

PROVISIONAL DRAFT

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STABILITY OF THE BICYCLE

From "Theorie des Kreisels" (Theory of Gyroscopes) by Sommerfeld and Klein, 1903 edition available from University Microfilms, Ann Arbor. [See also "Mechanics", A. Sommerfeld, Academic Press, p.156]

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Section 8: The Stability of Bicycles

The most essential question concerning the construction of modern bicycles, that of the energy which the rider must provide in order to achieve reasonably great velocities, has been treated theoretically in many ways.¹ Greater energy savings, and easier steerability, are essential advantages of the bicycle over the tricycle. However, with these advantages is associated a drawback, in that a vertical bicycle is in unstable equilibrium. To maintain this equilibrium in the presence of disturbing influences, it is necessary to learn to ride a bicycle, which is facilitated by the special construction of modern bicycles.

Is the gyroscopic action of the rotating wheels a significant factor in the stabilisation of the upright position? In the normal way of describing a gyroscope, we say that the axis of rotation tends to maintain a fixed direction in space. Considering the small mass of the wheels compared to the mass of the whole system formed by the rider and bicycle, one might doubt such an effect. And, it is obviously it is not the manufacturer's intention to increase such an effect, because his goal of saving energy leads him to build all the parts as light as possible, whereas an increase of the mass of the wheels would lead to better gyroscopic stabilisation.

In any case, we want to emphasise at this point that the gyroscopic action can be effective only if the system has sufficient degrees of freedom.

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For the unicycle, that is a single rolling disk, gyroscopic stabilisation is undoubtedly possible, as is shown by theoretical calculations in agreement with experience. [Note: for passive pitch stability with no trailing wheel the rider's c.m. should be below wheel axis.] At a sufficient velocity, the rolling motion of such a disk is stable in a vertical plane.² In this case, the motion can be viewed as an advancing motion of the center of mass, combined with a rotation about the center of mass. The latter was shown earlier in this book to be stable.³

The closest resemblance of a bicycle to a single disk occurs in the case of the high-wheeler, which consisted of a big front wheel (corresponding to the disk), and a small rear wheel which is there to support the seat of the rider and allow control of the steering. The rear wheel would reduce the system's degrees of freedom by one, thus making stabilisation impossible, if the front part of the frame had not been mounted on a pivoting steering axis which permits the plane of the front wheel to be rotated with respect to the plane of the rear wheel. If this pivot were fixed, the whole system would have only two degrees of freedom, that of tipping about the horizontal line of the bicycle track, and the motion in the forward direction connected to the rotation of the wheels. This would preclude the possibility of stabilisation by gyroscopic action. The contemporary bicycle is different from a high-wheeler only in its dimensional relations. Both wheels have the same size, and the mass of the wheels is much smaller compared to the total mass of the bicycle. This reduces the influence of gyroscopic action.

The third degree of freedom, that of rotation around the steering axis, not only allows the gyroscopic action to stabilise the bicycle, but it also allows the unconscious action of a trained cyclist to assist in keeping

¹ Enzyklopedie der Mathematischen Wissenschaften, Vol.IV, No.9, (Walker, Spiel, und Sport), p.149

² Carvallo, Journal de L'Ecole Polytechnique, Series 2, Vol.5, 1900. [Note: an ideal rolling disk wobbles forever; did Carvallo's disk stop wobbling?]

³ [Which pages, which volume?]

Noethers role described after title page in un-numbered pgs between 759 and 761.

the bicycle upright. The original theory of the bicycle, due to Rankine,⁴ considered only these contributions of rider control actions to bicycle stability. If for example the whole bicycle tilts to the right hand side, the rider must turn the front wheel towards this side, and force the bicycle to make a right hand turn. The centrifugal force generated by this turn, which acts on the system's center of mass, has a moment about the track line, which tends to restore the plane of the bicycle to the vertical. To avoid falling to the left side, the rider must similarly turn to the left. Since the existence of the steer degree-of-freedom is necessary for this active stabilisation, it is difficult to decide how much of the stabilisation is due to gyroscopic action, and how much is due to unconscious movements of the rider.

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Against the unconditional acceptance of this theory, the rider will object that he is not aware of a continuous guidance of the handlebars, that he is able to ride securely without touching the handlebars, and that he guides the handlebars more to control the direction of the front wheel than to stabilise his equilibrium. He could furthermore generate a moment by unconsciously tilting his body sideways.

