Reprinted from June, 1983, Vol. 105, Journal of Applied Mechanics

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Stability of Steady Frictional Slipping

The shear resistance of slipping surfaces at fixed normal stress is given by $\tau = \tau(V,state)$. Here V = slip velocity, dependence on "state" is equivalent to functional dependence with fading memory on prior $V(t)$, and $\partial\tau(V,state)/\partial V>0$. We establish linear stability conditions for steady-slip states ($V(t),\tau(t)$ constant). For single degree-of-freedom elastic or viscoelastic dynamical systems, instability occurs, if at all, by a flutter mode when the spring stiffness (or appropriate viscoelastic generalization) reduces to a critical value. Similar conclusions are reached for slipping continua with spatially periodic perturbations along their interface, and in this case the existence of propagating frictional creep waves is established at critical conditions. Increases in inertia of the slipping systems are found to be destabilizing, in that they increase the critical stiffness level required for stability.

Introduction

For many mechanical systems in sliding contact with an adjoining body, loading by the imposition of a constant relative displacement rate, directed parallel to the contact surface, is observed to lead to nonconstant slip motion at that surface. This unsteady motion is often referred to as "stick slip." It is exemplified by squeaking machinery, oscillating violin strings, and unstable fault slip on the boundaries of the Earth's crustal plates. On the other hand, motion with constant slip rate is often observed in situations that appear very similar. What distinguishes these two cases?

The simplest, although not complete, approach to this problem is to ask: Is steady sliding a possible stable motion? Classically this is analyzed by assuming the friction stress τ (at fixed normal stress σ) to be dependent on slip rate V only, i.e., $\tau = \tau(V)$. Then a one degree-of-freedom elastic system yields the following simple result, attributed to Rayleigh in his study of the violin string-bow interaction (Kosterin and Kragel'skii [1]): If $d\tau(V)/dV>0$, steady sliding is stable; if $d\tau(V)/dV<0$, steady sliding is unstable. If steady sliding is unstable, a nonlinear description, including full description of the function $\tau(V)$, possibly embodying the concept of higher static versus kinetic friction, leads to predictions of oscillations that may be very abrupt (relaxation oscillations) or nearly sinusoidal (Kosterin and Kragel'skii [1], Brockley and Ko [2]).

The simple stability result just mentioned contradicts the common experimental observation of steady slip in an adequately stiff machine even though the frictional stress τ is often less for greater steady sliding rates. The contradiction is resolved, however, by the analysis in this paper, which is based on a recently established constitutive framework for frictional slip, more comprehensive than that mentioned in the foregoing.

We derive conditions within this framework for the stability against small perturbations of steady frictional slipping in some mechanical systems. The analysis generalizes considerably the first results within this constitutive framework, obtained by Ruina [3] for a special class of frictional constitutive relations involving a single evolving state parameter. Implications for nonlinear analysis are mentioned at the end of the paper.

Constitutive Description of Frictional Slip

Recent experiments with rocks (Dieterich [4–8]; Ruina [3]) as well as earlier experiments with metals (Rabinowicz [9, 10]) suggest a constitutive framework, for sliding at fixed normal stress σ, in which the shear stress τ resisting unidirectional slip is regarded as being a function of both the slip rate V and the state of the surface, where the latter evolves with ongoing slip. We summarize this dependence by writing

$$\tau=\tau(V,\text{state}) \quad (\text{for } \sigma \text{ constant}) \qquad (1)$$

and regard the dependence on "state" as being equivalent to a functional dependence of τ on prior V. That is, assuming a loss of memory of slips in the distant past,

$$\tau(t)=F[V(t);\quad V(t'),-\infty<t'<t]. \quad (\sigma(t) \text{ constant}) \qquad (2)$$

A useful way of studying this functional dependence (Dieterich [6–8]; Ruina [3]), which will be illustrated shortly for a linearized perturbation version of equation (2), is to determine the response $\tau(t)$ to a suddenly imposed step change in $V(t)$. Such experiments as carried out thus far suggest a competition between the instantaneous dependence on rate and the dependence on the evolving state. Namely, τ increases (decreases) simultaneously with the suddenly im-

Contributed by the Applied Mechanics Division for presentation at the 1983 ASME Applied Mechanics, Bioengineering, and Fluids Engineering Conference, Houston, Texas, June 20–22, 1983 of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, May, 1982; final revision, August, 1982. Paper No. 83-APM-16.

Copies will be available until February, 1984.

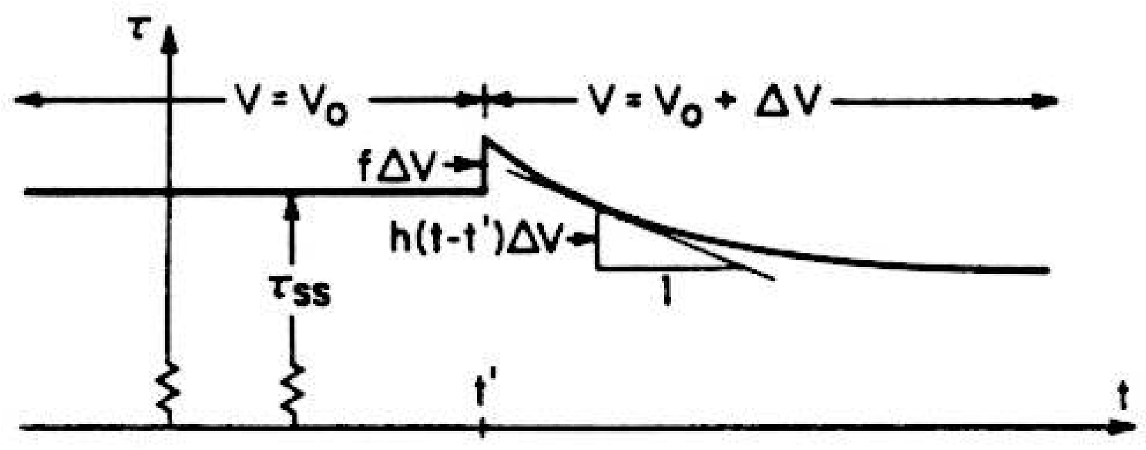

Fig. 1 Resistive shear stress τ in response to a sudden step ΔV in slip rate imposed at time t'

