

## RESEARCH ARTICLE

### The bricycle: A bicycle in zero gravity can be balanced or steered but not both

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*A bicycle or inverted pendulum can be balanced, that is kept nearly upright, by accelerating the base. This is done by steering on a bicycle. Simultaneously one can also control the lateral position of the base: changing of the track line of a bike or the position of hand under a balanced stick. We show here with theory and experiment that if the balance problem is removed, by making the system neutrally stable for balance, one can't simultaneously maintain balance and control the position of the base.*

*We made a bricycle, essentially a bicycle with springy training wheels. The stiffness of the training wheel suspension can be varied from near infinite, making the bricycle into a tricycle, to zero, making it effectively a bicycle. The springy training wheels effectively reduce or even negate gravity, at least for balance purposes. One might expect a smooth transition from tricycle to bicycle as the stiffness is varied, in terms of handling, balance and feel. Not so. At an intermediate stiffness, when gravity is effectively zeroed, riders can balance easily but no longer turn. Small turns cause an intolerable leaning.*

*Thus there is a qualitative difference between bicycles and tricycles, a difference that cannot be met halfway.*

**Keywords:** bicycle, tricycle, controllability, balance, lean, steer, dynamics.

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Figure 1. A ‘bricycle’: a vehicle which can transition between bicycle and a tricycle. It is a bicycle with two extra ‘training’ wheels at the back on a spring-loaded parallelogram mechanism. The spring resists bicycle lean, more or less depending on its adjustment. When the spring is removed the training wheels are mechanically invisible, effectively reducing the bricycle to a bicycle. When the spring is locked the bicycle is effectively a tricycle. The neutral point in-between is the main topic of study here.

## 1. Introduction

On a normal bicycle, gravity causes a capsizing moment that increases with lean. If gravity was removed but the rolling constraint retained (using, say, magnetic wheels) there would be no capsizing moment. We can mimic this zero-gravity bicycle, down here on earth, by adding a righting moment that cancels the gravity capsizing moment. We have done this using springy training wheels. Questions we address in terms of this vehicle include: What happens if you bicycle in zero gravity? What happens if you transition from tricycle to bicycle in a continuous manner? What is the role of gravity in bicycle control? The results are, perhaps, surprising. In short (spoiler alert): In zero gravity a bicycle could be easily balanced, but not steered at all (within the regime of small-lean, linearized equations); halfway between a tricycle and a bicycle is, effectively a zero-gravity bicycle; and the control of a bicycle depends essentially on gravity.

### 1.1. *Help from ‘the hand of god’*

One way to help someone learn a task is to help them do it, and then gradually reduce the help. For example, parents might give support to a child learning to walk, and then gradually withdraw it.

Rehabilitation of patients e.g. stroke victims, often uses physical guidance. Therapists or trainers may guide a motion, and then slowly withdraw the aid. This helping, and then withdrawal of help, is also used for robot-guided therapy (robots guiding people, [1, 2]).

The idea of helping and then withdrawing help extends to the machine-learning automatic control of simulations, for example in the simulations of running and walking by Van de Panne and Lamouret [3]. In these, a simulated external torque, “the hand of god” was initially used to hold their simulated characters upright. As the computer control algorithm learned to control the characters balance, the external righting torque was gradually reduced. The evolving controller in principle should gradually learn to balance the simulated characters without the help from external righting torques.

Richard Klein *et al.* [4] uses a related idea to help teach disabled children how to ride a bicycle. In his ‘lose the training wheels’ method, the rider progresses through a series of adapted bikes, none of which have the stigma of conventional training wheels. The first has, instead of wheels, rolling pins (slightly bulged narrow

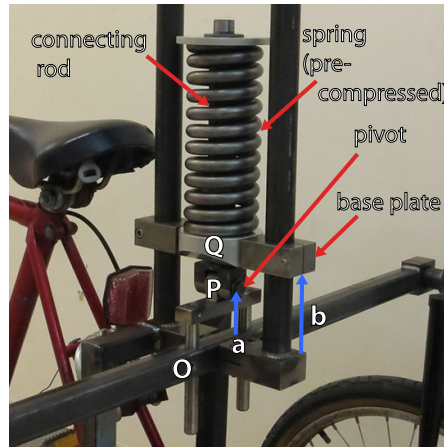


Figure 2. A **Zero rest-length spring** is the heart of the righting-moment mechanism. The spring is adjusted such that its force would be zero if the end of the connecting rod at P coincided with the pivot at Q; that would be the zero-length configuration. Because the spring compression is equal to the distance PQ a zero rest-length tension spring between P and Q is simulated. The dimensions  $a$  and  $b$  can be adjusted by sliding the spring and base plate assembly up or down and re-clamping, changing the effective torsional spring constant. See also the schematic in Fig. 3.

cylinders) having a center of lateral curvature well above the riders center of mass. Such a bike stays securely upright. As the rider learns to pedal and steer, the rollers are replaced with rollers with smaller rocking radius. Eventually regular bicycle wheels are used.