It is not yet decided to what extent the stability could be achieved by small motions of the rider, but this could be decided by an experiment. In any case, it would be interesting to investigate to what degree the self-stabilisation of the bicycle with an unmoving rider is possible, and how important gyroscopic actions are in that case. The process of stabilisation is then that one such that Rankine's steer-controlling actions are partly provided by gyroscopic effects, when bicycle construction is suitable, as we will discuss later. The question about the extent to which the bicycle is stabilised without rider actions (which means that we assume the rider to be connected rigidly to the frame of the bicycle and not touching the handlebars) was treated by Whipple⁵ and Carvallo.⁶ We will in the following investigate to what extent *gyroscopic* effects are important in stability. We will of course neglect all the peripheral issues (such as alternate-side loading of the pedals, flexibility of the tires which results in a finite area of contact, friction in the steering pivot, and twisting friction of the tire on the ground).

Whipple and Carvallo set out the general Lagrange equations of the first and second kind, the latter being suitably modified because it is not a holonomic system, and they are specialised to the case of small oscillations about straight-line upright motion. [In fact, Whipple gave a purely Newtonian analysis.] We hope to be able to make the mechanical connections more clear, if in the derivation of the approximate equations, we add to the forces on the system at rest the gyroscopic action and centrifugal forces generated by the motion. This was also done in the applications we treated earlier. To obtain the terms to first order in a small oscillation, it is in this case sufficient to use the simplified expression I of the gyroscope expression on page 764. [*** Get this!]

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If we neglect terms that are quadratic in small oscillations, we remark that the size of the deviation considered are completely within the limit for which this approximation is valid.

The equations obtained in this manner agree with the equations of Whipple and Carvallo. [We agree with Carvallo's equations, but have not yet been able to confirm Whipple's.] There is one remark to make. The motion is of course unstable for small velocities, whereas at certain intermediate velocities the motion becomes stable. This means that the oscillations can be written in the form

$$Ae^{\lambda t}$$

where λ is a complex quantity with a negative real part. Whipple, under numerical assumptions for bicycle parameters which agree somewhat better with modern practice than those of Carvallo, finds stability between 16 km/hr and 20 km/hr, which are easily achievable. In what appears to be a paradox, for higher velocities the motion becomes unstable, but this can easily be explained in terms of how the parts of the system are coupled. It is useful furthermore that the lateral instability is only weak, and can be suppressed by almost unnoticeable movements of the rider even when he is not touching the handlebars.

For us the contribution of gyroscopic action to these results is interesting. We will show something that was not followed up by the cited authors: that in the absence of gyroscopic actions, the speed range of

⁴ Rankine, The Engineer, Vol.28, pp.79,129,153,175; 1869.

⁵ Whipple, Quarterly Journal of Pure and Applied Mathematics, Vol.30, No.120, p.312, 1899

⁶ Carvallo, Journal de L'Ecole Polytechnique, Series 2, Vol.6, 1901

complete stability would vanish. The gyroscopic action, in spite of its smallness, is essential for autonomous (passive) stability.

The bicycle (Fig. 135) consists essentially of a frame which carries the rear wheel in its central plane, and a steering assembly with an axle similarly carrying the front wheel. The axis is supported by a tube in the plane of the frame, so that in effect we have two plane systems hinged together at their line of intersection. The rider is rigidly attached to the frame. The steering axis in modern bicycles is tilted backwards, and its geometry is such that the extension of the steering axis cuts the vertical diameter of the front wheel at a point between the wheel center and the ground contact point.

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As Bourlet⁷ stresses, this arrangement of the axis makes it more difficult for the front wheel to turn (collapse) sideways if the rear frame is held upright. [Is Bourlet saying that negative trail tends to make the wheel flip around when in motion?] A closer investigation of the consequences of this not unimportant arrangement is not necessary here, because its influence will be made clear naturally in the course of our analytical treatment.

Because of the kinematic connections between the front and rear wheels, we can at any instant define the position of the frame by the two co-ordinates at which the rear wheel touches the ground, plus the angles of heading and lean. As well, we have a co-ordinate defining the rotation of the steering assembly relative to the frame, that is the steering angle. Because of the additional condition that the front wheel must touch the ground, these five co-ordinates are sufficient to define the position and orientation of the complete bicycle. Ignoring the cyclic co-ordinates which define the rotational orientations of the wheels with respect to the bicycle, the freedom of motion is first a tilting of the plane of the frame, and then a rotation of the steering assembly about the steering axis. As the movement of both wheels must be one of rolling on the ground, the orientation of the plane of the front wheel determines its direction of motion. This in turn determines the motion of the handlebars, and also the motion of the plane of the rear wheel which is connected to the steering axis. Now, only the motion in the forward direction is not specified. The bicycle thus has three degrees of freedom.