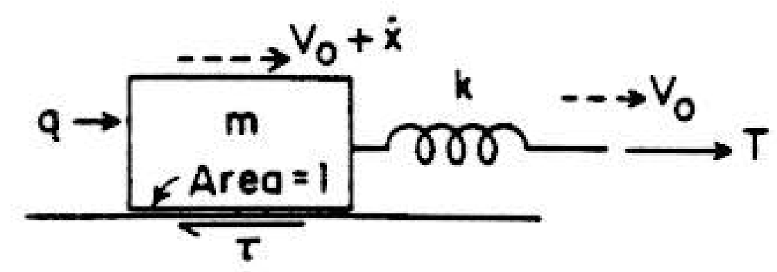

Fig. 2 One degree-of-freedom system represented by sliding block and attached spring. The velocity of the spring end is imposed as V_0 and stability of steady slipping is examined.

posed increase (decrease) of V. Presuming this instantaneous change to represent response of the surface at its current state, we have therefore that

$$\partial\tau(V,\text{state})/\partial V>0. \tag{3}$$

This may be contrasted with the constitutive framework often considered for straining of inelastic solids, in which the material exhibits an instantaneous elastic response and thus no jump in stress for a jump in deformation rate.

However, the instantaneous increase of τ resulting from an increase of V is not maintained. Rather, as slip progresses with, say, V held constant at its increased value, τ is found to decay in value, and we interpret this as meaning that the "state" of the surface is evolving toward a new one consistent with the increased V. Indeed, it is indicated in the experiments cited that independently of prior slip history, if V is maintained constant, then τ evolves toward a steady state value, τ_{ss}, which is a function only of V. We interpret this as meaning that the state term in equation (1) evolves to one that is dependent only on V, and therefore require that the constitutive relation exhibit the behavior

$$\tau(V,\text{state})\rightarrow\tau_{ss}(V), \quad \text{for } V \text{ constant.} \tag{4}$$

Furthermore, in most of the experimental studies cited in the foregoing, it is found that $\tau_{ss}(V)$ is a decreasing function of V,

$$d\tau_{ss}(V)/dV<0, \tag{5}$$

although studies at elevated temperature (Stesky [11, 12]), and on surfaces that have undergone relatively little total slip (Solberg and Byerlee [13]; Dieterich [8]), show that the inequality (5) need not always be met. We show subsequently that under plausible assumptions of the nature of the decay to steady state, inequality (5) is a necessary, but not sufficient, condition for instability (under small perturbations) of steady slip. Regarding the order of magnitude, experimental results [3, 5–8] suggest that the velocity derivatives in (3) and (5) are of order $\pm 0.01\bar{\sigma}/V$.

The existence of an instantaneous positive viscosity-like property of frictional response as in (3), with a long-term negative viscosity, as in (5), is a recent discovery in the work of Dieterich [5–8] and Ruina [3], although such a competition of effects was postulated by Tolstoi [14]. Classical descriptions of friction seem to recognize only inequality (5), often summarized as saying that "static" friction is greater than "kinetic" friction (e.g., Jenkin and Ewing [15]).

For our present purposes of examining stability within a small-perturbation theory, we linearize the dependence of $\tau(t)$ on $V(t)$ in equation (2). In particular, we perturb (2) about a steady state at slip rate V_0, writing

$$V(t)=V_0+\dot{x}(t) \quad (|\dot{x}(t)|/V_0<<1), \tag{6}$$

and express the result for $\tau(t)$ as

$$\tau(t)=\tau_{ss}+f\,\dot{x}(t)-\int_0^t h(t-t')\dot{x}(t')dt' \tag{7}$$

(assuming that $\dot{x}(t)=0$ for $t<0$ and that conditions at $t=-\infty$ have no effect). Here all of τ_{ss}, f, and $h(t)$ depend on V_0. The significance of f and $h(t)$ is illustrated in Fig. 1 for a small step increase in V, from V_0 to $V_0+\Delta V$, at time t'. It is evident that

$$f=\partial\tau(V,\text{state})/\partial V, \quad f-\int_0^\infty h(t)dt=d\tau_{ss}(V)/dV \tag{8}$$

where the derivatives of τ are evalated in the steady state with speed V_0. Hence the inequalities (3) and (5) are equivalent, respectively, to

$$f>0, \quad \text{and} \quad \int_0^\infty h(t)dt>f \tag{9a,b}$$

where, again, we expect (9a) to be a general property and (9b) is required for the instabilities to be described. We also assume later that $h(t)\geq 0$, i.e., that the relaxation in Fig. 1 is monotonic.

We close this section with a few further remarks on frictional constitutive relations. First, the discussion thus far is for σ = constant and the same condition is assumed in the subsequent stability analyses. We presume that a suitable generalization for nonconstant $\sigma(t)$ would be to include in F of equation (2) a direct dependence on $\sigma(t)$ and a functional dependence on $\sigma(t')$, $-\infty<t'<t$. A strong, approximately linear dependence of τ on σ is well known but, to our knowledge, experiments have not yet documented whether there are memory effects relating to $\sigma(t)$ analogous to those previously discussed for $V(t)$. Second, characteristic slip distances in the decay process of Fig. 1 are typically small, 0.3 to 200 μm representing the range of surfaces studied so far. Thus, we neglect the fact that points currently mating across the slip surface at time t had slightly different prior slip histories (if the adjoining solids are deformable).

Finally, a special form of the dependence of τ on state, adopted in description of experimental results (Ruina [3], Dieterich [5–8], Kosloff and Liu [16]), is to represent the latter by a set of variables $\theta_1, \theta_2, \ldots, \theta_n$, collectively the column $\{\theta\}$, subject to first-order rate equations. Thus, with σ again constant,

$$\tau=\tau(V,\{\theta\}), \quad \{\dot{\theta}\}=\{g(V,\{\theta\})\} \tag{10a,b}$$

The equation (7) may be thought of as a linearization of such a state variable description. In this case $h(t)$ would be given as a sum of (possibly complex) exponentially decaying functions. Here we take the form of equation (7), as in Fig. 1, to be the basic constitutive assumption independent of the nonlinear description, such as equation (1), (2), or (10), of which it is a linearization.