### 1.2. Signs of trouble

The gradually-weakening ‘hand of god’ approach to teaching might not always be effective. Domingo *et al.* [5] had healthy subjects on a narrow or wide balance beam, mounted on a treadmill. For some of the subjects assistance was given by springs attached to a hip belt that applied restoring forces towards the beam center. On both narrow and wide beams, subjects learning without assistance had greater performance improvements in maintaining balance while walking, compared to subjects in assisted groups (measured by failures per minute without assistance on the same beam). Physical assistance seems to hinder learning in this context.

Indeed, the “guidance hypothesis” [6] is that the provision of too much external, augmented feedback during practice may cause the learner to develop a harmful dependency on this source of feedback. The experience of errors and failures is important for learning. Physical guidance reduces the range of errors and hence hinders learning. Besides the experience of error, Domingo *et al.* [5] identify another factor important to learning, the task-specific dynamics: “Having task dynamics more similar to the desired task would allow subjects to explore the state-space of position and velocity parameters and develop the ability to better control balance.” That is, despite its obvious appeal and common use, one can foresee problems with the approach.

### 1.3. The Bricycle concept

Tricycles are easier to ride and steer compared to bicycles, at least for slow moving beginners. Based on the intuitive appeal of ‘the hand of god’ approach, and ignoring the forewarnings, we considered making a vehicle (a bricycle) which could, by adjustment, transition smoothly from a tricycle to bicycle. For example, a beginner could start in tricycle-mode and change into bicycle-mode as she gains proficiency. The idea was to avoid the discrete changes needed in the Klein sequence-of-thinner-

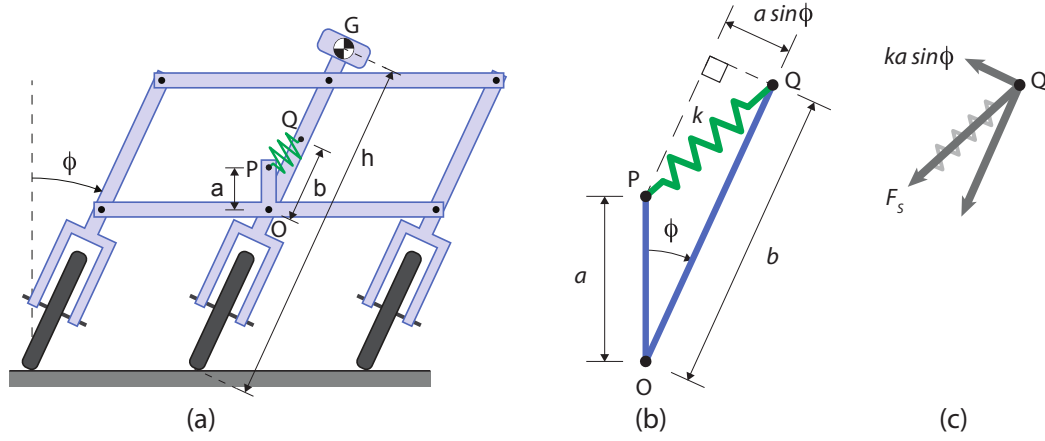


Figure 3. a) Training wheel suspension. The rear wheels are connected to each other via a parallelogram mechanism. The connection between points P and Q is effectively a zero rest-length spring. The lengths  $a$  and  $b$  are adjustable. b) Force generated by the zero rest-length spring, is proportional to the length PQ and the component of the force perpendicular to PQ is proportional to  $a \sin \phi$ . c) The restoring torque generated by the spring force  $F_s$  about point O is  $kba \sin \phi$ . This counters, less or more depending on parameters, the destabilizing gravitational torque  $mgh \sin \phi$ .

and-thinner-wheels approach.

The bricycle concept is related to that of narrow tilting vehicles (or narrow track vehicles, NTVs). NTVs are midway between a car and a motorcycle: they self-balance when stationary like a car and when moving they lean into turns like a motorcycle. Examples include the Ford Gyron and General motor's Lean Machine [7] and, more recently, the Mercedes F-3000 Life-Jet and Nissan land glider. The controllers for NTVs tilt the vehicle using some combination of two strategies: 1) Direct tilt control: an actuator on the longitudinal axis of the NTV providing torque to tilt the vehicle; 2) Steering tilt control: the steering angle applied by the driver is modulated to control the tilt angle using countersteering [8, 9]. The spring-righting on a bricycle is a special case of type (1) NTV control, thus the problems with bicycle control that we describe below need be circumvented in NTV controller design.

## 2. Bricycle design details

Figure 1 shows the bricycle. It is an ordinary bicycle with two extra 'training' wheels at the back. The rear wheels are mounted on a parallelogram mechanism. The mechanism is spring loaded, so as to provide a torque that resists lean. If the spring is rigid (or equivalently, if the parallelogram mechanism is locked) then the bricycle is effectively a tricycle. If the spring is removed then the bicycle is effectively a bicycle. In the same way that gravity causes a falling torque proportional to the sine of the lean angle, the spring mechanism is designed to give a righting torque also proportional to the sine of the lean angle, as described below. At an appropriate intermediate spring setting there can be angle-independent gravity cancellation, as described in the section 3.

