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1898-99

Write the first a, a_{m-1}, as unity, then the equations are

$$2.(4m-3)\,a_{m-2} = \{3\,(2m-1)^2 e_2 - B'\},$$

$$4.(4m-5)\,a_{m-3} = \{3\,(2m-3)^2 e_2 - B'\}\,a_{m-2},$$

....

$$(2m-2)\,(2m+1)\,a_0 = \{3.3^2 e_2 - B'\}\,a_1,$$

$$2m\,(2m-1)\,a_{-1} = \{3.1^2 e_2 - B'\}\,a_0,$$

but a_{-1} must be zero, therefore, multiplying the series of equations together,

$$O = (3.1^2.e_2 - B')\,(3.3^2.e_2 - B')\ldots\{3\,(2m-1)^2 e_2 - B'\},$$

therefore

$$B' = 3.1^2.e_2, \text{ or } 3.3^2.e_2, \ldots, \text{ or } 3.(2m-1)^2 e_2.$$

For these values in turn we get 0, 1, 2, ..., or $(m-1)$ a's in U_n', reckoning from a_0, zero, *i.e.* $U_n' = 0$, as an equation in $pu - e_2$, has then 0, 1, 2, ..., or $m-1$ zero roots, therefore when $e_1 = e_2$, and therefore when $e_1 \neq e_2$, for each $U_n/\sqrt{(pu - e_1)}$ of this type, we have a different arrangement in point of number of roots in the two places, between e_1 and e_2, and between e_2 and e_3.

Similar proofs will hold for U's of the other types.

THE STABILITY OF THE MOTION OF A BICYCLE.

By F. J. W. Whipple, B.A., Scholar of Trinity College, Cambridge.

THE Dynamics of a Bicycle is a subject which has not attracted much attention in this country. One branch alone has been studied, which may be called the Energetics of Velocipedes. Numerous experiments have been made to determine the work expended by the rider in overcoming the resistance of the air and of gravity on machines fitted with every variety of driving gear and tyre. The motion of the bicycle, as a machine on two wheels, has not received the same consideration. No satisfactory explanation on mathematical lines has been given of the practicability and ease of riding a bicycle. The instructions given to beginners in the art contain the information that, if the front wheel be turned

towards the side on which the rider is falling, centrifugal force will restore him to the normal position. An explanation is required of the ease with which the correct inclination is given to the handles, even when the rider's attention is devoted to other objects. There may be a certain amount of reflex action which enables the rider to balance himself without conscious attention, but such an action probably only comes into play after prolonged practice. To balance a stick on one's finger is an easy feat, but it would be a long while before a novice could do it automatically.

2. M. Bourlet, in his treatise on the Bicycle,* attacks most of the problems of the subject; his conditions are usually so artificial that the results have little interest. Thus (pp. 47–58), in discussing the restoration of equilibrium, he supposes that the rider turns the handles with a constant angular velocity for a certain time, and then holds them steady until the back frame is vertical. According to M. Bourlet's equations, the rider returns to the vertical position with a greater velocity than that with which he left it, so that a rider who obeys the given intructions will find the motion of his bicycle unstable. M. Bourlet's discussion of riding with hands off is based on a complicated differential equation, which he does not publish in his treatise. He decides that the efforts of the rider in balancing are continuous, and that a sudden movement of the body produces the opposite effect to a gradual movement.

3. The only other author I know of who has discussed the subject is Mr. McGaw,† who finds the condition of steady motion of the front wheel when the back-frame is vertical. This investigation has some interest in connection with the motion of a tricycle [see §§ 11, 23].

4. In the following pages I have discussed:—

The most general motion of a bicycle, §§ 5–9.

Motion in which the inclination of the wheels to the vertical is small, § 10.

The steady motion in a circle of large radius, § 11.

The oscillations about steady motion when the rider retains his fixed position on the machine and does not use the handles, §§ 12–16.

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* C. Bourlet, *Traité des Bicycles et Bicyclettes Gauthier-Villars.*

† *Engineer*, December 9, 1898.

will contain T_{1,τ^2} and therefore $T_{1,\alpha}$, where α is an arbitrary mark in the $GF[2^n]$. Further, $R_{1,2,\lambda}$ transforms $T_{1,\alpha}$ into $R_{1,2,\lambda(1+\alpha)}T_{1,\alpha}$. Hence, if $n > 1$, so that the $GF[2^n]$ contains a mark α neither zero nor unity, the group I contains a substitution $R_{1,2,\lambda(1+\alpha)}$ not the identity. But M_1M_2 transforms $R_{1,2,1}$ into $N_{1,2,1}$. With $N_{1,2,1}$, I contains M_i, $L_{i,1}$ $(i=1, 2)$, since it contained their products two at a time. Transforming $L_{i,1}$ and $N_{i,j,1}$ by $T_{i,\tau}$, we obtain L_{i,τ^2} and $N_{i,j,\tau}$ respectively. Hence I contains every $L_{i,\alpha}$ and $N_{i,j,\alpha}$. Finally I contains

$$M_iL_{i,\alpha}M_iL_{i,\alpha^{-1}}M_iL_{i,\alpha} = T_{1,\alpha}.$$

The invariant subgroup I therefore coincides with H, which is thus simple.

The simple group H has the order $2^{4n}(2^{4n}-1)(2^{2n}-1)$, which for $n=2$ and 3 is respectively

$$2^8.3^2.5^2.17 = 979,\ 200, \quad 2^{12}.3^4.7^2.5.13 = 1,\ 173,\ 836,\ 560.$$

Errata in the Paper, pp. 1–16.

p. 6, l. 19, *for* $Q_{2,3,\gamma_3}$ *read* $Q_{2,3,\alpha_3}$.

p. 14, l. 1, *omit* -1 *after* γ_2''.

p. 16, l. 3 of Errata, *for* γ_2' *read* γ_1'.

University of California,
January 6, 1899.

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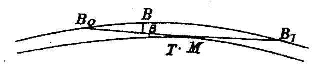
Fig. 4.



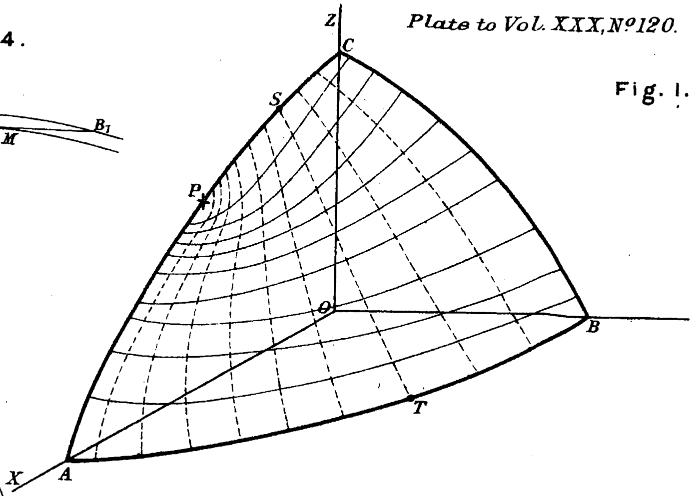


Fig. I.