Stability of One Degree-of-Freedom Elastic System

Consider a one degree-of-freedom elastic system, represented generically by the rigid block of unit base area in Fig. 2, loaded by a linear spring element whose end is constrained to move at speed V_0, namely, the steady slip speed for which stability is to be examined. Writing the slip speed as $V_0+\dot{x}$, x can be interpreted as the shortening of the spring from its steady state length, and thus the force (or stress, given the unit base area) exerted by the spring is

$$T=\tau_{ss}-kx. \tag{11}$$

The equation of motion is therefore

$$m\ddot{x}=T-\tau+q=\tau_{ss}-kx-\tau+q \tag{12}$$

where $q(t)$ is an arbitrary perturbing force, switched on at $t = 0$. Hence by equation (7) for τ,

$$kx(t)+m\ddot{x}(t)+f\dot{x}(t)-\int_0^t h(t-t')\dot{x}(t')dt'=q(t). \quad (13)$$

The Laplace transform of $x(t)$ is given by

$$\hat{x}(s)\equiv\int_0^\infty x(t)e^{-st}dt=\hat{q}(s)/D(s) \quad (14)$$

where

$$D(s)=k+ms^2+s[f-\hat{h}(s)] \quad (15)$$

A pole in $\hat{x}(s) = \hat{q}(s)/D(s)$ at $s = s_0$ corresponds to a term of the form $\exp(s_0t)$ in the inverse transform of $\hat{x}(s)$, $x(t)$. Evidently, then, the steady slipping state is stable if the equation $D(s) = 0$ has no solutions s_0 with $Re(s_0) > 0$. If $D(s) = 0$ for some $Re(s) > 0$ then steady sliding is unstable since $\hat{q}(s)$ is arbitrary. (An alternative, less rigorous approach to the stability analysis is to look for solutions of equation (15) of the form $x(t) = \exp(st)$, for large t. This again leads to $D(s) = 0$ and thus the stability condition $Re(s) < 0$ for $D(s) = 0$.)

We consider successively lower values of k and show next that as k reduces from ∞ to 0 one passes through a critical value, k_{cr}, at which two conjugate roots of $D(s) = 0$ cross the $Im(s)$ axis, say, at $\pm i\beta$, into the domain $Re(s) > 0$. Consequently, the steady slip state is stable for sufficiently stiff systems, i.e., if $k > k_{cr}$, and, at least in the vicinity of k_{cr}, the system exhibits flutter oscillations of frequency β whose amplitude grows in time if $k < k_{cr}$ and decays if $k > k_{cr}$.

To demonstrate the result just stated we first observe that due to the presumed integrability of $h(t)$, $\hat{h}(s)$ is bounded and $\hat{h}(\infty) = 0$ in the domain $Re(s) \geq 0$. Thus for $k\rightarrow\infty$, the equation $D(s) = 0$ can only possibly be satisfied in $Re(s) \geq 0$ by $s\rightarrow\infty$. But $\hat{h}(\infty) = 0$ so that $D(s) = 0$ leads to a quadratic equation for s that has roots with $Re(s) = -f/2m$, a contradiction if $f>0$ as required by (9a). We conclude that in the limit $k\rightarrow\infty$, $D(s)$ has no zeros with $Re(s) \geq 0$. Next, for $k = 0$, it can be observed that $D(s) = 0$ has at least one root in $Re(s) \geq 0$, on the positive real s axis. This follows because inequalities (9) and $\hat{h}(\infty) = 0$ show that

$$f-\hat{h}(0)=f-\int_0^\infty h(t)dt<0, \quad f-\hat{h}(\infty)>0, \quad (16)$$

and therefore that $D(s) < 0$ for small positive real s but $D(s) > 0$ for large positive s. Thus, assuming continuity, a real root or conjugate pair of complex roots must pass into the domain $Re(s) > 0$ as k reduces from ∞ to 0. A root cannot pass through the origin or infinity, because inspection shows that $D(0)$ and $D(\infty) \neq 0$ when $k > 0$ (and $m>0$ or $f>0$). By elimination, it is therefore the case that a conjugate pair of complex roots crosses the $Im(s)$ axis at critical conditions since $D(s)$ has real coefficients.

The crossing points $\pm i\beta$ are computed by setting

$$D(\pm i\beta)=k_{cr}-m\beta^2\pm i\beta[f-\hat{h}(\pm i\beta)]=0 \quad (17)$$

Separating (17) into real and imaginary parts yields two equations, one determining the critical frequency β by

$$\int_0^\infty \cos(\beta t)h(t)dt=f \quad (18)$$

and the other giving an expression for the critical spring constant as

$$k_{cr}=m\beta^2+\beta\int_0^\infty \sin(\beta t)h(t)dt. \quad (19)$$

Equation (18) shows that the frequency β of the flutter instability is determined solely by properties of the friction law, f and $h(t)$, and not the mass m or stiffness k. Equation (18) also results as an answer to the following question: For what frequency β does the friction force in steady oscillatory motion $x(t) = \cos\beta t$ not absorb any more work than the steady state work $\tau_{ss}V_0$? This is also equivalent to the question: For what frequency β is the oscillatory displacement $x(t) = \cos\beta t$ exactly out of phase with the excess friction force? Both statements follow because in steady oscillatory motion no energy is lost or gained by the spring or mass. Also, in steady sinusoidal motion both the force required to accelerate the mass and to cock the spring are in phase with the position of the mass and spring.

It seems reasonable to assume that $h(t) \geq 0$ because, by reference to Fig. 1, this assumption means that the decay of τ toward its steady state value is monotonic. If we do therefore assume that $h(t) \geq 0$, equation (18) will have a solution if and only if inequality (5), $d\tau_{ss}(V)/dV < 0$, which is equivalent to the second inequality of (9), is met. This is because the cosine transform of a positive function is bounded by the integral of that function,

$$\int_0^\infty h(t)dt\geq\int_0^\infty \cos\beta t\, h(t)dt, \quad (20)$$

with equality only at $\beta = 0$. Thus a necessary (and sufficient, since the cosine transform vanishes as $\beta\rightarrow\infty$) condition for equation (18) to have a solution with $\beta \neq 0$ is that $\int_0^\infty h(t)dt > f$, which is the second inequality of (9). Thus, if the decay process is monotonic, $h(t) \geq 0$, then $d\tau_{ss}(V_0)/dV_0 < 0$ is a necessary and sufficient condition for instability to be possible with some (sufficiently reduced) positive spring constant.

