A Walking Model with No Energy Cost

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We have found periodic collisionless motions for a rigid-body walking model. Unlike previous bipedal designs, this model can walk on level ground at non-infinitesimal speed with zero energy input. The model avoids collisional losses by using an internal mode of oscillation: swaying of the upper body which is coupled to the legs by springs. This avoidance of foot-strike collision losses by means of synchronized internal oscillations may help in the design of energy-efficient robots and may help to explain aspects of human and animal locomotion efficiency.

I. INTRODUCTION

When an object, robot, or animal traverses level ground at a constant (or steady) speed, the essential forces from gravity and support are orthogonal to the motion. Thus, the essential energetic cost is zero. For example, zero-cost locomotion is achieved by sliding on a frictionless surface or by rolling without slip on a frictional surface. Non-steady motion, *e.g.* a displacement from one stationary position to another, also has zero essential energy cost because the energy used for acceleration can, in principle, be recovered from deceleration.

Can legged transport over level ground be similarly energy-cost free? Nature and engineers have designed low friction hinges, and air-friction losses are small for walking. If one neglects these minor friction losses, is a zero-energy-cost walking mechanism possible?

Consider walking mechanisms made of frictionlessly-linked rigid objects (links) subject to gravity, and supported by a frictional level surface. The mechanisms can have torque actuators and conservative springs at the hinges (joints). If we limit attention to periodic motions, or at least motions with no average change in internal energy, then the positive work of the actuators (muscles) is balanced by negative actuator work (*eccentric* muscle contractions), frictional sliding, and collisions between the feet and the ground. Of central concern here are the collisional losses.

At a collision, energy would be lost by some combination of mechanisms (heat at the collision point, dissipation in muscles and soft tissues, acoustic radiation, etc.). If the rigid-object model is accurate for all parts before and after the collision then linear and angular momentum balance determine the energy lost, regardless of the mechanisms of dissipation [1]. In particular, energy is necessarily lost when any non-massless objects stick together with an abrupt change in velocity. These collision losses are the central energy consumption in passive dynamic walkers [2, 3] as well as some efficient powered robots [4]. Collisions also seem to play an important role in human and animal locomotion [5, 6]. However, no energy is lost in a collision if the new contacts are made in a *collisionless* manner, with zero relative velocity between the approaching objects.

It is possible to create a path for each link of a walking device to follow so that no collisions occur in the walking cycle. Then one could achieve that gait by providing the device with a controller and actuators to create the necessary joint torques to achieve that collisionless path. Blajer and Schiehlen designed and simulated a controlled collisionless walker that had an asymptotically stable gait [7]. However, the actuators in devices such as this must use energy to achieve their collisionless gait. Thus, the search for a zero-energy-cost walker is a search for a machine that has motions with no negative actuator work, with no frictional slip at the ground contacts, and with collisionless contacts. Because we will limit our attention to models with no ground slip and to models with no actuation, our search for a zero energy-cost motion will become a search for a collisionless walking motion.

In general, we will attempt to find a zero-energy-cost motion of a device made out of sticks (rigid bodies) and hinges (frictionless) which can locomote across level ground without any energy input. First we discuss various tricks that one can use to achieve this.

Kinematic mechanisms. Sometime between 1852 and 1894 Chebyshev designed a kinematic one-degree-of-freedom walking device, based on an approximate straight-line mechanism, that guided its endpoints (feet) in a manner that almost avoids ground collisions [8]. A slight change of design could eliminate the small residual collisions. The avoidance of collisions almost achieved by Chebyshev's design is by means of kinematic constraint.

Compliant contact. One could put massless springs on the bottom of the feet. These springs could be compliant in one direction (telescoping along the leg) or in two (2D). If a telescoping leg spring is used, zero-dissipation requires that the foot velocity at contact be parallel to the leg. McGeer found collisionless running motions using such a mechanism [9]. Walking motions of such a design may exist as well. If a 2D spring is used, collision losses at first contact are precluded. However, release of the contact can be dissipative through frictional sliding if the normal component of the spring force goes to zero while leaving a residual nonzero tangential spring force.

Singular-limit of Passive Walking. Garcia *et al.*

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[3, 10] found that some unactuated mechanisms can walk down arbitrarily small slopes with the gravitational power used scaling with v^4 (v = average forward speed). Thus these machines can use arbitrarily small energy for a given forward motion by moving arbitrarily slowly. Chatterjee *et al.* [11] proved that such a McGeer-type walker with no upper body cannot walk at non-vanishing speeds on vanishing slopes. Note that our model does have an upper body.