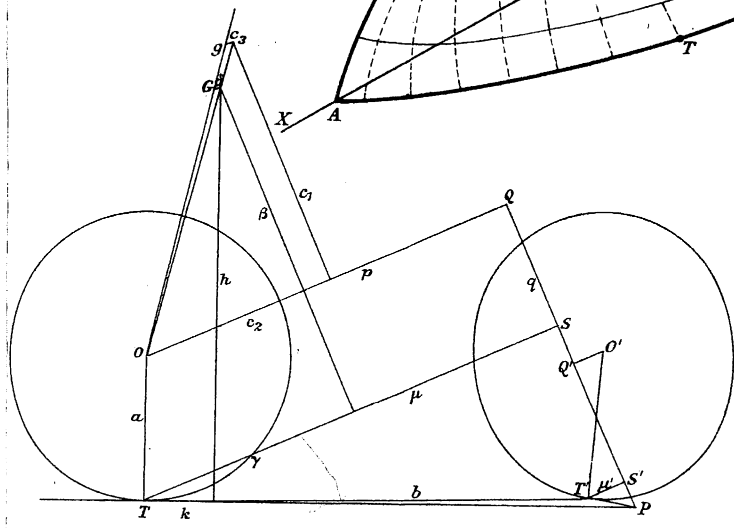


Fig. 2.

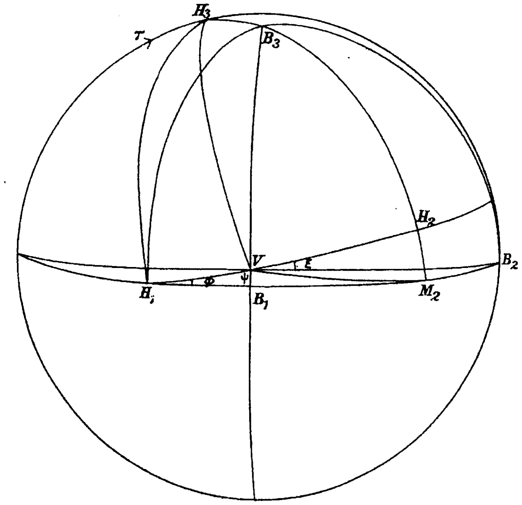


Fig. 3.

Use of handles in a restricted 'elastic' manner which may be automatic, § 17.

Movement of body in a similar automatic manner without the use of hands, § 18.

Notes on the effect of rolling and spinning friction are appended, §§ 20–22.

The general results are given in § 19.

5. For analytical purposes the bicycle consists of two frames, known as the front- and back-frames, hinged together along the 'head' QQ', in such a way that the only relative motion is round QQ' (fig. 2). The wheels are circular, and each is thought of as touching the ground at one point, the back wheel in T, the front in T'. The axle of the back wheel passes through O, and is at right angles to the plane OQQ'. OQ is perpendicular to QQ', and its length is denoted by p. The corresponding length for the front wheel is denoted by p'. The dashed notation is used throughout this paper for points or lengths in the front-frame. The symbol is usually defined in the case of the back-frame; the same symbol with a dash indicates the corresponding quantity in the front-frame.

$$a = \text{radius of back wheel,}$$

$$q = Q'Q,$$

$$q \text{ is odd, } i.e.\ q' = -q.$$

Fig. 3 indicates the angles between various directions.

V is the vertical.

B_1B_2 is the plane of the back wheel and frame, B_2 being horizontal.

B_3 is the direction of the axis of the back wheel.

$H_1H_2H_3$ is a system at right angles, of which H_1 is the direction of the head of machine.

H_3 is horizontal.

$H_1M_2B_3$ is a system at right angles.

$\theta = H_1V =$ angle head makes with vertical.

$\phi = VH_1B_1 =$ angle between the plane of the back wheel and the vertical plane through the head.

$$\eta = B_1H_1, \qquad \psi = VB_1,$$

$$\pi - \xi = H_1VB_3, \qquad \tfrac{1}{2}\pi - \chi = VM_2.$$

Of these angles only two are independent: taking θ and ϕ as those two, we have

$$\sin\psi = \sin\theta\sin\phi, \qquad \tan\xi = \cos\theta\tan\phi,$$

$$\tan\eta = \tan\theta\cos\phi, \qquad \sin\chi = \sin\theta\cos\phi.$$

Let the spin of H_3 about V be $(-\tau)$.

The spins of the system of axes $H_1M_2B_3$ may be denoted by $\theta_1\theta_2\theta_3$, where

$$\left.\begin{aligned} \theta_1 &= -\tau\cos\theta - \dot{\phi} \\ \theta_2 &= -\tau\sin\chi + \dot{\theta}\sin\phi \\ \theta_3 &= -\tau\sin\psi - \dot{\theta}\cos\phi \end{aligned}\right\}.$$

The first relation between the coordinates is the geometrical condition, that both wheels touch the same horizontal plane. Project $TOQQ'O'T'T$ on the vertical.

$$p\sin\eta - p'\sin\eta' + a - a' = q\cos\theta \ \ldots\ldots\ldots\ldots \text{I.}$$

The kinematical conditions indicate that the motion of the head QQ' is the same, whether determined by that of the back or that of the front wheel.

Using axes $H_1M_2B_3$, the spins of the back wheel are $\theta_1\theta_2\Omega$ and the vector TO is $(a\cos\eta,\ a\sin\eta,\ o)$. The conditions for rolling shew that the velocity of O is

$$\{-\Omega a\sin\eta,\ \Omega a\cos\eta,\ a(\theta_1\sin\eta - \theta_2\cos\eta)\}.$$

The velocity of Q relative to O is $(-\theta_3 p,\ o,\ \theta_1 p)$, therefore the velocity of Q in space is, when referred to axes $H_1M_2B_3$,

$$\{-\Omega a\sin\eta - \theta_3 p,\ \Omega a\cos\eta,\ \theta_1(p + a\sin\eta) - \theta_2 a\cos\eta\}.$$

Change to axes $H_1H_2H_3$, and write down the condition that the difference of the velocities of Q and Q' is due to the rotation of the vector (q, o, o) with spins $(-\tau\cos\theta, -\tau\sin\theta, -\dot{\theta})$ about these axes.

$$\left.\begin{aligned} &[-\Omega a\sin\eta - \theta_3 p] - [-\Omega' a'\sin\eta' - \theta_3' p'] = 0 \\ &[\Omega a\cos\eta\cos\phi + \{\theta_1(p + a\sin\eta) - \theta_2 a\cos\eta\}\sin\phi \\ &\quad - [\Omega' a'\cos\eta'\cos\phi' + \{\theta_1'(p' + a'\sin\eta') - \theta_2' a'\cos\eta'\}\sin\phi'] \\ &\qquad = -\dot{\theta} q \\ &[-\Omega a\cos\eta\sin\phi + \{\theta_1(p + a\sin\eta) - \theta_2 a\cos\eta\}\cos\phi] \\ &\quad - [\Omega' a'\cos\eta'\sin\phi' + \{\theta_1'(p' + a'\sin\eta') - \theta_2' a'\cos\eta'\}\cos\phi'] \\ &\qquad = \tau\sin\theta q \end{aligned}\right\} \ldots \text{II.}$$

Dynamics.