We remark further that equation (18) for β can have at most one solution if the cosine transform of $h(t)$ decreases monotonically with increasing β. Such monotonicity would result if $h(t)$ had a decaying exponential representation as would be the case in the state variable description if the equations (10b) could be decoupled (at least when linearized). Note that a one-state-variable constitutive law would necessarily generate a $h(t)$ satisfying this monotonicity condition. More generally, however, we cannot rule out the possibility that multisolutions for β may exist in some cases and, in such cases, the solution yielding the highest k_{cr} in (19) is to be taken for the instability criterion.

Equation (19) for k_{cr} shows that mass is always destabilizing, since increasing m increasees the threshold k_{cr} below which instability occurs. Note further that the right side of equation (19) is necessarily positive since the method of derivation has shown that inequalities (9) imply the existence of a k_{cr} between ∞ and 0. The derivation applies for a system with $m = 0$, if $f>0$, so at β given by equation (18),

$$\beta\int_0^\infty \sin\beta t\, h(t)dt>0. \quad (21)$$

One consequence of the last inequality is that no system is stable when β exceeds its natural vibration frequency $\omega \equiv (k/m)^{1/2}$, because then the result for k_{cr} in equation (19) is plainly in excess of k.

Single Decay Process

To illustrate the preceding formulas, consider the single exponential representation of the decay shown in Fig. 1, namely

$$h(t)=(1+\lambda)rf\,e^{-rt}, \quad r>0. \quad (22)$$

Such an exponential form necessarily results, for example, if the constitutive relation in the form of equations (10) involves only a single state variable as explained in Ruina [3]. Assuming net rate weakening [equations (5), (9b)], $\lambda > 0$.

Equations (18) and (19) then give the frequency and stiffness at critical conditions

$$\beta=r\sqrt{\lambda}, \quad k_{cr}=mr^2\lambda+fr\,\lambda. \quad (23)$$

Alternatively, in terms of the frequency ω of the spring-mass system at critical stiffness ($\omega^2 = k_{cr}/m$)

$$k_{cr} = fr\lambda/(1 - \lambda r^2/\omega^2). \tag{24}$$

Since

$$f = \partial\tau(V,\text{state})/\partial V, \quad \lambda f = -d\tau_{ss}(V)/dV,$$

both expressions being evaluated at $V = V_0$, and $1/r$ is the characteristic time of the decay process, the result for k_{cr} may be put into the more inspectable form

$$k_{cr} = -\frac{V d\tau_{ss}(V)/dV}{d_c}\left[1 + \frac{mV}{d_c \partial\tau(V,\text{state})/\partial V}\right] \tag{25}$$

with $V = V_0$. Here we have introduced $d_c \equiv V_0/r$ as the decay parameter expressed in terms of slip distance rather than time, i.e., with the decay in Fig. 1 proportional to $\exp(-V_0 t/d_c)$. Experiments (Dieterich [5-8]; Ruina [3]) suggest that the decay distance d_c is approximately independent of the slip rate V_0, and is thus closer to a material property than is $1/r = d_c/V_0$, which obviously depends on V_0. Ruina [3] derived the quasi-static version of (25), with $m = 0$, by an analysis based on constitutive laws of type (10) with a single state parameter. The dynamical result could have been derived from the static relation since, in steady sinusoidal oscillations $m\ddot{x}$ is in phase with $-kx$, the spring force. Thus any steady sinusoidal oscillation with frequency β found with some k and $m = 0$, as in Ruina [3], could be replaced by a motion with finite m and an increased spring constant $k + \beta^2 m$. This reasoning can lead to (25) directly from the results of Ruina [3]. Similarly the term $m\beta^2$ could have been added to equation (19) by this reasoning after the derivation was done with no inertia ($m = 0$). Note also that, at any fixed V, τ_{ss} is approximately proportional to σ. Hence $\lambda f \propto \sigma$ and thus $k_{cr} \propto \sigma$ (when $m = 0$), as has been emphasized by Dieterich [3, 4] based on a qualitative instability analysis.

Further Discussion

How necessary is inequality (3), i.e., the positive instantaneous viscosity property? We assumed in the analysis leading to equations (18) and (19) that at least one of the instantaneous viscosity f and the mass m is nonzero. Consider the case $m = 0$. The results of our analysis then carry through with $m = 0$ substituted in all equations containing m. Now if the instantaneous viscosity $f \to 0$ equation (18) shows that the frequency of flutter at neutral stability becomes infinite, $\beta \to \infty$, but equation (19) with $m = 0$ shows that k_{cr} tends to a finite value (lim $\beta \to \infty$ of $\beta \int_0^\infty \sin(\beta t) h(t) dt$). On the other hand, if we let $f \to 0$ with *any* finite m, equation (18) shows that $\beta \to \infty$ as in the foregoing, and now equation (19) shows that the critical stiffness becomes infinite, $k_{cr} \to \infty$. Equation (25) for k_{cr}, in the case of a single decay process, clearly shows the result just discussed. If $f \equiv \partial\tau(V,\text{state})/\partial V \to 0$ a quasi-static analysis, i.e., based on setting $m = 0$, gives a finite k_{cr}. But if $m \neq 0$ the limit $f \to 0$ of zero instantaneous viscosity gives an unbounded k_{cr}. Hence, presuming as implicit in the preceding discussion that there is ultimate velocity weakening, steady state slip should not be possible in any elastic system, no matter what its stiffness, if there is no instantaneous viscosity. One might reverse the argument and say that the experimental observation that steady state slip on a given surface is possible, in a system of adequate stiffness, implies a positive instantaneous viscosity (at least on surfaces that exhibit ultimate velocity weakening, inequality (5)).