In the passive-dynamic walkers discussed above, even if the step frequency increases by use of an interleg spring, the normal collision still vanishes as the step-length vanishes. A finite-speed vanishing-energy-cost walking machine could use an interleg spring with stiffness tending to ∞, step length tending to 0, and step frequency tending to ∞.

Our Goal. We search for collisionless motions which don't use the tricks listed above. We seek a collisionless motion of a device that, at least kinematically, allows collisions to occur. That device does not use compliant contact with the ground to avoid collisions, and we look for motions that have exactly (rather than in-the-limit) zero energy cost for non-infinitesimal speed walking.

Chatterjee *et al.* [11] conjectured that a passive-dynamic walking device with an upper body (see Fig.1), could walk at a non-infinitesimal speed on level ground. Here we check this conjecture using the model in Fig. 1.

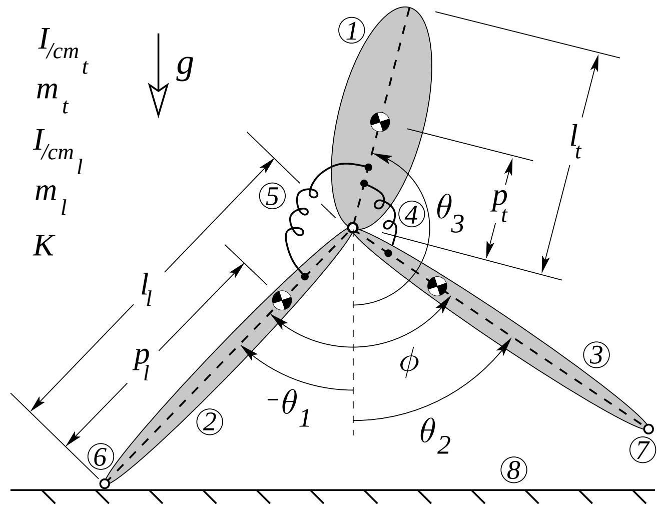

FIG. 1: Three link walking model with hip springs. Modified from [11]. ①torso, ②stance leg, ③swing leg, ④swing leg hip spring, ⑤stance leg hip spring, ⑥stance foot, ⑦swing foot, ⑧ground. Non-geometric parameters are: $I_{/cm_t}$=moment of Inertia of the torso about its center of mass, m_t=mass of the torso, $I_{/cm_l}$=moment of Inertia of the leg about its center of mass, m_l=mass of each leg, l_l=leg length, p_t=leg's center of mass location, l_t=length of the torso, p_t=torso's center of mass location, K= torsional spring constant. The spring constant is the same for both the stance and swing legs.

II. MODEL

The model has two identical legs and a torso connected by a common hinge (Fig. 1 see also [11]). Each leg is also connected to the torso by a torsional spring which is relaxed when the machine stands upright ($\theta_1 = \theta_2 = 0, \theta_3 = \pi$). Thus, if both legs are at rest and touching the ground (at any angle ϕ), the upper body has an equilibrium position (stable or not, depending on parameter values) at $\theta_3 = \pi$. We assume two dimensional motion, inelastic (no bounce) ground collisions, and arbitrarily high friction (no slip) at ground contacts.

The equations of motion when one foot is on the ground are determined by using angular momentum balance of the free leg about the hip, of the torso about the hip, and of the whole system about the ground contact point (see Fig. 1). This results in three second order, nonlinear, coupled, ordinary differential equations in the generalized coordinates θ_1, θ_2, and θ_3. These equations could also have been found using *e.g.* Lagrange equations. When a foot strikes the ground and sticks, the other foot is assumed to simultaneously lose contact.

For general motions, the discontinuity in $\dot{\boldsymbol{\theta}}$ due to the ground collision would be determined by angular momentum balance of the whole mechanism about the new foot contact point as well as angular momentum balance of the two non-ground-contacting parts about the hip. However, because our numerical search was only for collisionless motions (having no velocity discontinuities), we had no need to evaluate the collision-transition relations.

The equations of motion can be rearranged in the form: $[M]\ddot{\boldsymbol{\theta}} = \boldsymbol{v}$, where $[M]$ is a 3×3 matrix that depends on the state, $\ddot{\boldsymbol{\theta}}$ is a column vector of angular accelerations, and $\boldsymbol{v}$ is a column vector with gravity terms and terms quadratic in the angular rates. Due to the complexity of these equations, we handled them by repeatedly forming and solving them numerically as we integrated forward in time (see appendix A).

Previously found passive-dynamic walking motions [9–15] did not have an extended double stance; the solutions, including the collision-transition condition, assumed exactly one foot on the ground at a time. In contrast, the model here, because of its upper-body, is capable of motions with extended double stance. None-the-less, for simplicity, we limited our search here to motions where only one foot was on the ground at a time (see appendix B).