### 2.1. The zero-rest-length spring mechanism

The spring mechanism allows gravity, for balance purposes to be reduced, zeroed or negated. Figures 2 and 3 show the parallelogram linkage and spring. Key is an emulated 'zero rest-length' spring between the points P and Q (Fig 2 and 3). A zero rest-length spring has tension proportional to the distance between the points

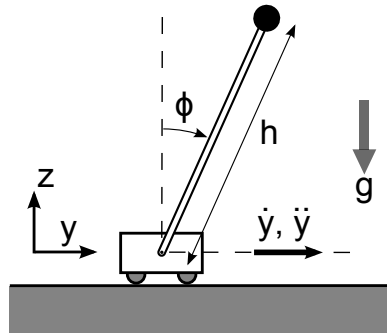


Figure 4. The classical cart and inverted pendulum. The acceleration of the cart ( $\ddot{y}$ ) is varied in time (say by action of a horizontal force) so as to control both the angle  $\phi$  of the pendulum (mass  $m$ ) and the position  $y$  of the cart.

it is connecting ( $P$  and  $Q$ ). That is, its rest length is zero and its stretch is its total length. Hence, the vector force it transmits is proportional to the relative position vector of its ends  $P$  and  $Q$ ; the components of the spring force along any direction are proportional to the components of  $PQ$  along that direction. The idea of a zero rest-length spring was used, in a mechanisms similar to this one, by Lacoste in 1934 for use in a long period gravity-meter for a seismograph [10] (shown on the cover of Scientific American on March 1959) and by George Carwardine in 1932 for his Anglepoise lamp, the design crudely copied in the two-parallelogram mechanism of modern common student lamps [11].

Using the geometry of figure 3, the restoring torque about the hinge  $O$ , generated by the spring mechanism, is  $kab \sin \phi$ . The angle  $\phi$  is the lean of the bricycle,  $k$  is the stiffness of the spring and  $a$  and  $b$  are the lengths shown. The mechanism gives a restoring torque ( $kab \sin \phi$ ) with the same angular dependence as the capsizing torque of gravity. Hence the springy mechanism reduces the effect of gravity for leaning. For small lean angles, the mechanism linearizes to being a torsional spring of stiffness  $kab$  acting at the hinge  $O$ .

The stiffness of the effective torsional spring (or equivalently the gravity for leaning) can be adjusted by mechanically adjusting the lengths  $a$  and  $b$  as described in Fig 2. When  $a$  is 0, the stiffness is 0 and the training wheel suspension acts like an bicycle. When  $a$  and  $b$  are large, the stiffness is high and the system is close to a tricycle. To get near-infinite stiffness, corresponding to standard stiff tricycle, the hinge joint  $O$  can be locked.

Next we consider the balance and steering control of bricycle by considering 3 models with progressively more complexity, in sequence: 1) An inverted pendulum with moving base, 2) A primitive point-mass bicycle, and 3) The full Whipple bicycle model.

### 3. Control of an inverted pendulum with moving base

The simplest analogue to bicycle-balance is the balancing of a stick on a hand, e.g. a classical cart and inverted pendulum from elementary controls classes. The tendency of the pendulum to fall is equivalent to the lean (tilt or roll) instability of the bicycle. Steering a forwards-moving bicycle causes lateral acceleration of the base, analogous to accelerating the cart at the base of the pendulum. Using an inverted pendulum as an analogue to bicycle balance was apparently first presented in detail by Rankine in 1869 [12], as described in Kooijman *et al.* supplementary material [13]. Figure 4, shows the cart with an inverted pendulum (point mass  $m$ , length  $h$ ). The linearized equation of motion around the unstable equilibrium point

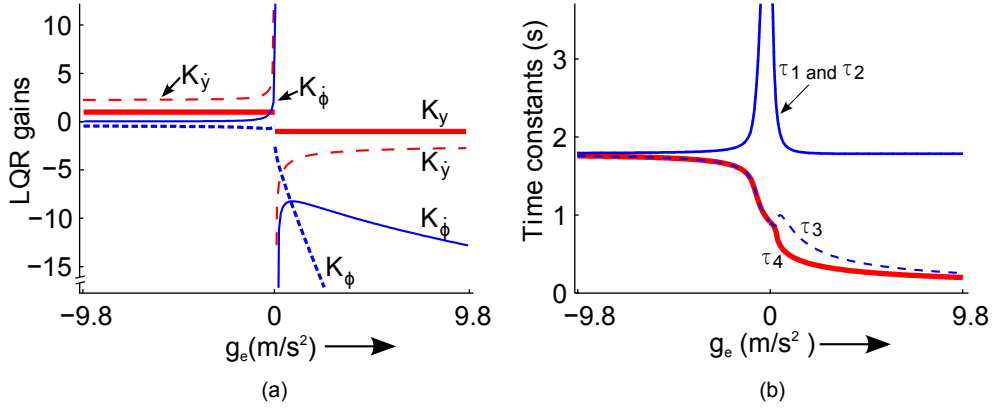


Figure 5. (a) LQR gains for the cart pendulum system as functions of gravity,  $g$  (and  $h = 1$ ). Note that around zero gravity ( $g = 0$ ) some of the controller gains go to infinity and others have a finite jump. System is uncontrollable at  $g = 0$ . (b) The time constants of the LQ Regulated, closed-loop system as function of  $g$ . Notice that two of them go to infinity at  $g = 0$  (closed-loop poles/eigenvalues have zero real parts.) This implies that disturbances in two directions of the state space, cannot be nullified by the linear controller gains. As  $g$  approaches zero, from above or below, two time constants go to infinity for any controller gains, not just the optimal LQR gains.