We now readdress the question of whether instability is possible at all if inequality (5) is *not* satisfied but rather reversed, with $d\tau_{ss}(V)/dV > 0$. This means that the surface exhibits ultimate velocity strengthening; the second of (9) then fails and instead

$$\int_0^\infty h(t)dt < f \quad (f > 0).$$

We have already shown that no instability is then possible if $h(t) \geq 0$, i.e., if the decay process in Fig. 1 is monotonic. Any case allowing instability must therefore show nonmonotonic decay. A specific mathematical form allowing such instability is that of oscillating exponential decay,

$$h(t) = Re\left[H(a+ib)e^{-(a+ib)t}\right],$$

$a > 0$, $b > 0$, in which case the foregoing inequality becomes $Re(H) < f$. Equation (18) can still be satisfied, so that instability is possible, if b is sufficiently large compared to a. But experimental observations as made thus far do not lend support to decay with such marked oscillations, that are not likely the result of machine-sample interaction (of the type predicted here for k slightly greater than k_{cr}). We thus propose that the inequality (5), that the steady state friction force is a decreasing function of slip rate, is a necessary condition for the instability of steady sliding.

Viscoelastic Effects

Consider now the same one degree-of-freedom system of Fig. 2 but suppose that the spring element is viscoelastic. Then we may express the force exerted by the spring as

$$T = \tau_{ss} - k\int_0^t \gamma(t-t')\dot{x}(t')dt' \tag{26}$$

where $k\gamma(t)$ is the viscoelastic relaxation function; $\gamma(t)$ is normalized so that $\gamma(0) = 1$ and hence k is the instantaneous spring constant and $k\gamma(\infty)$, with $0 < \gamma(\infty) \leq 1$, is the long-time or relaxed spring constant. If one has in mind a viscoelastic element that has an infinite instantaneous spring constant, as for example a spring and dashpot in parallel, the instantaneous viscosity can be subtracted from the viscoelastic element and added to the term f in the friction law. This then leaves the form of equation (26) with k finite. The possibility of instability is then determined by whether inequality (9*b*) is satisfied with this modified f. Writing equations of motion, it is seen that equation (14) applies for $\hat{x}(s)$ with

$$D(s) = k[s\hat{\gamma}(s)] + ms^2 + s[f - \hat{h}(s)]. \tag{27}$$

Observing that $s\hat{\gamma}(s) \to \gamma(\infty)$, 1, respectively, as $s \to 0, \infty$, a similar argument to that outlined earlier can be followed to show that instability occurs by flutter oscillations of frequency β when k is reduced to a critical value, k_{cr}. The critical conditions are again given by $D(\pm i\beta) = 0$ and we find

$$k_{cr}\int_0^\infty \cos(\beta t)[\gamma(t) - \gamma(\infty)]dt + f = \int_0^\infty \cos(\beta t)h(t)dt,$$

$$k_{cr}\left\{\gamma(\infty) + \beta\int_0^\infty \sin(\beta t)[\gamma(t) - \gamma(\infty)]dt\right\} \tag{28}$$

$$= m\beta^2 + \beta\int_0^\infty \sin(\beta t)h(t)dt.$$

These equations are difficult to solve and we do not present explicit results. However, we remark that now β is dependent on the viscoelastic properties of the spring and on the mass m, and not merely on characteristics of the friction law, as it is for an elastic system. We can also see from the first of equations (28) that when the cosine transforms of $\gamma(t) - \gamma(\infty)$ and $h(t)$ are monotonic, as in the typical case when both are represented by a sum of decaying exponentials in t with positive coefficients, the effect of viscoelasticity is to reduce β from the value for an elastic system, equation (18). This is as expected, since for steady oscillatory motion the friction surfaces changes from an energy sink to an energy source (once the steady state sink $V_0\tau_{ss}$ is subtracted out) when β decreases through the value given by equation (18). Thus the viscoelastic energy absorbed is accommodated through the decrease in β.

Slipping Elastic Continua and Creep Waves

In this section we assume that the sliding bodies are identical elastic continua with interface along the ξ_1,ξ_2 plane of a ξ_1,ξ_2,ξ_3 cartesian coordinate system, Fig. 3. For simplicity we neglect inertia here. It is considered in a special version of the problem taken up in the next section. The sliding bodies are assumed to be translationally homogeneous in the ξ_1 direction, and steady relative slip at speed V_0 in either the ξ_1 or ξ_2 direction is enforced by displacement boundary conditions imposed at $\xi_3 = \pm H$. The perturbation from steady slip along the interface is represented by a relative displacement δ, in the same direction as V_0, between $\xi_3 = 0^+$ and 0^- of the form

$$\delta(\xi_1,t) = V_0 t + x(t)\cos(\kappa\xi_1 + \phi) \qquad (29)$$

where κ is the spatial wave number of the disturbance and ϕ is any constant. More general perturbations may be obtained by Fourier superposition with various κ, ϕ. The relative slip speed at the interface is

$$V(\xi_1,t) = V_0 + \dot{x}(t)\cos(\kappa\xi_1 + \phi). \qquad (30)$$

Because of the translational homogeneity, the associated variation in resistive shear stress τ along the interface, computed from elasticity theory in terms of the given displacement nonuniformity, must have the same spatial dependence but be exactly out of phase with it. Hence

$$\tau(\xi_1,t) = \tau_{ss}(V_0) - k(\kappa)\,x(t)\cos(\kappa\xi_1 + \phi), \qquad (31)$$

where the coefficient $k = k(\kappa)$ can evidently be interpreted as an effective spring constant for disturbances with wave number κ (compare equations (6) and (11) with (30) and (31)).

The stiffness $k(\kappa)$ can be found simply for some representative models. For example, if the two elastic bodies are isotropic, homogeneous half spaces of shear modulus G and Poisson ratio ν, then one can derive from elementary elasticity theory that

$$k = G|\kappa|/2(1-\nu), \quad G|\kappa|/2 \qquad (32)$$

for the respective cases of plane strain (V in ξ_1 direction) and antiplane strain (V in ξ_2 direction). For finite layers of height H (in the ξ_3 direction) in contact, the preceding formulas remain valid in the short wavelength limit $\kappa H >> 1$, but in the long wavelength limit, $\kappa H << 1$, both expressions for k approach the limiting value

$$k = G/2H, \qquad (33)$$

which corresponds to uniform ($\kappa - 0$) shearing of the layers. In fact, the complete expression for $k(\kappa)$ in the antiplane strain mode (see the next section) is

$$k(\kappa) = G|\kappa|/[2\tanh(|\kappa|H)]. \qquad (34)$$

Analogously to the treatment of the sliding block, we assume that a perturbing load distribution, generating shear stress