III. SEARCH FOR ZERO ENERGY-COST WALKING MOTIONS

Given mass and length parameter values and initial values for the state ($\boldsymbol{q} \equiv [\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}]$), the equations of motion determine the subsequent motion. We sought the mathematical result that this system has a periodic solution. The complexity of the governing equations seems to put mathematical proof out of reach, so we sought firm numerical evidence.

As is now common for such periodic locomotion problems ([16]), we treated one walking step as a Poincaré map. Given the state just after one foot fall $\boldsymbol{q}_n$, the solution of the governing equations defines the state after the next foot fall, thus determining a map $F(\boldsymbol{q}_n) = \boldsymbol{q}_{n+1}$. Our goal is to find a fixed point, $\boldsymbol{q}^*$; *i.e.* $F(\boldsymbol{q}^*) = \boldsymbol{q}^*$. This is equivalent to finding a root, q^*, of $G(\boldsymbol{q}^*) = \boldsymbol{0}$ where $G(\boldsymbol{q}) \equiv F(\boldsymbol{q}) - \boldsymbol{q}$.

By simple equation counting it seems plausible to find such a root. Finding a root of G is the same as finding a solution of 5 scalar equations in 5 unknowns (the Poincaré section is $6 - 1 = 5$ dimensional). Previous passive-dynamics studies sought, and often found, fixed points of similar maps. However, rather than using a brute force search for solutions of the five dimensional map, we made additional symmetry assumptions, further limiting the search space.

The model has both temporal and spatial symmetry. Between foot contacts the equations of motion are time reversible but, for general motions, the collision transition equations are not. However, the collisionless solutions we seek have no discontinuities and are thus time reversible; any conjectured solution running backward in time is also a solution. Further, the fore-aft physical symmetry of the device means that any solution can be spatially reflected to obtain a second solution.

We restricted our search to *symmetric* solutions which were unchanged by being simultaneously reflected and time-reversed. Thus half of a walking step fully characterizes the full periodic motion (a full step is shown in Fig. 2.

We used the spatially-symmetric ground contact switching point as an initial condition, now restricted to $\dot\theta_1 = \dot\theta_2 = 0$ (zero-velocity impact), $\theta_1 = -\theta_2$ (both feet on the ground), and $\theta_3 = \pi$ (symmetry condition). Defining $\phi = \theta_1 - \theta_2$ at that point results in a 2 dimensional initial condition space $(\phi, \dot\theta_3)$. Our target is to have the swing leg straight down and the torso straight up on the map section where the stance leg is vertical $\theta_1 = 0$, $\theta_2 = 0$, and $\theta_3 = \pi$. In other words, we seek special values of the input variables$(\phi,\ \dot\theta_3)$ to the map, H:

$$H(\phi, \dot\theta_3)|_{(\dot\theta_1=\dot\theta_2=0, \theta_1=-\theta_2, \theta_3=\pi)} = (\theta_2, \theta_3)|_{(\theta_1=0)} \qquad (1)$$

so that the output is $(\theta_2, \theta_3) = (0, \pi)$. The map is $\mathbb{R}^2 \to \mathbb{R}^2$ and our counting argument reduces to $2-2=0$. Thus for these restricted symmetric solutions it would again be non-degenerate to find isolated solutions for the initial conditions (ϕ and $\dot\theta_3$ at $\dot\theta_1 = \dot\theta_2 = 0, \theta_1 = -\theta_2, \theta_3 = \pi$). We sought a solution to this map by making a guess and doing numerical root finding on equation 1 using Newton's method.

IV. RESULTS AND DISCUSSION

As made plausible by the $n = n$ $(2 = 2)$ equation counting argument, using the mass and geometry parameters given in Fig. 2 we found a fixed-point solution (Fig. 2). Because we are trying to demonstrate a mathematical result, and not an artifact of numerical approximation, we wanted to find the root as accurately as we could. We trust the 12 significant figures shown (see appendix C).