$\phi = 0$  is:

$$h\ddot{\phi} = g\phi - \ddot{y}. \quad (1)$$

Alternatively, using the state vector  $x = [y, \phi, \dot{y}, \dot{\phi}]^T$  state is controlled by  $u = \ddot{y}$ . Thus we get the state space form  $\dot{x} = Ax + Bu$ :

$$\begin{bmatrix} \dot{x} \\ \ddot{y} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{g}{h} & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y \\ \phi \\ \dot{y} \\ \dot{\phi} \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{h} \end{bmatrix}}_B \underbrace{u}_{\ddot{y}}. \quad (2)$$

### 3.1. The pendulum loses controllability at zero gravity

As is well known in the controls community (Mark Spong – private communication) various inverted pendulum systems, when linearized, are not controllable at  $g = 0$  ([14] and example 6.2 on page 170 of [15]). When there is no gravity ( $g = 0$ ) the balance problem (i.e. the tilt instability) is gone. However, the control authority is lost for some directions in state-space.

Plugging  $g = 0$  and rearranging the equation 1, we find that for any acceleration of the base:

$$\ddot{y} + h\ddot{\phi} = \frac{d^2}{dt^2}(y + h\phi) = 0. \quad (3)$$

The variable combination  $y + h\phi$  is always

$$y + h\phi = C_1 + C_2 t \quad (4)$$

where  $C_1$  and  $C_2$  are constants. Thus the variable combination  $y + h\phi$  is unaffected by the control  $u = \ddot{y}$ . Note that  $y + h\phi$  is the  $y$ -position of the pendulum's bob (the point mass  $m$ ), and for small angles this is unaffected by motions of the base.

Alternatively, the angular momentum of the system about any point on the  $y$ -axis,  $\vec{H} = -mh(\dot{y} + h\dot{\phi})\hat{z}$ , is conserved when  $g = 0$ . This implies that the velocity of mass  $m$  remains constant independent of the base acceleration. If  $m$  was at rest initially, it will remain at rest for all time.

Hence, for the case of zero gravity, assuming, say, a stationary vertical initial condition, the position  $y$  and the angle  $\phi$  (balance) cannot be controlled independently. If one is changed the other has to change to keep the  $y + h\phi = 0$ .

More simply, in the presence of gravity one possible motion of the pendulum system is constant acceleration  $a$  to, say, the right at fixed lean angle  $\phi$  with  $\tan\phi = a/g$ . When gravity is set to zero, there is no small-angle solution (the constant acceleration is only at a lean of  $\phi = \pm\pi/2$ ).

[Note, the uncontrollability at zero gravity is only true for the linearized inverted pendulum. The arguments precluding control are heavily dependent on the linearity of the system, and the non-linear inverted pendulum in zero gravity seems to be controllable. Indeed, simulations by Philip James (private communication — tilt-vehicle@bigpond.com) demonstrate that with appropriate wiggles of the base, a constant average acceleration of the base can be maintained while holding the pendulum angle in a bounded range.]

### 3.2. LQR Control of inverted pendulum

The lack of controllability can also be understood in terms of classical control theory [16]. Consider a stabilizing linear feedback controller  $u = -Kx$ , which drives a system to the equilibrium  $x = 0$ , while minimizing a quadratic cost functional,

$$J = \int_0^\infty (x^T Q x + u^T R u) dt. \quad (5)$$

The controller gains  $K = [K_y \ K_\phi \ K_{\dot{y}} \ K_{\dot{\phi}}]$  for such a controller can be found using the LQR approach (MATLAB command 'lqr'). Figure 5 shows the controller gains  $K$  and time constants  $\tau_i$  of the closed-loop system, as functions of gravity  $g$  (and  $h = 1$ ).  $Q$  and  $R$  are chosen to be identity matrices of appropriate dimensions. The time constants  $\tau_i$  are inverses of the real parts of eigenvalues of the closed-loop system  $\dot{x} = (A - BK)x$ .

As we vary the gravity,  $g$  from negative to positive, the point  $x = 0$  changes from stable to unstable equilibrium. Expectedly it is easy to stabilize the pendulum with negative gravity, it is just like a normal (non-inverted) pendulum. For positive gravity, as  $g$  increases the pendulum becomes 'more' unstable (it is 'easier' to fall). Hence, the gains  $K_\phi$  and  $K_{\dot{\phi}}$  which correspond to  $\phi$  and  $\dot{\phi}$  increase with  $g$ , and the corresponding time constants become smaller.