$$q(t)\cos(\kappa\xi_1 + \phi)$$

along the interface, is switched on at $t = 0$. Precisely, the preceding expression gives the shear stress that the considered load perturbation would cause along the interface if $x(t)$ were constrained to be zero. Hence this term must be added to equation (31) for $\tau(\xi_1,t)$. The resulting τ at each location ξ_1 must satisfy the constitutive relation (7) when $\dot{x}(t)\cos(\kappa\xi_1 + \phi)$ is read-in for $\dot{x}(t)$. Hence we find that

$$k(\kappa)x(t) + f\dot{x}(t) - \int_0^t h(t-t')\dot{x}(t')dt' = q(t), \qquad (35)$$

which is the same as equation (13) for the sliding block without inertia. Thus we may again write $\hat{x}(s) = \hat{q}(s)/D(s)$ as in equation (14), but now with

$$D(s) = k(\kappa) + s[f - \hat{h}(s)]. \qquad (36)$$

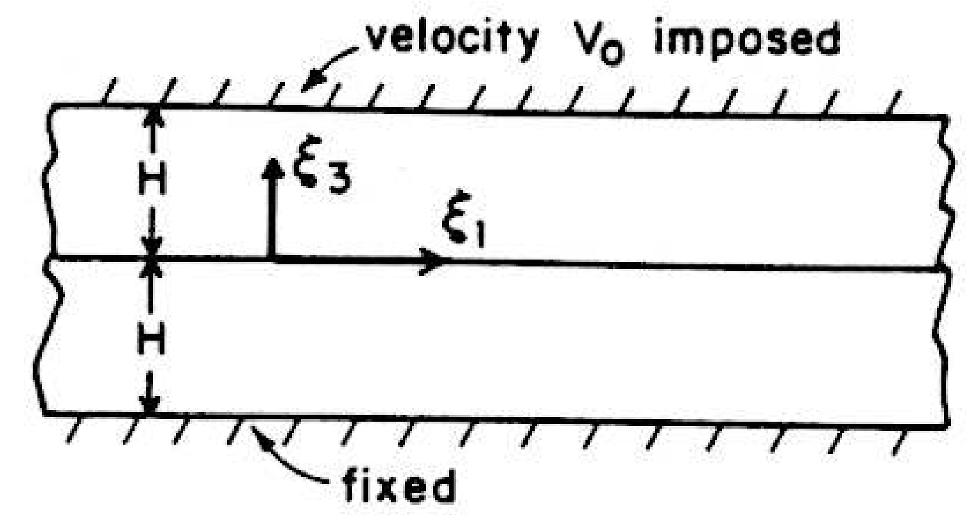

Fig. 3 Elastic continua in slipping contact. Bottom of lower layer is fixed. Top of upper level has imposed velocity V_0 in slip direction.

Hence, following the earlier discussion, instability occurs, if at all, by a flutter mode of frequency β satisfying $D(\pm i\beta) = 0$ and given by equation (18). Disturbances decay or grow in amplitude according to whether (by equation (19) with $m = 0$)

$$k(\kappa) \gtrless k_{cr} \equiv \beta\int_0^\infty \sin(\beta t)h(t)dt. \qquad (37)$$

One expects that k varies monotonically with κ so that one of the following two cases occur: As κ decreases from ∞ to zero, k decreases from ∞ to either: (*i*) a value in excess of k_{cr} (possible when finite H gives a long wavelength cutoff); or, (*ii*) a value smaller than k_{cr} (and equal to 0 when the bodies are unbounded half spaces, $H = \infty$). In case (*i*) all perturbations are stable. In case (*ii*) perturbations with sufficiently high wave number (short wavelength) are stable, but those with lower wave number (long wavelength) are unstable.

The result at critical conditions has an interesting interpretation. Since $x(t)$ then varies as $Re(e^{i\beta t})$ or $Im(e^{i\beta t})$, the combination of space and time dependence as in equation (30) leads to disturbances with

$$V(\xi_1,t) - V_0 \propto \cos(\kappa_{cr}\xi_1 \pm \beta t). \qquad (38)$$

This represents propagating quasi-static waves that move along the interface with speed β/κ_{cr}. The existence of such waves was first noted by Ruina [3] in analysis of a simple model of a continuous elastic system with a sliding surface described by a one-state variable form of equations (10).

As an example, for the friction law with a single exponential decay process, equations (22) and (23), $\beta = r\sqrt{\lambda}$ and the speed of the creep waves is

$$\text{speed} \equiv \beta/\kappa_{cr} = r\sqrt{\lambda}/\kappa_{cr} = V\sqrt{\lambda}/(\kappa_{cr}d_c) \qquad (39)$$

where d_c is the decay distance, V/r. The critical wave number depends on details of the elastic continua. But for isotropic half spaces under antiplane slip we obtain from (32) that $G\kappa_{cr}/2 = k_{cr}$, where k_{cr} is evaluated from (25) with $m = 0$. Thus the critical wavelength λ_{cr} is

$$\lambda_{cr} \equiv 2\pi/\kappa_{cr} = \pi\, d_c G/[-V d\tau_{ss}(V)/dV] \qquad (40)$$

Further, using this κ_{cr} and the interpretation of λ given before, the speed of creep waves is found from equation (39) to be

$$\text{speed} = G/2\sqrt{[-d\tau_{ss}(V)/dV][\partial\tau(V,\text{state})/\partial V]} \qquad (41)$$

According to the results presented by Ruina [3] and Dieterich [5–8], the bracketed terms in the last expression are each of order 0.01 σ/V, where σ is the normal stress. In that case we obtain

$$\lambda_{cr} \approx 300\, d_c G/\sigma, \quad \text{speed} \approx 50\, VG/\sigma. \qquad (42)$$

If we choose σ as the overburden pressure in the earth from a 1 to 10 km depth range, one estimates $G/\sigma = 10^3$ to 10^2 for faults under crustal earthquake conditions. Thus the creep wave speed is $5\cdot10^3$ to $5\cdot10^4$ times the nominal steady slip speed V. This is still much slower than seismic shear wave speeds if V is of the order of a cm/sec. or less. The corresponding wavelengths λ_{cr} are then of order $3\cdot10^5$ to $3\cdot10^4$ times d_c, resulting in $\lambda_{cr} \approx 0.1$ to 1 m if d_c is of order 3μm (representative of laboratory studies on polished quartzite surfaces, Ruina [3]), but of the order $\lambda_{cr} \approx 30$ to 300 m if, for example, a d_c of the order 1 mm is postulated (which is somewhat larger than the largest results of Dieterich [8] for laboratory fault gauge).