6. Consider the back wheel only and use axes $H_1M_2B_3$. Let the reaction of the ground be merely the force $F_1F_2F_3$. In making this assumption the couples due to rolling and spinning friction are neglected. These will be considered later. The reaction of the back-frame on the back wheel consists of a force $(-Q_1, -Q_2, -Q_3)$ through O, and a couple $(-L_1, -L_2, o.)$ Here we assume that no driving couple is applied to the wheel.

Let m be the mass of the wheel, B, B, B_3 its moments of inertia. The velocity of O is

$$-\Omega a\sin\eta,\ \ \Omega a\cos\eta,\ \ a(\theta_1\sin\eta-\theta_2\cos\eta).$$

Resolving the forces on the back wheel, and applying d'Alembert's principle, we find

$$\left.\begin{aligned}
&F_1-Q_1-mg\cos\theta=\\
&m.\left[-\frac{d}{dt}(\Omega a\sin\eta)-\theta_3(\Omega a\cos\eta)+\theta_2 a(\theta_1\sin\eta-\theta_2\cos\eta)\right]\\
&F_2-Q_2-mg\sin\chi=\\
&\quad m.\left[\frac{d}{dt}(\Omega a\cos\eta)-\theta_1 a(\theta_1\sin\eta-\theta_2\cos\eta)-\theta_2\Omega a\sin\eta\right]\\
&F_3-Q_3-mg\sin\psi=\\
&\quad m.\left[a\frac{d}{dt}(\theta_1\sin\eta-\theta_2\cos\eta)+(\theta_2\sin\eta+\theta_1\cos\eta)a\Omega\right]
\end{aligned}\right\}\text{III.}$$

The moments of momentum of the wheel about O are $B\theta_1$, $B\theta_2$, $B_3\Omega$. Take moments about O

$$\left.\begin{aligned}
-L_1-F_3a\sin\eta&=B\dot\theta_1-(B\theta_3-B_3\Omega)\theta_2\\
-L_2+F_3a\cos\eta&=B\dot\theta_2+(B\theta_3-B_3\Omega)\theta_1\\
-a(F_2\cos\eta-F_1a\sin\eta)&=B_3\dot\Omega
\end{aligned}\right\}\text{......IV.}$$

7. Now consider the motion of the frame. Using the same axes assume as constants of the rigid body formed by frame and rider:—Mass M. Moments and products of inertia about g, $A_1A_2A_3A_{23}A_{31}A_{12}$, Coordinates relative to O of g the centre of mass, $c_1c_2c_3$. Let the reaction of the front-frame on the back-frame consist of a force $P_1P_2P_3$ and a couple $C_1C_2C_3$. The reaction of the back wheel on the frame is the force $F_1F_2F_3$, and the couple L_1L_2o.

The velocity of g is

$$(-\Omega a\sin\eta-\theta_3c_2+\theta_2c_3),\ \ (\Omega a\cos\eta-\theta_1c_3+\theta_3c_1),$$
$$(a\sin\eta+c_2)\,\theta_1-(a\cos\eta+c_1)\,\theta_2.$$

The equations obtained by resolving parallel to the axes $H_1M_2B_3$ are

$$\left.\begin{aligned}
&Q_1+P_1-Mg\cos\theta=M\Big[\frac{d}{dt}(-\Omega a\sin\eta-\theta_3c_2+\theta_2c_3)\\
&-\theta_3(\Omega a\cos\eta-\theta_1c_3+\theta_3c_1)+\theta_2\{(a\sin\eta+c_2)\theta_1-(a\cos\eta+c_1)\theta_2\}\Big]\\
&Q_2+P_2-Mg\sin\chi=M\Big[\frac{d}{dt}(\Omega a\cos\eta-\theta_1c_3+\theta_3c_1)\\
&-\theta_1\{(a\sin\eta+c_2)\theta_1-(a\cos\eta+c_1)\theta_2\}+\theta_3(-\Omega a\sin\eta-\theta_3c_2+\theta_2c_3)\Big]\\
&Q_3+P_3-Mg\sin\psi=M\Big[\frac{d}{dt}\{(a\sin\eta+c_2)\theta_1-(a\cos\eta+c_1)\theta_2\}\\
&\quad+(\theta_2\sin\eta+\theta_1\cos\eta)\Omega a-\{(\theta_1^2+\theta_2^2)c_3-\theta_3(c_2\theta_2+c_1\theta_1)\}\Big]
\end{aligned}\right\}\text{V.}$$

The equations found by taking moments about g are

$$\left.\begin{aligned}
&L_1+C_1+(Q_2+P_2)c_3-(Q_3+P_3)c_2+P_3p\\
&\qquad=\frac{D}{Dt}[A_1\theta_1-A_{12}\theta_2-A_{13}\theta_3]\\
&L_2+C_2+(Q_3+P_3)c_1-(Q_1+P_1)c_3\\
&\qquad=\frac{D}{Dt}[A_2\theta_2-A_{23}\theta_3-A_{21}\theta_1]\\
&\quad C_3+(Q_1+P_1)c_2-(Q_2+P_2)c_1-P_1p\\
&\qquad=\frac{D}{Dt}[A_3\theta_3-A_{31}\theta_1-A_{32}\theta_2]
\end{aligned}\right\}\text{...VI,}$$

where the operator $\frac{D}{Dt}$, acting on the x component of a vector xyz, gives $\dot x-\theta_3y+\theta_2z$.

8. From the twelve equations, III. to VI., $F_1F_2F_3$, $Q_1Q_2Q_3$, L_1L_2 must be eliminated. The four resulting equations will

give the wrench, which must be exerted by the front-frame on the back-frame to produce the motion.

The first of these equations is that which would be obtained by taking moments about an axis in the direction H through T. The values of the moments of inertia about axes $H_1 M_2 B_3$ through T are

$$\begin{aligned}
\Gamma_1 &= A_1 + B + ma^2 \sin^2\eta + M.[(a \sin\eta + c_2)^2 + c_3^2],\\
\Gamma_2 &= A_2 + B + ma^2 \cos^2\eta + M.[(a \cos\eta + c_1)^2 + c_3^2],\\
\Gamma_3 &= A_3 + B_3 + ma^2 \qquad + M.[(a \sin\eta + c_2)^2 + (a \cos\eta + c_1)^2],\\
\Gamma_{23} &= A_{23} \qquad + M.c_3 (a \cos\eta + c_1),\\
\Gamma_{31} &= A_{31} \qquad + M.c_3 (a \sin\eta + c_2),\\
\Gamma_{12} &= A_{12} + ma^2 \sin\eta \cos\eta + M(a \cos\eta + c_1)(a \sin\eta + c_2).
\end{aligned}$$

Mass of back frame and wheel $= M + m = W$; and if G, the C. G. of frame and wheel have coordinates β, γ, δ when the axes are $H_1 M_2 B_3$ through T,

$$\begin{aligned}
W\beta &= a(M+m)\cos\eta + c_1 M,\\
W\gamma &= a(M+m)\sin\eta + c_2 M,\\
W\delta &= \qquad c_3 M.
\end{aligned}$$

N.B.—Γ_1 β, γ, etc. are all variables.