When  $g = 0$  the system is uncontrollable in certain directions of the state space, i.e. the deviations in those directions cannot be corrected using any control action (at least in the limit of linear system approximation). The uncontrollable directions are the eigenvectors corresponding to zero eigenvalues of the controllability Gramian of the system,  $W_C$ :

$$W_C = \int_0^T e^{At} B B^T e^{A^T t} dt. \quad (6)$$

Given a control input  $u(t)$  and zero initial condition  $x(0) = \vec{0}$ , the state at time  $T$  can be written as:  $x(T) = \int_0^T e^{A(T-t)} B u(t) dt$ . If one considers all possible input profiles of unit or lesser energy ( $\int_0^T u^2(t) dt \leq 1$ ), all the states that can be reached

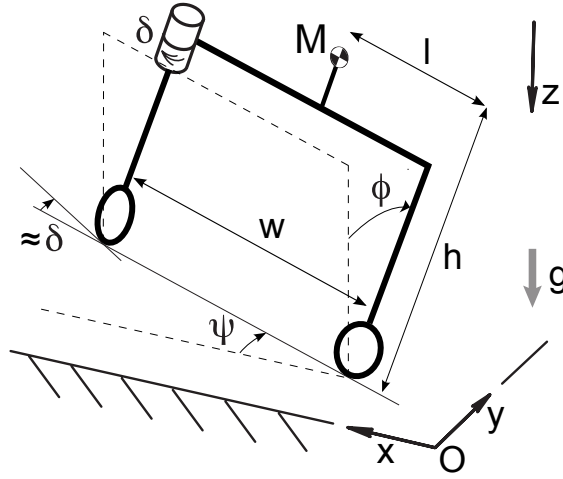


Figure 6. Primitive model of a bicycle is obtained from the full model described in figure 7 by simplifying assumptions. The bicycle is assumed to be massless and the rider is modeled as a point mass fixed with respect to the body frame. The steering assembly has no tilt, trail or caster. The rate of change of the steering angle  $\dot{\delta}$  is taken to be the control input.

in time  $T$  or less, lie in an ellipsoid in state space. The axes of this ellipsoid are along the eigenvectors of the Gramian and axes lengths are square roots of its eigenvalues. Directions corresponding to the zero eigenvalues are the directions in the state space which the control inputs cannot control. One is stuck with the motions that follow from the initial conditions in those directions.

For the cart and inverted pendulum, under zero gravity, the uncontrollable directions/ eigenvectors are:  $[1/h, 1, 0, 0]^T$  and  $[0, 0, 1/h, 1]^T$ . These correspond to the (uncontrollable) variables  $\bar{x}_1 = y + h\phi$  and  $\bar{x}_2 = \dot{y} + h\dot{\phi}$ . Note that  $\bar{x}_1$  is the same variable combination found before using simpler reasoning, and  $\bar{x}_2$  is its first derivative.

The gains as seen in figure 5 can be transformed to the gains in the eigendirections and it can be noticed that only the gains corresponding to the uncontrollable directions  $\bar{x}_1$  and  $\bar{x}_2$  are the ones that go infinite ( $\bar{x}_2$ ) or have a jump discontinuity ( $\bar{x}_1$ ) at  $g = 0$ .

#### 4. Primitive bicycle model

First we present the simpler of two bicycle models, a primitive point mass model (Fig. 6). This seems to be the most extreme simplification of a bicycle for understanding its control, used for example, by Boussinesq, 1899 [17] and Getz and Marsden, 1995 [18]. All of the mass is concentrated at a point a distance  $l$  in front, and  $h$  above the rear contact point. The relevant configuration of the bike is described by the  $x$  and  $y$  coordinates of the rear contact, the heading/yaw angle  $\psi$  of the body frame, the lean/roll angle  $\phi$  of the body frame and the steer angle  $\delta$  of the handle bar. The wheels are massless and infinitesimal (equivalent to small skates). The steering has no tilt, trail nor caster.

Consider the nominal motion as the upright bike moving along  $x$ -axis at constant speed  $v$  i.e  $\phi = \dot{\phi} = \delta = \dot{\delta} = \psi = 0$ ,  $v = \text{constant}$ . The equations of motion for small perturbations from the nominal motion, including the torque from the spring

mechanism  $kab\phi$ , can be written as:

$$\begin{aligned}
 h\ddot{\phi} &= \underbrace{\left(g - \frac{kab}{mh}\right)}_{g_e} \phi - \frac{v^2}{w}\delta - \frac{vl}{w}\dot{\delta}, \\
 &= g_e\phi - \frac{v^2}{w}\delta - \frac{vl}{w}\dot{\delta}, \\
 \dot{\psi} &= \frac{v}{w}\delta, \\
 \dot{y} &= v\psi.
 \end{aligned} \tag{7}$$

where  $w$  is the wheel base. Note that the effect of spring mechanism is equivalent changing the gravity. The effective gravity  $g_e = g - kab/mh$ .

While a direct derivation is possible, these equations follow from simplification of the equations for the full Whipple model [19].