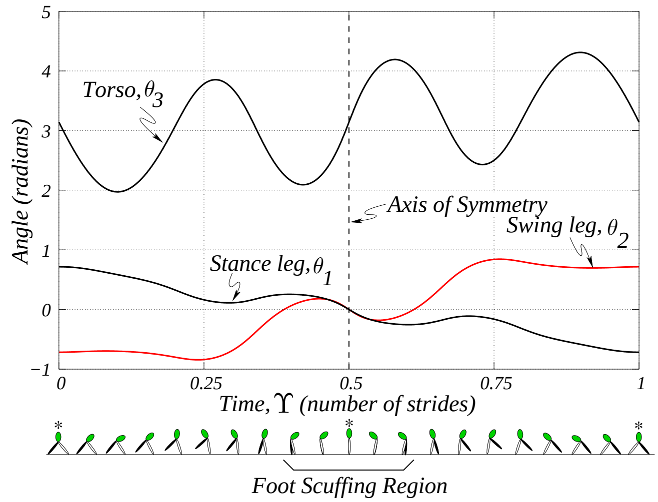

FIG. 2: A collisionless periodic solution, without initial ground penetration, for the three link walking model. The snapshots shown in this figure are spaced evenly in time. The "*" label denotes points in the motion about which the whole stride is symmetric. This solution uses the following non-dimensional parameters: $M_l = m_l/m_l = 1, L_l = l_l/l_l = 1, P_l = p_l/l_l = 0.5, R_l = \sqrt{I_{leg_{cm}}/m_l}/l_l = 0.3, M_t = m_t/m_l = 0.7, L_t = l_t/l_l = 0.6, P_t = p_t/l_l = 0.3, R_t = \sqrt{I_{torso_{cm}}/m_l}/l_l = 0.14, G = g/g = 1, \kappa = K/(m_l l_l g) = 1.5$. The associated initial conditions (the result of the numerical search after convergence tests and discussed in appendix C) are: $\phi/2 = 0.716749728386$, $\theta_3' =$ -7.43120601953, where $(')$ denotes derivatives taken with respect to non-dimensional time, $\tau = \mathrm{t}/\sqrt{l_l/g}$. Despite graphical appearances, θ_1' and θ_2' are not equal at the symmetry point $(\Upsilon = 0.5)$ $(\theta_1' \approx -2.2$ and $\theta_2' \approx -2.5)$. The non-dimensional period of motion shown above is about 2.621.

Although solutions do not exist for all possible mass and length parameter values, solutions seem to exist quite generally.

An unanticipated aspect of the solution in Fig. 2 is the large number of upper body oscillations per step. We searched for collisionless motions with fewer upper body oscillations, but found that they all had the swing foot immediately pass down through the ground when weight shift occurred (see appendix B). We found that in order for the new swing foot to lift off the ground at support transfer, a relatively large angular velocity of the upper body was required. Because of this increased velocity, for the parameters that we used, the computer only found solutions with a higher number of oscillations. A non-systematic study with approximately 10 parameter sets was done but all of the collisionless solutions found, which did not violate our restriction on immediate ground penetration by the swing foot, had multiple oscillations of the upper body.

The stability of the solutions found still need further study. They cannot be formally stable; small perturbations can reduce the energy of the system from which it has no means to recover. However it is possible that the gaits have a one-sided stability, like the collisionless motions of the hopping block of Chatterjee *et al.* [17], allowing stable motion if all perturbations add energy. Alternatively, the solution could also be unstable. However, although the solution is not asymptotically stable, a device built to implement this motion could probably be stabilized by using a proportional controller (with zero energy cost on the target trajectory) for the two map variables with control action taken at each map section.

We note that other zero-energy-cost steady walking motions may exist. The motions we found are symmetric, but we cannot rule out the possible existence of pairs of non-symmetric collisionless motions. We also cannot rule out solutions with a periodicity of more than one step, or even non-periodic solutions. Finally, our solutions assumed one foot leaves the ground when a new one makes contact but we cannot exclude the possibility of solutions which have a period of extended double stance (both feet on the ground for an extended period).

This paper claims a mathematical result, the existence of a root of a given function, as inferred by reasonably accurate numerical search. It seems possible to make the claim mathematically rigorous by using interval arithmetic, generating an interval that necessarily contains a root.

Real, as opposed to ideal, wheels take some locomotory power to overcome various small non-ideal effects (air friction, bearing friction, and contact dissipation). Similarly this paper shows by means of an example that the only essential energetic costs for walking are those associated with non-ideal effects (joint friction, air friction, imperfections in the trajectories, etc). Energy losses through collisions is not essential for walking, even idealized passive dynamic walking.

We have demonstrated the possibility of using internal oscillations as a means of eliminating collisional dissipation in forward walking. This complements the related hopping-in-place solution of Chatterjee et al. [17]. As opposed to all other passive dynamic walking designs to date (*e.g.* [10, 12, 15, 18–20]), this device walks on *level* ground (in the ideal sense that a wheel rolls steadily on level ground), by eliminating collisions. Perhaps the general concept of collision avoidance will be of use in energy efficient legged robot locomotion. The collision-reduction concept might partially explain aspects of animal and human coordination patterns. For example, Maloiy *et al.* [21] conjecture that some African women might have developed a technique to achieve a higher than normal efficiency in walking when carrying a load and we surmise it is possible that these women coordinate the motions of their load to reduce their collision losses with the ground, thereby reducing their metabolic effort.