The equation is obtained by multiplying VI. (1) by 1, IV. (1) by 1, III. (3) by $a \sin\eta$, V. (2) by $-c_3$, VI. (3) by $(c_2 + a \sin\eta)$, and adding

$$\left.\begin{aligned}
&C_1 + (p + a \sin\eta) P_3 - [\gamma \sin\psi - \delta \sin\chi] g W\\
&\quad = \frac{D}{Dt}[\Gamma_1\theta_1 - \Gamma_{12}\theta_2 - \Gamma_{13}\theta_3] - \dot\eta\,[a \cos\eta\, (\gamma_1\theta_1 - \beta\theta_2)\, W]\\
&+ (\Omega - \theta_3)\,[B_3\theta_2 + (\theta_2 \sin\eta + \theta_1 \cos\eta)\, a\gamma W + \theta_3 \sin\eta\, a\delta W]\\
&\qquad + \frac{d}{dt}[(\Omega - \theta_3)(-Wa \cos\eta)]
\end{aligned}\right\}\ \text{VII.}$$

The second equation could be obtained by taking moments about an axis in the direction M_2 through T.

It is found by multiplying VI. (2) and IV. (2) by 1, III. (3) by $-a \cos\eta$, V. (3) by $-(\cos\eta + c_1)$, V. (1) by c_3, and adding

$$\left.\begin{aligned}
&C_2 - a \cos\eta P_3 + g(\beta \sin\psi - \delta \cos\theta)\, W\\
&\quad = \frac{D}{Dt}[\Gamma_2\theta_2 - \Gamma_{23}\theta_3 - \Gamma_{21}\theta_1] - \dot\eta\,[a \sin\eta\,(\gamma_1\theta_1 - \beta\theta_2)\, W]\\
&+ (\Omega - \theta_3)[-B_3\theta_1 - (\theta_2 \sin\eta + \theta_1 \cos\eta)\, a\beta W - \theta_3 \cos\eta\, a\delta W]\\
&\qquad - \delta \frac{d}{dt}[(\Omega - \theta_3)\, Wa \sin\eta]
\end{aligned}\right\}\ \text{VIII.}$$

The third equation could be obtained by taking moments about an axis in the direction B_3, through O, of the forcive on the frame.

The following expressions, for the moments of inertia of the frame about O, are required

$$E_1 = A_1 + M(c_2^2 + c_3^2), \quad E_{23} = A_{23} + Mc_2c_3,$$

etc. etc.

To obtain the equation multiply VI. (3) by 1, V. (1) by $(-c_2)$, V. (2) by c_1, and add

$$\left.\begin{aligned}
&C_3 - pP_1 + Mg(c_2 \cos\theta - c_1 \sin\chi) = \frac{D}{Dt}[E_3\theta_3 - E_{31}\theta_1 - E_{32}\theta_2]\\
&+ Ma\left[\frac{d}{dt}\{\Omega(c_2 \sin\eta + c_1 \cos\eta)\} + \theta_3\Omega\,(c_2 \cos\eta - c_1 \sin\eta)\right]
\end{aligned}\right\}\ \text{IX.}$$

Finally, multiply IV. (3) by $\frac{1}{a}$, III. (2) and V. (2) by $\cos\eta$, III. (1) and V. (1) by $-\sin\eta$, and add

$$\left.\begin{aligned}
&P_2 \cos\eta - P_1 \sin\eta = \frac{B_3}{a}\dot\Omega\\
&\qquad + Wa[\dot\Omega - (\theta_1^2 - \theta_2^2) \cos\eta \sin\eta + \theta_1\theta_2 \cos 2\eta]\\
&+ M\Big[\frac{d}{dt}\{-(\theta_1 \cos\eta + \theta_2 \sin\eta)\, c_3 + \theta_3 (c_1 \cos\eta + c_2 \sin\eta)\}\\
&\qquad - (\theta_1 \cos\eta + \theta_2 \sin\eta)(-\theta_2 c_1 + \theta_1 c_2)\\
&+ (\theta_3 + \dot\eta)\{-\theta_3 (c_2 \cos\eta - c_1 \sin\eta) + (\theta_2 \cos\eta - \theta_1 \sin\eta)\, c_3\}\Big]
\end{aligned}\right\}\ \text{X.}$$

9. The action of the front-frame on the back-frame, and the reaction of the back-frame on the front-frame, together form a system of forces in equilibrium.

Resolving in the directions H_1, H_2, H_3, we find

$$\left.\begin{array}{l} P_1 = -P_1' \\ (P_2 \cos\phi + P_3 \sin\phi) = -(P_2' \cos\phi' + P_3' \sin\phi') \equiv R_2 \\ (P_3 \cos\phi - P_2 \sin\phi) = -(P_3' \cos\phi' - P_2' \sin\phi') \equiv R_3 \end{array}\right\} \text{...XI.}$$

Take moments about straight lines in the directions H_1, H_2, H_3,

$$\left.\begin{array}{l} C_1 + C_1' = 0 \\ (C_2 \cos\phi + C_3 \sin\phi) + (C_2' \cos\phi' + C_3' \sin\phi') - qR_3 = 0 \\ (C_3 \cos\phi - C_2 \sin\phi) + (C_3' \cos\phi' - C_2 \sin\phi') + qR_2 = 0 \end{array}\right\} \text{...XII.}$$

10. All the equations required to determine the motion in any circumstances have now been obtained. In the general case the elimination of the quantities P_1, P_2, P_3; R_1, R_2, R_3; C_2, C_3 would be very cumbersome, and could not lead to an intelligible result.

Accordingly, the particular case in which ϕ and ϕ' are small will alone be considered.

In this case $\phi^2 : 1$ is neglected, and

$$\left.\begin{array}{r} \psi = \phi \sin\theta \\ \eta = \chi = \theta \\ \xi = \phi \cos\theta \end{array}\right\}, \quad \left.\begin{array}{l} \theta_1 = -\tau\cos\theta - \dot\phi \\ \theta_2 = -\tau\sin\theta \\ \theta_3 = -\tau\phi\sin\theta - \dot\theta \end{array}\right\}.$$

The geometrical condition I. reduces to

$$(p - p')\sin\theta + (a - a') = q\cos\theta \text{..............I.}$$

This equation shews that, when $\phi^2 : 1$ and $\phi'^2 : 1$ are neglected, θ is independent of ϕ, ϕ' and is constant. Hence $\dot\theta = 0$ to this order of approximation. II. (3) reduces to

$$\begin{aligned} &[-\Omega a \cos\theta.\phi + \Omega' a' \cos\theta.\phi'] \\ &\quad - [(p + a\sin\theta)(\tau\cos\theta + \dot\phi) - \tau a \sin\theta\cos\theta] \\ &\quad + [(p' + a'\sin\theta)(\tau\cos\theta + \dot\phi') - \tau a' \sin\theta\cos\theta] = \tau q \sin\theta. \end{aligned}$$

This equation shews that τ is of the same order as $\dot\phi$ and $\dot\phi'$. III. (1) and (2) now reduce to $\Omega a = \Omega' a' \equiv V$.

Write $\qquad (p + a\sin\theta) = \mu,$

$$(p - p')\cos\theta + q\sin\theta = b = (\mu - \mu')/\cos\theta, \text{ by I.}$$

II. (3) takes the simple form

$$\mu\dot\phi - \mu'\dot\phi' + V\cos\theta(\phi - \phi') = -\tau b \text{.........XIII.}$$

$b = TT'$ is the wheel-base of the bicycle.