#### 4.1. Primitive bicycle loses controllability at zero gravity

Unlike the Whipple bicycle model discussed later, the primitive bicycle model cannot be stable without controls. In this paper we are concerned with controlled stability rather than self-stability so the primitive bicycle is still appropriate. Because the steering has no mass we cannot be concerned with the dynamics of the steering angle. We think of the steer angle as the control variable. Lean and heading are controlled by steering  $(\delta, \dot{\delta})$  as governed by the equations above.

When  $g_e = 0$  we can substitute the second of equations 7 into the first, and use the third to eliminate  $\psi$  to get:

$$v\dot{\psi} + h\ddot{\phi} + \frac{vl}{w}\dot{\delta} = \frac{d}{dt}(v\psi + h\dot{\phi} + \frac{vl}{w}\delta) = \frac{d}{dt}(\dot{y} + h\dot{\phi} + \frac{vl}{w}\delta) = 0. \tag{8}$$

Hence the uncontrollable direction is  $v\psi + h\dot{\phi} + \frac{vl}{w}\delta$ , which can be written as  $\dot{y} + h\dot{\phi} + \frac{vl}{w}\delta$ . Note that

$$\dot{y} + h\dot{\phi}$$

is the lateral velocity of the center of mass of the bicycle.

As for the inverted pendulum, 7 can be rearranged into state space form  $\dot{x} = Ax + Bu$  where the state vector  $x = [\phi, \delta, \psi, \dot{\phi}]^T$  and the control input  $u = \dot{\delta}$ . And, as for the pendulum, the system is uncontrollable when  $g_e = 0$ . Figure 8(a) shows the time constants as functions of gravity for an LQR controller. The uncontrollable mode corresponds to the collection of variables  $\dot{y} + h\dot{\phi} + \frac{vl}{w}\delta$  described above.

#### 4.2. What is the uncontrollable mode

Assume a person is initially riding in a straight line i.e.  $\phi_i = \delta_i = \psi_i = \dot{\phi}_i = 0$ . Hence

$$\dot{y} + h\dot{\phi} + \frac{vl}{w}\delta = 0 \tag{9}$$

initially and it will remain 0 for all time. Thus

- (1) **Heading can't be corrected without falling.** If the rider tries to steer by giving some steering angle profile  $\delta(t)$  for some time and then finally makes the steering handle straight again to stop steering:  $\delta_f = 0$ . So  $v\psi_f + h\dot{\phi}_f = 0$ . Staying upright means the lean angle is not be changing i.e.  $\dot{\phi}_f = 0$ . This implies  $\psi_f = 0$ . Hence the heading cannot be changed permanently.
- (2) **Center of mass cannot be moved laterally.** Using the kinematics eqn  $\dot{y} = v\psi$ , and using eq. 7  $\delta = \frac{w}{v}\psi$ , we get

$$l\dot{\psi} + \dot{y} + h\dot{\phi} = \frac{d}{dt}(l\psi + y + h\phi) = 0$$

Hence we have another conserved quantity whose initial value was zero (taking  $y = 0$ ) and final value is also zero. This, along with  $\psi_f = 0$ , implies that:  $y_f + h\phi_f = 0$ .

Noting the geometry in fig. 6, we see that  $y + h\phi$  is the  $y$  co-ordinate of the center of mass. That is, the mass remains at the same lateral location in a steering maneuver. Hence the bicycle steering problem is really strictly analogous to the cart and inverted pendulum problem. The uncontrollability of both the primitive bike model and the point-mass cart pendulum, in zero gravity, boils down to inability to move the center of mass sideways, by any control efforts (assuming small angle approximations).

- (3) **Leaning causes sideways displacement.** We have  $y_f + h\phi_f = 0$ , this can be rewritten as  $y_f = -h\phi_f$ . Hence after a steering maneuver the rider can get sidetracked and have a resulting lean angle that is proportional to the distance sidetracked,  $y_f$ .
- (4) **If you keep steering you'll fall.** If  $\delta$  is not zero by the end of a maneuver, then  $\dot{\psi} = \frac{v}{w}\delta$  will cause the heading  $\psi$  to increase (or decrease) continuously. Thus conservation of  $\bar{x}$  will lead to a non zero lean rate  $\dot{\phi}$ . The rider will progressively fall.

## 5. Whipple model

Figure 7, shows the more sophisticated Whipple bicycle model as presented by Meijaard *et al.*[19]. The relevant configuration of the bike is described by the  $x$  and  $y$  coordinates of the rear contact  $P$ , the heading/yaw angle  $\psi$  of the rear frame, the lean/roll angle  $\phi$  of the rear frame, the steer angle  $\delta$  of the handle bar, and the rotation of the front and rear wheels  $\theta_F$  and  $\theta_R$ . The equations of motion as described in equation 5.3 of Meijaard *et al.* are:

$$\begin{aligned} M \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} + vC_1 \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix} + (gK_0 + v^2K_2) \begin{bmatrix} \phi \\ \delta \end{bmatrix} &= \begin{bmatrix} T_\phi \\ T_\delta \end{bmatrix}, \\ \dot{\psi} &= \left( \frac{v}{w}\delta + \frac{c}{w}\dot{\delta} \right) \cos \lambda. \end{aligned} \tag{10}$$