The analysis predicts that disturbances with $\lambda < \lambda_{cr}$ are stable and decay in time but that those with $\lambda > \lambda_{cr}$ exhibit oscillatory growth in amplitude, at least for λ in the vicinity of λ_{cr}.

Inertia Effects in Antiplane Perturbations of Slipping Elastic Continua

Consider isotropic, homogeneous elastic bodies as in Fig. 3, with enforced relative motion in the ξ_2 direction, and let the relative displacement δ along the interface be given as in equation (29) of the preceding section. This loading causes an antiplane strain deformation and, if $u(\xi_1,\xi_3,t)$ is the antiplane displacement field (in the ξ_2 direction) measured relative to the steady sliding state, we have a boundary value problem described by the following equation of motion (43), and antisymmetry and boundary conditions (44):

$$\partial^2 u/\partial \xi_1^2 + \partial^2 u/\partial \xi_3^2 = (1/c^2)\partial^2 u/\partial t^2; \qquad (43)$$

$$u(\xi_1,\xi_3,t) = -u(\xi_1,-\xi_3,t), \quad u(\xi_1,H,t)=0,$$
$$u(\xi_1,0^+,t) = \frac{1}{2}x(t)\cos(\kappa\xi_1+\phi), \qquad (44)$$

where c is the shear wave speed.

The Laplace transform of the solution is (for $\xi_3 > 0$)

$$\hat{u}(\xi_1,\xi_3,s) = \frac{1}{2}\hat{x}(s)\cos(\kappa\xi_1+\phi)\sinh[\sqrt{\kappa^2+s^2/c^2}\,(H-\xi_3)]/\sinh(\sqrt{\kappa^2+s^2/c^2}\,H). \qquad (45)$$

The stress $\tau(=\tau_{32})$ along the interface due to this elastodynamic loading can be written as

$$\tau_1(t)\cos(\kappa\xi_1+\phi) = G[\partial u(\xi_1,\xi_3,t)/\partial\xi_3]_{\xi_3=0} \qquad (46)$$

The last equation shows that

$$\hat{\tau}_1(s) = -K(s,\kappa)\hat{x}(s), \text{ where}$$
$$K(s,\kappa) \equiv G\sqrt{\kappa^2+s^2/c^2}/[2\tanh(\sqrt{\kappa^2+s^2/c^2}\,H)]. \qquad (47)$$

Now the friction stress τ must equal the steady state stress τ_{ss} in addition to the elastodynamic stress from equation (46) and the perturbation stress:

$$\tau = \tau_{ss}(V_0) + \tau_1(t)\cos(\kappa\xi_1+\phi) + q(t)\cos(\kappa\xi_1+\phi). \qquad (48)$$

The perturbation amplitude $q(t)$ is again zero for $t<0$ but otherwise arbitrary.

The expression for τ in equation (48) must equal the value required by the constitutive law of equation (7) with the slip perturbation $x(t)\cos(\kappa\xi_1+\phi)$. The Laplace transform of equation (48) with equation (47) then gives $\hat{x}(s) = \hat{q}(s)/D(s)$, as in equation (14), but now with

$$D(s) = K(s,\kappa) + s[f-\hat{h}(s)] \qquad (49)$$

and $K(s,\kappa)$ given by equation (47). Again stability of steady sliding requires no poles in $\hat{x}(s)$ for $Re(s)>0$ and thus no zeroes of $D(s)$ in $Re(s)>0$. The subsequent analysis of this case follows the pattern established earlier. As $|\kappa|$ is reduced in value from ∞ to 0, roots of $D(s)=0$ first pass into $Re(s)>0$, if at all, by crossing the $Im(s)$ axis. Hence, setting $D(\pm i\beta)=0$ we obtain the pair of equations to be met at critical conditions

$$\int_0^\infty \cos(\beta t)h(t)\,dt = f,$$
$$K(i\beta,\kappa_{cr}) = \beta\int_0^\infty \sin(\beta t)h(t)\,dt \qquad (50)$$

where

$$K(i\beta,\kappa) = \frac{G\sqrt{\kappa^2-\beta^2/c^2}}{2\tanh(\sqrt{\kappa^2-\beta^2/c^2}\,H)} \quad \text{for} \quad \kappa^2>\beta^2/c^2$$
$$= \frac{G\sqrt{\beta^2/c^2-\kappa^2}}{2\tan(\sqrt{\beta^2/c^2-\kappa^2}\,H)} \quad \text{for} \quad \kappa^2<\beta^2/c^2 \qquad (51)$$

The first of the pair of equations (50) is now familiar (see equation (18)) and gives the critical frequency β at flutter instability, if such instability can occur. The second of equation (50) is analogous to equation (19) with $m\beta^2$ moved to the left-hand side.

Instability can occur if the second of equations (50), whose right side is positive by (21), has a solution for some $|\kappa|$ between ∞ and 0. To analyze the second equation, let us observe that the equation $K(i\omega,\kappa)=0$ implies no traction at $\xi_3=0$ and thus gives the natural frequencies of clamped-free vibrations of either layer in Fig. 3, compatible with spatial periodicity of wave number κ. These frequencies are given by

$$\sqrt{\omega_n^2/c^2-\kappa^2}\,H = (2n-1)\pi/2, \quad n=1,2,3,\ldots,$$
$$\text{or } \omega_n = \sqrt{\kappa^2c^2+(2n-1)^2(\pi c/2H)^2} \qquad (52)$$

The lowest frequency of all is ω_1 for $\kappa=0$; calling this $\omega_1{}^*$, we have $\omega_1{}^* = \pi c/2H$.