We learn here about the characteristics of non-holonomic systems⁸,

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which include all (?) rolling systems: the Bicycle can be transferred to each of its ∞^5 possible positions with a sequence of allowed motions, but it is not possible at every instant to move directly to all nearby configurations within an infinitesimal neighborhood via an infinitesimal motion. Expressed analytically, the equations relating the variations of the five configuration co-ordinates form a non-integrable system of differential equations.

The parameters of the bicycle are as follows: M_1 is the mass of the steering assembly, and we can assume that its center of gravity S_1 is at the center of the wheel, without much error. [Note: this is badly wrong if there is luggage or a motor on the steering assembly.] Its height h_1 is thus the radius of the front wheel, as shown in Fig.135. We are ignoring the handlebars and front fork, which would displace the center of mass from the center of the wheel, though of course we will at least add their mass to the mass of the wheel at S_1 . M_2 designates the mass of rear wheel, frame, and rider. The center of gravity S_2 is at height h_2 , and is at a distance r in front of the point B_2 where the rear wheel touches the ground.

Further, let A_v be the moment of inertia of the steering assembly about the vertical axis through the point B_1 where it contacts the ground, and let A_h be its moment of inertia about its track-line (the horizontal line where its plane intersects the ground). B_v , B_h are the corresponding quantities of the system consisting of the rear wheel+frame+rider, about the rear contact point B_2 , and B_{hv} is its product of inertia.

The steering axis has a tilt of σ with respect to the vertical, and the point where it intersects the ground lies a distance c_1 in front of the front contact point, and a distance c_2 in front of the rear contact point. So, $c_2 - c_1 = l$, where l is the length of the wheelbase.

Furthermore let θ_2 be the tilt of the plane of the rear wheel away from the vertical (positive towards the right hand of the rider), θ_1 is the corresponding tilt of the front wheel, and γ is the angle between front and rear wheel measured around the steering axis, (positive if the front wheel is turned to steer to the left side

⁷ Bourlet's interesting small book Nouveau Traité des Bicycles et Bicyclettes, Paris 1898, p.87

⁸ Hertz, Die Prinzipien der Mechanik, I. Buch, Abschn.4, Nr. 123-133

of the rider). Let ϕ_1 be the angle of the line formed by the intersection of the plane of the front wheel with the ground plane, measured in the same sense as γ ; ϕ_2 is the corresponding angle for the rear wheel. Here we assume that these angles ($\theta_1, \theta_2, \phi_1, \phi_2, \gamma$) are small, as only in this case do these definitions have an immediate meaning. Under these limitations, we have to establish the kinematic formulas for the motion.⁹

We think of the front wheel being brought to its tilted position θ_1 in two steps.

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First the whole bicycle is tilted to the angle of the rear wheel θ_2 , with no steering allowed, then the rotation γ about the steer axis is added. [Note: this causes the front of the frame to rise.] The latter we can decompose into its horizontal and vertical components (see Fig. 135) to give a tilt of $-\gamma \sin(\sigma)$ about the track-line (taking account of the above-defined senses of rotation). This can be added, according to the theorems of small rotations, to the first tilt, yielding

$$\theta_1 = \theta_2 - \gamma \sin(\sigma).$$

The vertical component of the rotation γ then lets us write

$$\phi_1 - \phi_2 = \gamma \cos(\sigma).$$

To these geometric relations between the orientation co-ordinates there is added according to the above remarks a nonholonomic relation for their time derivatives. Let the average velocity of the forwards motion be u . We assume that the orientation of the bicycle, the velocity u , and the rate of rotation $\frac{d\phi_1}{dt}$ of the front wheel about its contact point are given. The desired relations are obtained if we observe that these two conditions determine the motion of the point where the steering axis intersects the ground. The motion of this intersection is determined, as is the motion of the plane of the rear wheel, which follows the steering axis. The intersection point F (fig. 136) has along the trace of the front wheel the velocity u . Perpendicular to this line it has the velocity $c_1 \frac{d\phi_1}{dt}$. We form the component of the motion perpendicular to the track of the rear wheel which forms the angle $\phi_1 - \phi_2 = \psi$ with the track of the front wheel. So this component becomes:

$$c_1 \frac{d\phi_1}{dt} \cos(\phi_1 - \phi_2) + u \sin(\phi_1 - \phi_2),$$

or, to first order:

$$c_1 \frac{d\phi_1}{dt} + u(\phi_1 - \phi_2).$$

As a result of the rotation $\frac{d\phi_2}{dt}$ of the rear wheel the point F also has the perpendicular component $c_2 \frac{d\phi_2}{dt}$, so setting them equal we finally arrive at

$$c_2 \frac{d\phi_2}{dt} = c_1 \frac{d\phi_1}{dt} + u(\phi_1 - \phi_2)$$

as the nonholonomic relation we seek.