Where  $T_\phi$  is the lean torque,  $T_\delta$  is the steering torque,  $v$  is the speed,  $c$  is the trail,  $w$  is the wheel base,  $\lambda$  is the steer axis tilt,  $M$ ,  $C_1$ ,  $K_0$ , and  $K_2$  are matrices as described in equation 6.1-4 in Meijaard *et al.*2007 [19]. For the case of bricycle, the lean torque is provided by the spring mechanism:  $T_\phi = kab\phi$  as in Fig. 3. Now instead of steer torque,  $T_\delta$  we can take  $\ddot{\delta}$  (which is fully controllable by  $T_\delta$ ) to be the input to the system. This way the steer dynamics can be neglected, and the

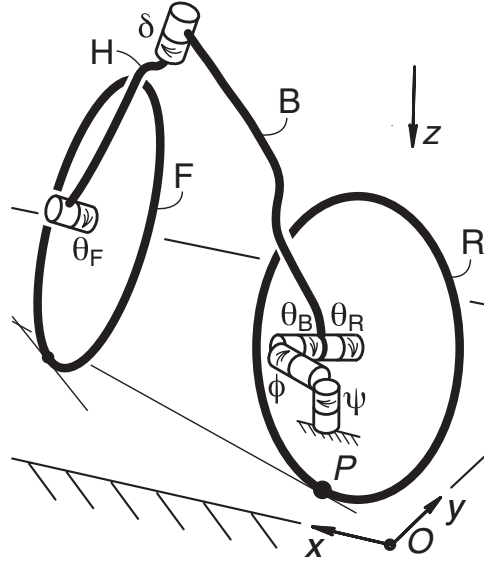


Figure 7. 7-D configuration of a Whipple bicycle (courtesy Meijaard *et al.* [19]). The  $x$  and  $y$  are coordinates of the rear contact  $P$ . Angles are represented by a sequence of hinges drawn as a pair of cans rotated with respect to each other. The clockwise (from top view) heading/yaw of the rear frame  $B$  is  $\psi$ . The  $\psi$ -can is grounded in orientation but not in location. The lean ('roll' in aircraft terminology) of the rear frame to the right is  $\phi$ . The rear wheel  $R$  rotates with  $\theta_R$  relative to the rear frame, with forward motion being negative. The steer angle is  $\delta$  with right steer of the Handle bar  $H$ , as positive. The front wheel  $F$ , rotates with  $\theta_F$  relative to the front frame. The three velocity degrees of freedom are parameterized by  $\dot{\phi}$ ,  $\dot{\delta}$  and  $\dot{\theta}_R$ . The rates-of-change of the remaining variables are determined by 4 non-holonomic constraints.

reduced equations of the system are:

$$\begin{aligned} M_{11}\ddot{\phi} + vC_{1,12}\dot{\delta} + (g_eK_{0,11})\phi + (gK_{0,12} + v^2K_{2,12})\delta &= -M_{12}\ddot{\delta} \\ \dot{\psi} &= K_3\delta + K_4\dot{\delta} \end{aligned} \quad (11)$$

The terms  $C_{1,11}$  and  $K_{2,11}$  are zero as shown in Meijaard *et al.* 2007 [19]. Note that the term  $(gK_{0,11} - kab)$  has been replaced by the effective gravity term  $g_eK_{0,11}$ . However the gravity  $g$ , unaffected by the spring mechanism still appears in the  $\delta$  term (that is, in the full Whipple model the system center of mass position is directly affected by steering).

Equations 11 can be rearranged into the state space form  $\dot{z} = Az + B\ddot{\delta}$  where the state vector  $x = [\phi, \delta, \psi, \dot{\phi}, \dot{\delta}]^T$ . The zero gravity case  $g_e = 0$ , is when the bricycle is in neutral equilibrium for leaning when the steer angle is also zero. As expected this is an uncontrollable system. The uncontrollable direction is:

$$\bar{x} = \left( vC_{1,12} - (gK_{0,12} + v^2K_{2,12})\frac{K_4}{K_3} \right) \delta + \frac{(gK_{0,12} + v^2K_{2,12})}{K_3} \psi + M_{11}\dot{\phi} + M_{12}\dot{\delta}, \quad (12)$$

Similar conclusions can be drawn using the uncontrollable mode, as for the case of primitive bike, except that the point which preserves its horizontal position is not exactly the center of mass. Figure 8(b) shows the closed-loop time constants as functions of gravity for an LQR controller (with identity  $Q$  and  $R$  matrices and  $v = 6$  m/s). The qualitative behavior, and even the quantitative behavior, is the same as in the case of the primitive model.

That is, the Whipple bicycle model and primitive bicycle model show essentially the same lack of controllability when the effective gravity is set to zero.