Acknowledgments

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APPENDIX A: EQUATIONS OF MOTION AND MAP

The equations of motion could be explicitly stated for this device, but we did not form them as we deemed them too lengthy. Even if generated with computer algebra, we have found that manipulating such large equations adds no insight yet increases chances for error in text manipulation. Rather, we carry out our integration without forming the state derivatives explicitly, generating the numeric values of the derivatives of the state vector four times (4th order RK integration) for each integration time step. The use of this procedure, as opposed to an analytic formula, makes every line of the program short and interpretable. The code is available upon request.

The equations for the two dimensional map, being based on the solution of these non-linear ordinary differential equations, cannot be written down in a closed form expression of any length. Such an expression would require an analytic solution for the time, t^*, when $\theta_1(t^*) = 0$. Even for linear problems with closed form solutions, evaluation of this time (t^*) generally involves solution of a transcendental equation [10].

APPENDIX B: GROUND CONTACT ISSUES

Lift off. When one foot makes contact with the ground we assume the other loses contact. The new swing foot may move up or down through the floor, depending on model parameters and initial conditions. In the solution we have presented the newly-swinging foot has a positive (upward) initial vertical component; we exclude solutions where the foot's first motion is through the floor.

Scuffing. Straight legged walkers in 2D will scuff their feet when walking. Scuffing refers to the swing foot passing through the ground when the relative angle between the legs is small. In ours, as well as other straight legged passive-dynamics research, scuffing is ignored. Such scuffing is viewed as a different decoupled problem (solved by various means: walking on spaced tiles, retracting ankles, or bending knees). Implementation of such anti-scuffing mechanisms in a physical device will have a small effect on the dynamics.

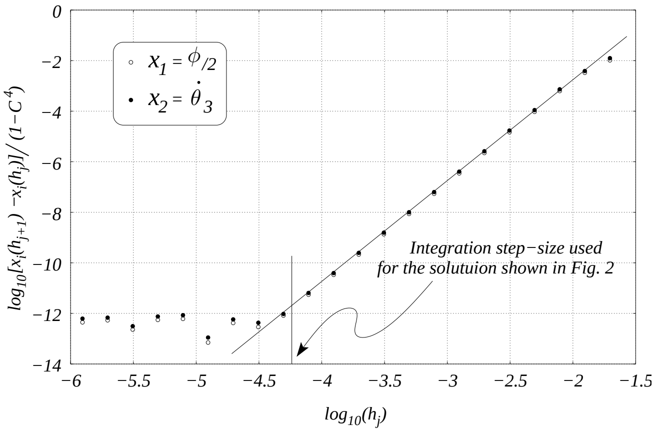

FIG. 3: Convergence plot for the 2 input map variables of the walker resulting in a zero of the map, using a fourth order explicit Runge-Kutta integration routine. The error of the solution can be estimated in the absence of the true solution by using two approximate solutions with known constant integration step-sizes [22]. The x-axis is the logarithm of the integration step size(h_j) used for a simulation number j. Note that for this convergence plot, there are five evenly spaced integration step sizes per unit on the x-axis, so $h_{j+1} = 10^{-0.2} h_j = C h_j$, $C = 10^{-0.20}$. The y-axis of the plot is the logarithm of the estimated relative error in $\phi/2$ and $\dot{\theta}_3$ which result in a symmetric motion.

APPENDIX C: ACCURACY OF INTEGRATION, MAP AND ROOT

We used a fixed step-size (step size picked judiciously, see below) fourth-order explicit Runge-Kutta routine to numerically integrate our equations of motion. Henon's method (a change of variables that replaces a traditional root finding procedure) was used to accurately determine the location of the event [23] (*e.g.* the halfway point of a stride, when $\theta_1 = 0$). A two-dimensional Newton's method was used to to find the fixed points of the numerically integrated map. Finite differences (on initial states) were used to approximate the required Jacobian.

The integration step size determines whether round-off errors (due to the finite precision of computer arithmetic) or truncation errors (due to the Taylor series of the solution being of infinite order while the integration method is only accurate to 4th order) dominate in the approximate solution. The convergence plot shown (Fig. 3 [22]) shows that round-off error begins to dominate, at a relative error of about 10^{-12}, when the step size is less than about $10^{-4.5}$. The slope of 4 in the straight-line portion follows from the method being a 4th order method.

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