THE SIMPLEST WALKING MODEL: STABILITY, COMPLEXITY AND SCALING.

Derivation of the transition rule at heelstrike:

**Just Before Heel Strike**

**Just After Heel Strike**

To get the transition rule we will conserve ANGULAR MOMENTUM of the system about the point on the ground and coincident with A.

From the figure it can be seen that the only force giving rise to an angular impulse during the time \( \Delta t \) (during which the heelstrike occurs) is \( N_1 \). This is because \( N_2 \) and \( F_2 \) are acting at A while \( F_1 \) passes through A.
However, we will assume that the angular impulse due to $N$, is negligible as compared to the ones due to the forces at A. We can attempt to justify this by the fact that C is "LIFTING OFF" as opposed to A which is "STRIKING".

Making the above assumption enables us to equate the angular momentum of the system before and after heelstrike.

Now,

Angular Momentum of the system = Angular Momentum of mass at A + that of mass at B + that of C.

We further assume that the hip mass, $M$ is much greater than the leg mass, $m$, i.e.

$M \gg m$

Thus, the angular momentum of C can be neglected in comparison to B's.

As the angular momentum is being taken about A (outer point), angular momentum of A is identically zero.
Hence,
\[ M \cdot V_B^- \cdot \gamma_{BA1} = -M \cdot V_B^+ \cdot \gamma_{BA1} \]  

where,
\[ V_B^- \] : Velocity of \( B \) before heelstrike
\[ V_B^+ \] : Velocity of \( B \) after heelstrike
\[ \gamma_{BA1} \] : Distance of \( A \) from the line of action of \( V_B^- \)
\[ \gamma_{BA1}^+ \] : Distance of \( A \) from the line of action of \( V_B^+ \)

To obtain \( V_B^- \) \( \gamma_{BA1} \) consider:

\[ \text{Before heelstrike} \]
\[ \text{BC is rotating about C with angular velocity} \]
\[ \frac{d }{dt}(\theta^-) \]
\[ = -\dot{\theta} \]

hence,
\[ V_B^- = -[\dot{\theta}^-] \]
\[ \gamma_{BA1}^- = [\cos \phi^-] \]  

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For $V_B^+$ and $V_{B,AL}$

After heel strike $AB$
starts rotating about
$A$ with an angular velocity $= -\dot{\theta}^+$.

Note the negative sign,
it has been used because
decreasing $\theta^+$ corresponds to the
direction of $V_B^+$ shown in the fig.

Thus,

$$V_B^+ = -l\dot{\theta}^+$$
$$V_{B,AL} = l$$

$$\begin{align*}
\text{Substituting (2) and (3) in (1)} \\
-Ml \ddot{\theta}^- \cos\phi^- &= -Ml \dot{\theta}^+ \\
\text{or,} \\
\dot{\theta}^+ &= \dot{\theta}^- \cos\phi^-
\end{align*}$$

To complete the process, we isolate $Bc$
and write the conservation of angular
momentum equation about a point
coincident with $B$ but fixed in space.

Angular impulse due to contact forces
at $C$ are again neglected.
Angular momentum of M about C will be zero. And so we will get,

$$m V_c^- Y_{cB}^- = m V_c^+ Y_{cB}^+$$  \( \text{Eqn. 5} \)

where the terms have meanings similar to the terms in eqn 1.

Note, before strike \( V_c^- = 0 \), hence the LHS of (5) is zero.

After strike,

The figure also shows the velocity diagram of point C. It graphically represents the equation

$$V_c^+ = V_b^+ + V_{cB}^+$$

where \( V_{cB}^+ \) is the velocity of \( C \) w.r.t \( B \).

Note that the only way (5) can be satisfied is when \( Y_{cB}^- = 0 \).

This in turn means that the line of action of \( V_c^+ \) passes through \( C \).

So,

$$V_c^+ = V_b^+ \cos \left( \frac{\pi}{2} - \varphi^+ \right)$$

from (3) \( V_c^+ = l \hat{\theta}^+ \sin \varphi^+ \)  \( \text{Eqn. 6} \)
Also,

\[ V_{c^+} = V_{c^0} \cot \left( \frac{\pi}{2} - \phi^+ \right) \]

\[ = 1 \left( \phi^+-\theta^+ \right) \tan \phi^+ \quad \ldots \quad (7) \]

Using (6) in (7) and rearranging,

\[ \phi^+ = \left( 1 + \cos \phi^+ \right) \theta^+ \quad \ldots \quad (8) \]

Now for some geometrical considerations,

Noting that the transition at helitake is practically instantaneous we can write down the following equations,

\[ \phi^+ = -\phi^- = \phi \quad \ldots \quad (9) \]

\[ \theta^+ = -\theta^- = \theta \]

And since the length of the legs is equal,

\[ \phi = 2\theta \quad \ldots \quad (10) \]

(4) and (8) together give

\[ \phi^+ = (1 + \cos \phi^+) \cos \phi^- \theta^- \quad \ldots \quad (11) \]
and (4) and (11) taken together with (9) and (10) give,

\[ \dot{\theta}^+ = \cos(2\theta) \dot{\theta}^- \]
\[ \dot{\phi}^+ = \cos(2\theta) [1 + \cos(2\theta)] \dot{\phi}^- \]

Now we can put (7), (10) and (12) in matrix form to get the desired transition rule:

\[
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix}^+ =
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & \cos(2\theta) & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & \cos(2\theta)[1 - \cos(2\theta)] & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix}^-
\]