$\mu = TR$ is the perpendicular from T on the head, and if P be the point in which the head cuts the ground,

$\mu/\cos\theta = TP$. I propose the name back-trail for this length, $\mu'/\cos\theta = T'P$ being the front-trail.

When $\phi^2 : 1$ is neglected, Γ_1, &c., β, γ, are constants, and to this order of approximation the dynamical equations are

$$\begin{aligned} C_1 + \mu P_3 = Wg.(\gamma\phi - \delta)\sin\theta - \Gamma_1(\ddot\phi + \dot\tau\cos\theta) + \Gamma_{12}\dot\tau\sin\theta \\ - V[mr\tau\sin\theta + W\gamma(\tau + \dot\phi\cos\theta)] - \dot V W\delta\cos\theta \text{...VII.,} \end{aligned}$$

where mr is written instead of B_3/a.

$$\begin{aligned} C_2 - a\cos\theta P_3 = -Wg.(\beta\phi\sin\theta - \delta\cos\theta) \\ - \Gamma_2\dot\tau\sin\theta + \Gamma_{12}(\dot\tau\cos\theta + \ddot\phi) \\ + V[mr(\dot\phi + \tau\cos\theta) + W\beta(\tau + \dot\phi\cos\theta)] - \dot V W\delta\sin\theta V \text{...III.} \end{aligned}$$

$$\begin{aligned} C_3 - pP_1 = -Wg.(\gamma\cos\theta - \beta\sin\theta) \\ + E_{13}(\tau\cos\theta + \dot\phi) + E_{23}(\tau\sin\theta) + M\dot V(c_2\sin\theta + c_1\cos\theta) \text{...IX.} \end{aligned}$$

$$P_2\cos\theta - P_1\sin\theta = \dot V[B_3/a^2 + W] + (\dot\tau + \ddot\phi\cos\theta)\, W\delta \text{...X.}$$

It will appear, in the course of the following analysis, that a solution of these equations in which ϕ is small is only possible if $\delta : \gamma$ is small. When this assumption is made VII. and VIII. shew that $C_1 C_2 P_3$ are small. Making this assumption, XI. and XII. take the forms

$$\left\{\begin{array}{l} P = -P_1,\ P_2 = -P_2' \\ P_3 - \phi P_2 = R_3 = -(P_3' - \phi' P_2') \end{array}\right. \text{...........XI.}$$

$$\left\{\begin{array}{l} C_1 = -C_1' \\ C_2 + \phi C_3 + C_2' + \phi' C_3' - qR_3 = 0 \text{.....XII.} \\ C_3 + C_3' + qP_2 = 0 \end{array}\right.$$

On adding X. and the corresponding equation for the front frame, it is seen that $\dot V$ is of the same order as $\phi\delta$. Neglecting small quantities of this order, it is possible to satisfy

$$P_1 = -P_1' = \Pi g\cos\theta,$$

$$P_2 = -P_2' = \Pi g\sin\theta.$$

IX. is now $\quad C_3 - p\Pi g \cos\theta = -Wg\kappa,$

and the corresponding equation is

$$C_3' + p'\Pi g \cos\theta = -\ W'g\kappa',$$

where h, κ, δ are the coordinates of G relative to T, when the axes are in directions $B_1 B_2 B_3$.

XII. (3) is $\quad C_3 + C_3' + q\Pi g \sin\theta = 0,$

and the last three equations give

$$b.\Pi = \kappa W + \kappa' W.$$

The remaining equations are VII. and VIII. with

$$\left.\begin{array}{l} P_3 - R_3 = \phi\Pi g \sin\theta \\ P_3' + R_3 = -\phi'\Pi g \sin\theta \\ C_2 + C_2' - qR_3 = -\phi C_3 - \phi' C_3' \\ C_1 + C_1' = 0 \end{array}\right\}.$$

One method of eliminating R_3 is suggested by the fact that, if moments be taken about TT' for the whole bicycle, neither internal nor external mechanical reactions appear. Accordingly we find the difference between the moment about an axis B_2 through T and an axis B_2' through T′ of the internal reactions.

The difference is

$$\begin{aligned}
&(C_2 - a\cos\theta\, P_3)\cos\theta - (C_1 + \mu P_3)\sin\theta \\
&\qquad + (C_2' - a'\cos\theta\, P_3')\cos\theta - (C_1' + \mu' P_3')\sin\theta \\
&= \cos\theta.[qR_3 - \phi C_3 - \phi' C_3'] \\
&\qquad - (\mu\sin\theta + a\cos^2\theta)(R_3 + \phi\Pi g\sin\theta) \\
&\qquad + (\mu'\sin\theta + a'\cos^2\theta)(R_3 + \phi'\Pi g\sin\theta) \\
&= -\phi\,[C_3\cos\theta + (\mu - p\cos^2\theta)\,\Pi g] \\
&\qquad + \phi'\,[-C_3'\cos\theta + (\mu' - p'\cos^2\theta)\,\Pi g] \\
&= -\phi\,[-Wg\kappa\cos\theta + (\mu/b)(\kappa W + \gamma' W')g] \\
&\qquad + \phi'\,[-W'g\kappa'\cos\theta + (\mu'/b)(\kappa W + \kappa' W')g] \\
&= (\phi' - \phi)\frac{\kappa W\mu' + \kappa' W'\mu}{b}g = (\phi' - \phi)\,\Lambda g.
\end{aligned}$$

Substitute in this equation from VII. and VIII., and write $I = (\Gamma_1 - \Gamma_2)\cos\theta\sin\theta + \Gamma_{12}\cos 2\theta$, for the product of inertia of the back-frame, wheel, and rider about axes $B_1 B_2$ through T.

$$\begin{aligned}
&\Gamma_1\sin\theta + \Gamma_{12}\cos\theta)\ddot\phi + (mr + Wh)\cos\theta V\dot\phi + (\Lambda - Wh\sin\theta)g\phi. \\
&\qquad + \text{the same expression with dashes} \\
&\qquad + (I + I')\dot\tau + (mr + Wh + m'r' + W'h')V\tau, \\
&\qquad + (W\delta + W'\delta')g = 0 \text{ ....XIV.}
\end{aligned}$$

The coefficient of ϕg in XIV. is

$$\begin{aligned}
&-(Wh\sin\theta - \Lambda) \\
&= -Wh\sin\theta + \frac{(\mu - b\cos\theta)\kappa W + \mu\kappa' W'}{b} \\
&= -(W\gamma - \mu\Pi).
\end{aligned}$$

XIV. is true whatever be the value of C_1, the couple exerted by the rider on the handles. The next equation depends on C_1; it is found from VII. with

$$P_3 + P_3' = (\phi - \phi')\,\Pi g\sin\theta.$$

$$\begin{aligned}
&\frac{1}{\mu}[\Gamma_1\ddot\phi + W\gamma\cos\theta V\phi - (W\gamma - \mu\Pi)\sin\theta g\phi] \\
&+ \frac{1}{\mu'}[\Gamma_1'\phi' + W'\gamma'\cos\theta V\phi' - (W'\gamma' + \mu'\Pi)\sin\theta g\phi'] \\
&+ \left[\frac{\Gamma_1\cos\theta - \Gamma_{12}\sin\theta}{\mu} + \frac{\Gamma_1'\cos\theta - \Gamma'_{12}\sin\theta}{\mu'}\right]\dot\tau + \left[\frac{mr\sin\theta + W\gamma}{\mu}\right. \\
&+ \left.\frac{m'r'\sin\theta + W'\gamma'}{\mu'}\right]V\tau + \left(\frac{W\delta}{\mu} + \frac{W'\delta'}{\mu'}\right)g\sin\theta = \left(\frac{1}{\mu'} - \frac{1}{\mu}\right)C_1. \\
&\qquad \text{...............XV.}
\end{aligned}$$

It will be convenient to re-write XIII. here:

$$\mu\dot\phi - \mu'\dot\phi' + V\cos\theta\,(\phi - \phi') = -\tau b \text{........XIII.}$$

The equations XIV., XV., XIII. are sufficient to determine the motion as long as ϕ and ϕ' are small.