We now distinguish two cases: (*i*) $\beta<\omega_1{}^*$, and (*ii*) $\beta>\omega_1{}^*$. For case (*i*) it is possible to show by simple analysis that $K(i\beta,\kappa)$ decreases monotonically with κ as the latter decreases from ∞ to 0; $K(i\beta,\infty)=\infty$ and the least value of K is

$$K(i\beta,0) = \frac{G}{2H}\,\frac{(\pi\beta/2\omega_1{}^*)}{\tan(\pi\beta/2\omega_1{}^*)}\left(<\frac{G}{2H}\right) \qquad (53)$$

If this value of $K(i\beta,0)$ is less than the right side of the second of equations (50), then the equation (50) has a solution and instability occurs by flutter oscillations for wave numbers $|\kappa|<|\kappa_{cr}|$. Again, conditions in the vicinity of neutral stability, $|\kappa|=|\kappa_{cr}|$, can be described as the propagation of frictional creep waves along the interface, which grow in amplitude when $|\kappa|<|\kappa_{cr}|$. On the other hand, if $K(i\beta,0)$ exceeds the right side of the last of equations (50), then no solution exists and slip is stable to perturbations of all wavelengths. This is analogous to the cutoff described in the preceding section with inertia neglected, and the results of that section are approached when the lowest vibration frequency of the layer is much higher than the critical frequency for slip instability, i.e., $\beta/\omega_1{}^* \to 0$. Also, just as for the one degree-of-freedom system, the inclusion of inertia is destabilizing; the critical wave number $|\kappa_{cr}|$ in the analysis with inertia always exceeds that of the quasi-static analysis although, of course, the difference is negligible when $\beta << \omega_1{}^*$.

For case (*ii*), $\beta>\omega_1{}^*$, it is evident from equation (52) that at least one $|\kappa|>0$ exists such that β coincides with a natural frequency for that κ, and hence that $K(i\beta,\kappa)=0$. The largest $|\kappa|$ satisfying that condition, say $|\kappa_1|$, is readily seen to be that κ for which β coincides with frequency ω_1. Hence, from (52) with $n=1$ and $\omega_1=\beta$ we find

$$|\kappa_1| = \sqrt{\beta^2-\omega^2_1{}^*}/c. \qquad (54)$$

Then from equation (51) one can see that $K(i\beta,\kappa)$ decreases monotonically from ∞ to 0 as $|\kappa|$ decreases from ∞ to $|\kappa_1|$. Thus in this case, for which $\beta>\omega_1{}^*$, there always exists a κ_{cr} satisfying the second of equations (50), and $|\kappa_{cr}|$ is necessarily greater than $|\kappa_1|$. That is, the system is unstable to perturbations of long enough wavelength if β exceeds its lowest vibration frequency.

Since the analysis of this section has relied only on rather general properties to be expected of any function $K(s,\kappa)$ relating nonuniformity of slip to nonuniformity of stress, it seems likely that similar conclusions would be reached for other modes of perturbing sliding continua of the general class introduced in the last section.

Concluding Discussion

The systems discussed in our paper exhibit a common general pattern. In particular, if $x(t)$ denotes the displacement perturbation from steady state slip and $q(t)$ the perturbing force, then $\hat{x}(s) = \hat{q}(s)/D(s)$, as in equation (14), where

$$D(s) = Q(s) + s[f - \hat{h}(s)]. \tag{55}$$

Here f and $h(t)$ are defined by the friction law (7) whereas $Q(s)$ is a transfer function. It relates the displacement perturbation to corresponding changes in stress τ induced by the system (e.g., by its elastic or viscoelastic springiness and inertia) on the slip surface; i.e., if $\tau_1(t) = \tau - \tau_{ss}$, then $\hat{\tau}_1(s) = -Q(s)\hat{x}(s)$.

In the various cases that we have examined, subject to (9), the form of the transfer function has assured that instability occurs by the flutter mode when an appropriately defined stiffness is reduced to a critical value. This contrasts with analyses that neglect the memory effects in (7) and thus deduce that rate weakening is a sufficient condition for instability and that oscillations depend on nonlinear effects (e.g., Brockley and Ko [2, 17]).

Indeed, this universality of the flutter instability, meaning that roots of $D(s) = 0$ inevitably pass to $Re(s) > 0$ by traversing the $Im(s)$ axis in conjugate pairs, means that the bifurcation is of the Hopf type (e.g., Howard [18]). We have presented only a linear analysis here, but the generic behavior of the nonlinear solution in the vicinity of critical conditions is understood. In particular, in a one-sided neighborhood of $k = k_{cr}$ (in terms of the spring-block analysis) there exists finite amplitude periodic oscillations of amplitude that increases with $|k - k_{cr}|$. When the neighborhood is that for which $k < k_{cr}$, the growing oscillations of linear instability theory grow into a stable periodic limit cycle, at least close to $k = k_{cr}$. When the neighborhood is that for which $k > k_{cr}$, the periodic oscillation is unstable, and the decaying oscillations of linear stability theory may, in fact, not be realized if the perturbation of the system is of too great an amplitude. Exceptionally, it may occur that the finite amplitude periodic oscillations occur with $k = k_{cr}$. This is precisely what we have found recently (Gu et al. [19]) for the nonlinear stability analysis of a certain one-state variable constitutive law proposed by Ruina [3], namely that for which equation (10) has the form

$$\tau = \tau_1 + A\ ln(V/V_1) - \theta,$$
$$\frac{d\theta}{dt} = -\frac{V}{d_c}[\theta - B\ ln(V/V_1)] \tag{56}$$

where τ_1, d_c, A, and $B(>A)$ are all positive constants. We remark that there is ample experimental evidence for the type of flutter instability that we predict here (Ruina [3], Scholz et al. [20], Teufel [21]). The flutter is of such low frequency in these experiments that inertia is negligible and classical calculations of the Rayleigh type cannot apply. Whether or not our results are appropriate to the type of fast oscillations observed by Brockley and Ko [2, 17] is not clear. Their results do show that τ is not a function of V alone (although they neglect this in their analysis). Also, the experimental results show much richer nonlinear behavior than thus far discussed. For example, signs of period doubling are visible in the experiments of Ruina [3] as k is decreased from k_{cr}.

It is plain that there remains much to be learned about nonlinear stability analysis in the framework of the rate and state-dependent frictional constitutive laws discussed here. The topic is of interest not only as an extension of studies of the type that we have reported, but also as foundation for a more general and realistic fracture mechanics of slip propagation (shear cracking) along fault surfaces.

Acknowledgment

This work was supported by the NSF Geophysics Program and USGS Earthquake Hazards Reduction Program. Pleasant and informative discussion by A. R. with Herbert Hui is appreciated.

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