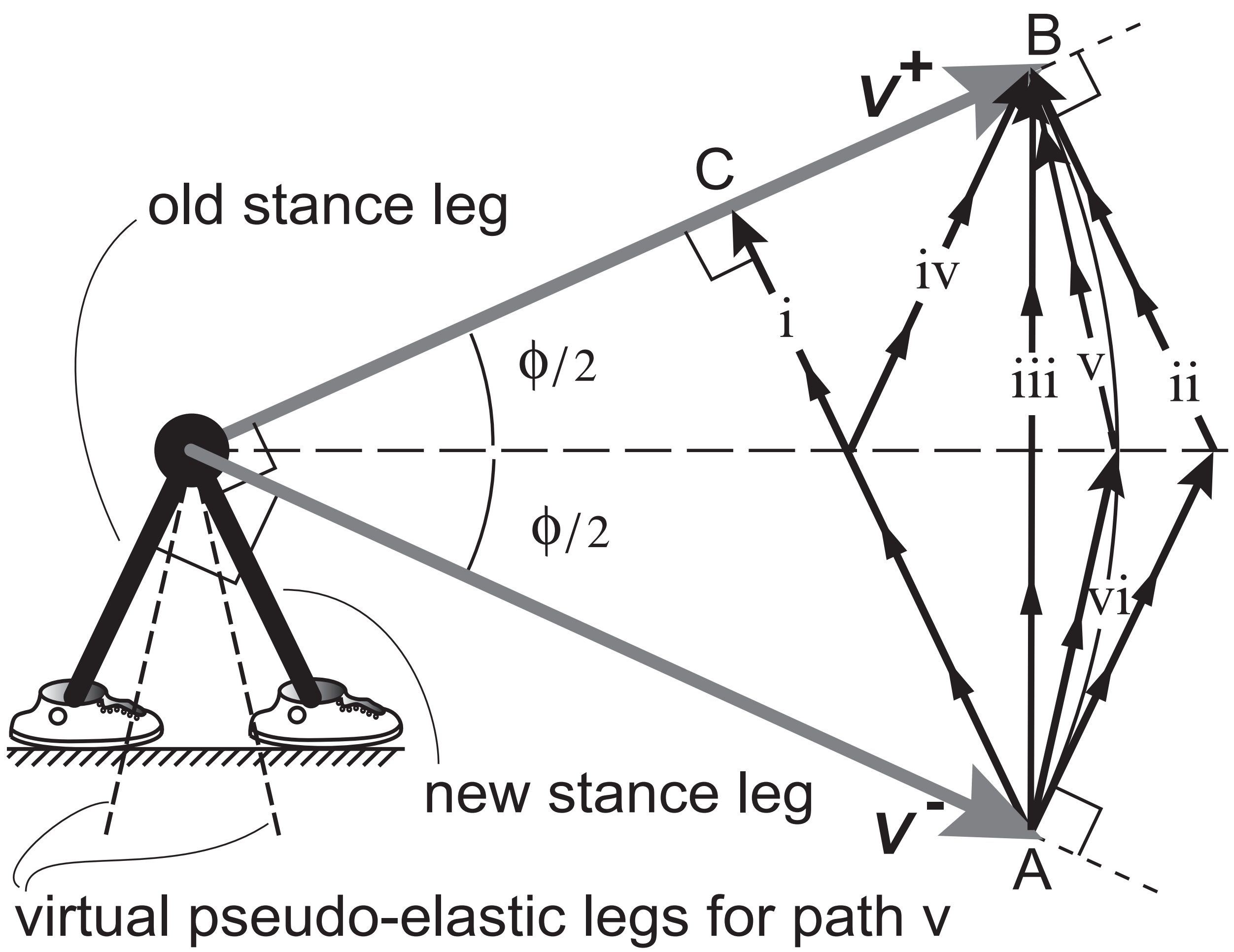
A distinguished limit.

Punchline



path	collision reduction factor $= J$ $= E_m / (bmv^2\phi^2/2)$
i	*1*
iv	*3/4*
iii	*1/2*
vi	*1/3*
ii	*1/4*
v	*1/8*

Punchline



path	collision reduction factor = J = $E_m / (bmv^2\phi^2/2)$
i	1
iv	3/4
iii	1/2
vi	1/3
ii	1/4
v	1/8

Even within collisional/rolling model, energetics is sensitive to details.