Steady motion with hands off.

11. When the motion is steady, $\dot{\phi} = \dot{\phi}' = \dot{\tau} = 0$; and, if the machine trace a large circle of radius ρ,

$$\frac{1}{\rho} = \frac{\tau}{V}.$$

XIII. may now be written

$$\phi' - \phi = \frac{b}{\rho\cos\theta} \quad \text{..................XVI.}$$

XIV. when multiplied by $\sin\theta$ is

$$(mr\sin\theta + W\gamma + m'r'\sin\theta + W'\gamma' - \Pi b\cos\theta)\, V\tau$$
$$= g\sin\theta . [(W\gamma - \mu\Pi)\phi + (W'\gamma' + \mu'\Pi)\phi' - (W\delta + W'\delta')] \text{...XVII.}$$

XV. reduces, since $C_1 = 0$, to

$$\left(\frac{mr\sin\theta + W\gamma}{\mu} + \frac{m'r'\sin\theta + W'\gamma'}{\mu'}\right) V\tau$$
$$= g\sin\theta\left[\frac{W\gamma - \mu\Pi}{\mu}\phi + \frac{W'\gamma' + \mu'\Pi}{\mu'}\phi' - \left(\frac{W\delta}{\mu} + \frac{W'\delta'}{\mu'}\right)\right] \text{...XVIII.}$$

The only case of interest is that in which $\delta' = 0$.

Eliminate δ, by subtracting XVII. multiplied by $\frac{1}{\mu}$ from XVIII., and dividing out by $\left(\frac{1}{\mu'} - \frac{1}{\mu}\right) = \frac{b\cos\theta}{\mu\mu'}$,

$$[m'r'\sin\theta + W'\gamma' + \mu'\Pi]\, V\tau = g\sin\theta . (W'\gamma' + \mu'\Pi)\phi'.$$

$$\phi' = \frac{V^2}{\rho g\sin\theta} \cdot \left[1 + \frac{m'r'\sin\theta}{\mu'\Pi + W'\gamma'}\right] \text{.........XIX.}$$

From XVI. we have

$$\phi = -\frac{b}{\rho\cos\theta} + \frac{V^2}{\rho g\sin\theta}\left[1 + \frac{m'r'\sin\theta}{\mu'\Pi + W'\gamma'}\right] \text{......XX.}$$

The result of eliminating ϕ' from XVII. and XVIII. is

$$\sin\theta[(W\gamma - \mu\Pi)\phi - W\delta] = \frac{V^2}{\rho g}[mr\sin\theta + W\gamma - \mu\Pi],$$

therefore

$$W\delta = (W\gamma - \mu\Pi)\left[\left\{\frac{m'r'}{W'\gamma' + \mu'\Pi} - \frac{mr}{W\gamma - \mu\Pi}\right\}\frac{V^2}{g\rho} - \frac{b}{\rho\cos\theta}\right] \text{ XXI.}$$

Let $V = V_1$ be the solution of this equation when $\delta = 0$.

For steady motion, with hands off, at velocities less than V_1, the C. G. is displaced from the mean plane of the frame towards the centre of the circle described. For velocities greater than V_1 the C. G. is displaced in the opposite direction. For $V = V_1$, $\delta = 0$, whatever be the value of ρ. This indicates, that, for this particular velocity, the rider may sit symmetrically and describe a circle of any radius. When the motion is disturbed in such a way as to start him off on another circle, he will continue in that other circle. In other words, the stability of the steady motion for disturbances of this character is neutral. See § 16.

The motion in which the back-frame is vertical is given by $\phi = 0$,

$$\varepsilon \equiv \frac{bg}{V^2} = \cot\theta + \frac{m'r'\cos\theta}{\mu'\Pi + W'\gamma'} \text{ ....XXII.}$$

For the machine whose dimensions are given in § 14, XXII. gives $\varepsilon = \frac{bg}{V^2} = 2{\cdot}879$.

The steady motion for velocities of this order is shewn to be unstable in § 15.

This case is of interest, because it has been investigated by Mr. G. T. McGaw,* and used by him to determine the most suitable value for μ' when V is given. In his paper he does not state clearly that the criterion he adopts is the vertical position of the back-frame. As a solution of XXII. has been obtained, which denotes an unstable motion, the disadvantage of adopting the criterion is obvious. Owing to an unjustifiable application of the principle of virtual work, Mr. McGaw's result differs from XXII. principally by the omission of the term $\cot\theta$.

Small oscillations about steady motion.

12. Denote the values of ϕ, ϕ', and τ for steady motion by ϕ_0, ϕ_0', and τ_0 respectively, and assume

$$\phi = \phi_0 + \Sigma K e^{\lambda t},$$
$$\phi' = \phi_0' + \Sigma K' e^{\lambda t},$$
$$\tau = \tau_0 + \Sigma T e^{\lambda t}.$$

* *Engineer*, December 9th, 1898.

Substitute these values in XIV., XV., XIII., and pick out the terms in $e^{\lambda t}$ from each of these equations. The result of eliminating K, K', T from the expressions thus obtained is

$$\Delta = 0 \quad \text{...XXIII,}$$

where

$$\Delta \equiv \begin{vmatrix} [(\Gamma_1 \sin\theta + \Gamma_{12}\cos\theta)\lambda^2 & [(\Gamma' \sin\theta + \Gamma'_{12}\cos\theta)\lambda^2 & [(I+I')\lambda \\ \quad + (mr + Wh)\cos\theta V\lambda & \quad + (m'r' + W'h')\cos\theta V\lambda & \quad + (mr + Wh + m'r' + W'h')V] \\ \quad\quad -(W\gamma - \mu\Pi)g], & \quad\quad -(W'\gamma' + \mu'\Pi)g], & \\ \frac{1}{\mu}[\Gamma_1\lambda^2 + W\gamma\cos\theta V\lambda & \frac{1}{\mu'}[\Gamma_1'\lambda^2 + W'\gamma'\cos\theta V\lambda & \left[\left(\frac{\Gamma_1\cos\theta - \Gamma_{12}\sin\theta}{\mu} + \frac{\Gamma_1'\cos\theta - \Gamma'_{12}\sin\theta}{\mu'}\right)\lambda\right. \\ \quad -\sin\theta(W\gamma - \mu\Pi)g], & \quad -\sin\theta(W'\gamma' + \mu'\Pi)g], & \quad \left. + \left(\frac{mr\sin\theta + W'\gamma)}{\mu} + \frac{m'r'\sin\theta + W'\gamma'}{\mu'}\right)V\right] \\ \mu\lambda + V\cos\theta, & -(\mu'\lambda + V\cos\theta), & b \end{vmatrix}.$$

This is the equation by which the four values of λ are determined.