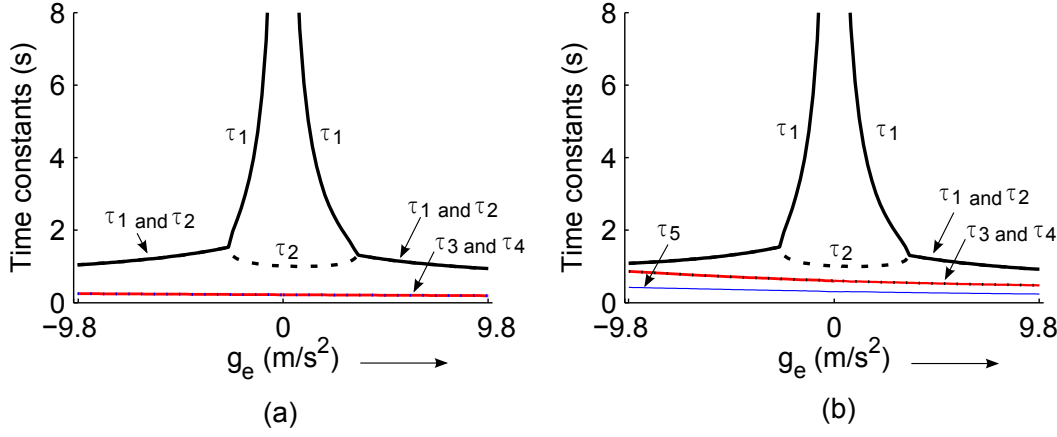


Figure 8. The time constants of the LQ Regulated closed-loop system as function of  $g_e$ : a) for the primitive bicycle model as described in figure 6 and b) the complete model as described in figure 7. Notice that in each case one of the time constants goes to infinity at  $g_e = 0$  (closed-loop poles/eigenvalues have zero real parts). This implies an uncontrollable direction in the state-space, deviations in that direction cannot be corrected.

## 6. Riding the bricycle

The bricycle was built to test the above theory. A video showing the rider's experience in various spring setting is available at <http://www.youtube.com/watch?v=rNQdSfgJDNM>.

When the spring is removed or has low stiffness, the bricycle behaves like a normal bicycle. Riders lean into the turns and a component of gravity provides the centripetal acceleration. Moment balance gives a lean angle  $\phi$ , in a steady turn of radius  $r$  at speed  $v$ , as  $\tan \phi = \frac{v^2}{rg_e}$ . To initiate this leaning, some counter steering is needed. When the spring is stiff, or the mechanism is clamped, the bricycle behaves like a tricycle. Because there is then little leaning, the rider experiences the centrifugal push away from the center of turn. Moments are balanced by the asymmetrical normal reactions on the rear wheels. Figure 9, shows the lean angles in a steady turn for various speeds and stiffnesses (i.e. effective gravities  $g_e$ ).

When the spring is set to cancel gravity ( $g_e = 0$ ) it was observed that a rider initially moving straight, cannot perform a steady turn. As expected based on the discussion in section 4, he/she ends up sidetracking and is left with a constant lean in the opposite direction. If counter-steering is done to initiate the turn, the rider is unable to change the heading, and ends up side-tracking away from the desired turn with a fixed lean angle into the desired turn, and vice-verca if no counter-steering is used. If the rider keeps steering (e.g. with a constant steer angle  $\delta$ ) he/she will fall. Lean angle  $\phi$  goes towards 90 degrees (bike moves flat on the ground) and  $\tan \phi$  is infinite, as seen in figure 9. This is why, riders after a few initial trials, learn (perhaps unconsciously) to give up on the steering and instead just preserve balance.

## 7. Discussion and conclusions

The theory and experiments in this paper show that there is a qualitative difference in the dynamics and control of a bicycle and of a tricycle. A smooth transition between them cannot be achieved using a bricycle-type design. At some transition point (effectively zero gravity) the system is uncontrollable. The idea of providing aid to help learn a motion or activity works for many situations as described in section 1, but may not be a good idea in this case, at least if steering is to be

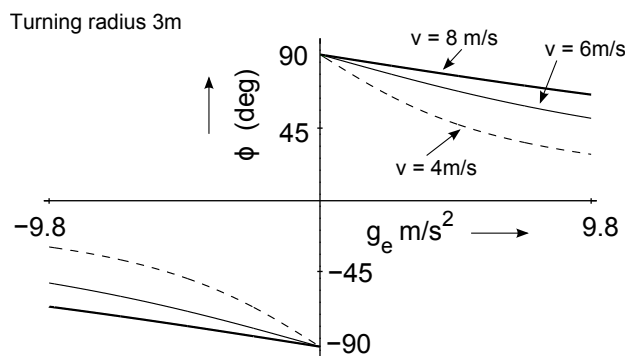


Figure 9. Lean angles in a steady turning radius of 3m for various speeds and gravitational accelerations. For a bricycle the effective gravity can be changed by adjusting the stiffness of the spring mechanism. Note that for zero gravity the steady state turning angle is 90 degrees, i.e. the bike is flat on the ground. For negative gravity (i.e. tricycle like situations) the lean angle is negative (i.e. away from the turn).

learned as well as balance. A bricycle cannot be used to transfer the steering skills from a tricycle to a bicycle in a continuous fashion.

Balance and control are not independent issues for a controller to solve. Gravity, the force which causes instability and loss of balance, is also the force that facilitates the control of position and heading. Without gravity, balance and control maneuvers, like steering and navigation, cannot be performed independently.

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