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Because c_1 is small compared to c_2 , the above condition expresses the rear wheel in general having to follow to the side towards which the front wheel is turned. [Note: if they are equal, steering becomes impossible — get tractrix convergence only.] In consideration of the component of the point F's velocity in the track of the rear wheel, another condition for the forward velocity of the rear wheel will follow, which shows that it differs from u by terms of second order. For our purposes, this condition is unimportant.

Now we have to determine the forces and reactions acting on the bicycle (fig. 137). At the center of gravity S_1 acts the force $-M_1 g$, while at the center of gravity S_2 acts the much larger force $-M_2 g$. Reaction forces act at both contact points and at the steering axis connecting the front and rear parts of the frames.

⁹ For an exact treatment of the kinematics, see Whipple or Carvallo loc. cit. [Note: it would be surprising to find an exact treatment because of the difficulties in calculating precisely the drop of the front of the frame due to the turning of the handlebars.]

Let's consider first the vertical reactions. We only need to consider reactions which balance the gravity forces on an upright bicycle travelling in a straight line, as these are terms of finite size. Their changes in cases of small deviations from straight riding are small terms of higher order and can be neglected.

So we are dealing only with the vertical forces which support both parts of the bicycle at rest. According to the general laws, one has to assume a reaction force at the steering axis, and a reaction moment with an axis perpendicular to the plane of the bicycle. But a force and moment can be written as a single force acting at another point (fig. 137): [Note: they are saying that the front contact force is transferred vertically to act on the steering axis of the frame.]

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First if the system were completely rigid, a force

$$Z_2 = M_2 g \frac{l-r}{l}$$

acts at the rear contact point, while

$$Z_1 = M_1 g + M_2 g \frac{r}{l}$$

acts at the front one. The component $M_2 g \frac{r}{l}$ in Z_1 is transmitted through the pressure of the rear frame onto the steering assembly. Therefore we have a balance in both parts of the system connected with the steering hinge, if we add another reaction at the steering axis vertically over the point where the front wheel touches the ground:

$$-Z = -M_2 g \frac{r}{l} \text{ steering assembly,}$$

$$Z = M_2 g \frac{r}{l} \text{ frame.}$$

These are the forces acting respectively on the steering assembly and on the rear frame. The height of the point of action is $c_1 \cot(\sigma)$, as can be seen in Fig. 137.

We now come to the horizontal reactions. First a reaction $\pm Y$ at the steering axis perpendicular to the plane of the wheel, and therefore also perpendicular to Z , which stems from a transfer of sideways motion between both parts of the system. Its size, which is also of first order, is not a static quantity but depends on the state of motion. The determination of its point of action is unimportant for our purposes.

Finally, we must introduce the reactions at the steering axis in the direction of motion, as they transmit the drive from the rear wheel to the front wheel. But these are absent in a uniform straight-line ride if there is no rolling friction and in the case of small deviations they are of second order. As their lever arm around all axes in the vertical plane containing the point of contact are of first order, their moments in all the upcoming moment equations will be of third order, and can therefore be disregarded.

Furthermore we don't have to take into account that by turning the front wheel around the steering axis its contact point is displaced sideways, as geometric intuition shows. Because this displacement is first order, its influence on the considered moments around the point of contact is of second order.

We have to add a kinematic remark concerning the position of the center of gravity.¹⁰

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It is easy to show geometrically, that in our assumptions about the position of the handlebars and the steering axis, and the center of gravity S_2 , in particular $c_1 > 0$, $r > 0$, by rotating the front wheel while the rear wheel is held vertical the height of the center of gravity S_2 , in a first approximation, is unchanged for reasons of symmetry, but in calculations to second order its height is reduced. So that the position $\gamma = 0$ means a maximum of the height of the center of gravity. This means that in addition to the gravity potential due to the simple tilt to one side of the front and rear assemblies

$$\text{Const} - \frac{g}{2} (M_1 h_1 \theta_1^2 + M_2 h_2 \theta_2^2),$$

also terms with the factor $c_1 r g M_2 \gamma^2$ and $c_1 r g M_2 \theta \gamma$ have to be added. We don't have to calculate them exactly because the force terms corresponding to these enter the final equations through the reaction Z (see eqs. 6) acting on the steering assembly.

¹⁰ See Bourlet, page 91.