If the determinant in $\Delta = 0$ were developed, a quartic in λ would be found. The coefficients are very complicated in the general case, and the roots could not be found. In most cases the roots will not occur in pairs, and in favourable cases the roots will all have their real parts negative. The disturbed motion will in those cases be rapidly dissipated. This is not inconsistent with the conservation of energy, for when the kinetic energy of translation is increased by the addition of the energy of the oscillations, the increase in V is of the second order. Such changes of V have been ignored in the above analysis.

The coefficients of the higher powers of λ in Δ cannot be reduced to simple forms. The coefficients of gV^2, g^2, λVg, λV^3 are reduced here.

The coefficient of gV^2

$$= \begin{vmatrix} -(W\gamma - \mu\Pi) & , -(W'\gamma' + \mu'\Pi) & , \frac{W\gamma + mr\sin\theta + W'\gamma' + m'r'\sin\theta - \Pi b\cos b}{\sin\theta} \\ -\frac{W\gamma - \mu\Pi}{\mu}\sin\theta, & -\frac{(W'\gamma' + \mu'\Pi)}{\mu'}\sin\theta, & \frac{W\gamma + mr\sin\theta}{\mu} + \frac{W'\gamma' + m'r'\sin\theta}{\mu'} \\ \cos\theta, & -\cos\theta, & 0 \end{vmatrix},$$

$$= -\cos\theta . \left[(W\gamma - \mu\Pi)\left\{\left(\frac{1}{\mu'} - \frac{1}{\mu}\right)(W'\gamma' + m'r'\sin\theta) + \Pi b\cos\theta) + \frac{\Pi b\cos\theta}{\mu}\right\}\right.$$

$$\left. -(W'\gamma' + \mu'\Pi)\left\{\left(\frac{1}{\mu} - \frac{1}{\mu'}\right)(W\gamma + mr\sin\theta) - \frac{\Pi b\cos\theta}{\mu'}\right\}\right],$$

$$= -\frac{b\cos^2\theta}{\mu\mu'} . [(W\gamma - \mu\Pi)m'r'\sin\theta - (W'\gamma' + \mu'\Pi)mr\sin\theta] \quad \text{..................................XXIV.}$$

The coefficient of g^2 in Δ is

$$\frac{b^2\cos\theta\sin\theta}{\mu\mu'} . (W\gamma - \mu\Pi)(W'\gamma' + \mu'\Pi) \quad \text{..................................XXV.}$$

Let J_1 be the moment of inertia of back-frame and rider about the axis B_1 through T.

The coefficient of λVg is found, on writing

$$\Gamma_1\cos\theta-\Gamma_{12}\sin\theta=J_1\cos\theta+I\sin\theta,$$

to be

$$-\cos\theta\begin{vmatrix} W\gamma-\mu\Pi, & W'\gamma'+\mu'\Pi, & I+I' \\ \dfrac{W\gamma-\mu\Pi}{\mu}\sin\theta, & \dfrac{W'\gamma'+\mu'\Pi}{\mu'}\sin\theta, & \dfrac{J_1\cos\theta+I\sin\theta}{\mu}+\dfrac{J_1'\cos\theta+I'\sin\theta}{\mu'} \\ 1, & -1, & 0 \end{vmatrix}$$

$$-\sin\theta\begin{vmatrix} (mr+Wh)\cos\theta, & (m'r'+W'h')\cos\theta, & (mr+Wh+m'r'+W'h') \\ \dfrac{W\gamma-\mu\Pi}{\mu}, & \dfrac{W'\gamma'+\mu'\Pi}{\mu'}, & 0 \\ \mu, & -\mu', & b \end{vmatrix}$$

$$-\begin{vmatrix} (W\gamma-\mu\Pi), & W'\gamma'+\mu'\Pi, & 0 \\ \dfrac{W\gamma\cos\theta}{\mu}, & \dfrac{W'\gamma'\cos\theta}{\mu'}, & \dfrac{mr\sin\theta+W\gamma}{\mu}+\dfrac{m'r'\sin\theta+W'\gamma'}{\mu'} \\ \mu, & -\mu', & b \end{vmatrix}.$$

The term in mr is found to vanish in this expression, which accordingly

$$=(W\gamma-\mu\Pi)\left[-I'\frac{b\cos^2\theta\sin\theta}{\mu\mu'}-\left(\frac{J_1}{\mu}+\frac{J_1'}{\mu'}\right)\cos^2\theta+\left\{\frac{1}{\mu}\left(\frac{Wh}{\mu}+\frac{W'h'}{\mu'}\right)\sin\theta-\left(\frac{W\gamma}{\mu^2}+\frac{W'\gamma'}{\mu'^2}\right)\right\}\mu\mu'\right]$$

$$+(W'\gamma'+\mu'\Pi)\left[I\frac{b\cos^2\theta\sin\theta}{\mu\mu'}-\left(\frac{J_1}{\mu}+\frac{J_1'}{\mu'}\right)\cos^2\theta+\left\{\frac{1}{\mu'}\left(\frac{Wh}{\mu}+\frac{W'h'}{\mu'}\right)\sin\theta-\left(\frac{W\gamma}{\mu^2}+\frac{W'\gamma'}{\mu'^2}\right)\right\}\mu\mu'\right]$$

$$=-\sin\theta\cos^2\theta\left[\{-I(W'\gamma'+\mu'\Pi)+I'(W\gamma-\mu\Pi)\}\frac{b}{\mu\mu'}+\left(\frac{J_1}{\mu}+\frac{J_1'}{\mu'}\right)(Wh+W'h')+WW'.\left(\frac{h\gamma'}{\mu}-\frac{h'\gamma}{\mu}\right)\frac{b}{\cos\theta}\right] \quad \text{XXVI.}$$

The coefficient of λV^3 in Δ is

$$\begin{vmatrix} (mr+Wh)\cos\theta, & (m'r'+W'h')\cos\theta, & (mr+Wh)+(m'r'+W'h') \\ \dfrac{W\gamma}{\mu}\cos\theta, & \dfrac{W'\gamma'}{\mu'}\cos\theta, & \dfrac{mr\sin\theta+W\gamma}{\mu}+\dfrac{m'r'\sin\theta+W'\gamma'}{\mu'} \\ \cos\theta, & -\cos\theta, & 0 \end{vmatrix},$$

$$=\sin\theta\cos^2\theta.\left[\frac{mr}{\mu}+\frac{m'r'}{\mu'}\right][mr+Wh+m'r'+W'h'] \quad \text{.....................XXVII.}$$

13. We proceed to the discussion of a particular bicycle. It is worth noticing that μ, μ', θ, a can be found by measurement of the machine. The position of the C.G. of the rider can be estimated with fair accuracy. The values of the moments of inertia A_1, &c. are not required to be very accurate, as they are not the most important terms in Γ_1, &c.

As a numerical example, the following values of the quantities involved are taken. The unit of length is b, which may be thought of as one metre. The unit of mass is denoted by w, it may be thought of as 1 kgm. The numbers in brackets refer to the pages on which the symbols are introduced.

$$\tan\theta = \cdot 4, \qquad \theta = 21^\circ 48' \qquad (314)$$

$$\mu = 1\cdot 05 b \cos\theta, \qquad \mu' = \cdot 05 b \cos\theta \qquad (321),$$

$$a = \cdot 35 b, \qquad a' = \cdot 35 b \qquad (314),$$

$$\begin{cases} \kappa = \cdot 25 b, \\ h = 1\cdot 0 b, \end{cases} \qquad \begin{matrix} \kappa' = 0, \\ h' = \cdot 4 b \end{matrix} \qquad (322),$$

whence

$$\begin{cases} \gamma = \cdot 65 b \cos\theta, \\ \beta = \cdot 9 b \cos\theta, \end{cases} \qquad \begin{matrix} \gamma' = \cdot 16 b \cos\theta, \\ \beta' = \cdot 4 b \cos\theta \end{matrix} \qquad (318),$$

$$W = 80 w, \qquad W' = 2w \qquad (318),$$

$$A_1 + B = 5 w b^2, \qquad A_1' + B' = \cdot 075 w b^2 \qquad (316),$$

$$A_2 = 5 w b^2, \qquad A_2' = \cdot 075 w b^2,$$

$$A_{12} = 0, \qquad A_{12}' = 0,$$

$$mr = \cdot 5 w b, \qquad m'r' = \cdot 5 w b \qquad (321).$$

From these values the following can be deduced:

$$J_1 = 10 w b^2, \qquad J_1' = \cdot 075 w b^2 \qquad (327),$$

$$J_2 = 85 w b^2, \qquad J_2' = \cdot 395 w b^2,$$

$$I = 20 w b^2, \qquad I' = 0 \qquad (323);$$

$$\Gamma_1 \sin\theta + \Gamma_{12} \cos\theta = J_2 \sin\theta + I \cos\theta = 54 w b^2 \cos\theta,$$

$$\Gamma_1' \sin\theta + \Gamma_{12}' \cos\theta = \cdot 158 w b^2 \cos\theta,$$

$$\Gamma_1 \cos\theta - \Gamma_{12} \sin\theta = J_1 \cos\theta + I \sin\theta = 18 w b^2 \cos\theta,$$

$$\Gamma_1' \cos\theta - \Gamma_{12}' \sin\theta = \cdot 075 w b^2 \cos\theta,$$

$$\Gamma_1 = 99 w b^2 \cos\theta \sin\theta,$$

$$\Gamma_1' = \cdot 345 w b^2 \cos\theta \sin\theta,$$

$$\Pi = 20 w \qquad (322),$$

$$W\gamma - \mu\Pi = 31 w b \cos\theta, \qquad W'\gamma' + \mu'\Pi = 1\cdot 32 w b \cos\theta.$$

14. Take wb out from the first row, and $\frac{wb \sin\theta}{\mu}$ from the second row, $\cos\theta$ from the first two columns,

$$\Delta = \frac{w^2 b^2 \cos^2\theta \sin\theta}{\mu} \begin{vmatrix} [54\lambda^2 b + 80\cdot 5\lambda V - 31g], & [\cdot 158\lambda^2 b + 1\cdot 3\lambda V - 1\cdot 32g], & [20\lambda b + 81\cdot 8V] \\ [99\lambda^2 b + 130\lambda V - 31g], & [7\cdot 25\lambda^2 b + 16\cdot 8\lambda V - 27\cdot 7g], & [48\cdot 94\lambda b + 157\cdot 8V \\ 1\cdot 05\lambda b + V, & -(\cdot 05\lambda b + V), & b \end{vmatrix},$$

$$= \frac{w^2 b^2 \cos^2\theta \sin\theta}{\mu} [264\cdot 7\lambda^4 b^3 + 1071\cdot 5\lambda^3 V b^2 + 987\cdot 7\lambda^2 V^2 b - 1116\cdot 6\lambda^2 b^2 g + 899\cdot 8\lambda V^3$$

$$- 903\cdot 3\lambda b V g - 296\cdot 8 V^2 g + 818\cdot 4 b g^2].$$

Write

$$\frac{\lambda b}{V} = \zeta, \quad \frac{gb}{V^2} = \epsilon,$$

$$\Delta = 86\cdot 9 w^2 V^4 . [\zeta^4 + 4\cdot 05\zeta^3 + (3\cdot 73 - 4\cdot 6\epsilon)\zeta^2 + (3\cdot 40 - 3\cdot 41\epsilon)\zeta - (1\cdot 12\epsilon - 3\cdot 09\epsilon^2)] \text{......XXVIII.}$$

The steady motion is stable, if the equation $\Delta = 0$ has four roots, whose real parts are negative.

When this is the case the coefficients of powers of ζ are all positive. This condition is satisfied when ϵ lies between the limits $\frac{3\cdot 73}{4\cdot 6}$ and $\frac{1\cdot 12}{3\cdot 09}$, or $\cdot 81$ and $\cdot 363$, which will be denoted by ϵ_0 and ϵ_1 respectively.

For the upper of these limits $\Delta = 0$ reduces to

$$\zeta^4 + 4\cdot 05\zeta^3 + \cdot 64\zeta + 1\cdot 12 = 0,$$

which has roots

$$-4\cdot 076, \ -\cdot 6, \ \cdot 312 \pm \cdot 60i.$$

For $\epsilon = \epsilon_0$, $\Delta = 0$ reduces to

$$\zeta^4 + 4{\cdot}05\zeta^3 + 2{\cdot}06\zeta^2 + 2{\cdot}16\zeta = 0,$$

which has roots

$$-3{\cdot}65,\ 0,\ -{\cdot}2 \pm {\cdot}75i.$$

There must be some value of ϵ for which the real part of the complex roots changes sign. This value must make $\{4{\cdot}05\zeta^2 + (3{\cdot}40 - 3{\cdot}41\epsilon)\}$ a factor of Δ.

The two factors are

$$[\zeta^2 + 4{\cdot}05\zeta + (2{\cdot}89 - 3{\cdot}76\epsilon)]\,[\zeta^2 + ({\cdot}839 - {\cdot}84\epsilon)]$$

if the absolute term agrees, *i.e.* if

$$(2{\cdot}89 - 3{\cdot}76\epsilon)(839 - {\cdot}84\epsilon) = 3{\cdot}09\epsilon^2 - 1{\cdot}12\epsilon,$$

$${\cdot}068\epsilon^2 - 4{\cdot}46\epsilon + 2{\cdot}425 = 0.$$

The root lying between ϵ_0 and ϵ_1 is

$$\epsilon_2 = {\cdot}505.$$

For $\epsilon = \epsilon_2$ the factors of Δ are

$$(\zeta^2 + {\cdot}415)(\zeta^2 + 4{\cdot}05\zeta + {\cdot}99),$$

and the roots of $\Delta = 0$ are

$$-3{\cdot}8,\ -{\cdot}26,\ \pm {\cdot}64i.$$

For values of ϵ between ϵ_0 and ϵ_2 the roots have their real parts negative. Thus for $\epsilon = {\cdot}4$, the equation $\Delta = 0$ reduces to

$$\zeta^4 + 4{\cdot}05\zeta^3 + 1{\cdot}89\zeta^2 + 2{\cdot}04\zeta + {\cdot}046 = 0,$$

and the roots of this equation are

$$-3{\cdot}689,\ -{\cdot}023,\ -{\cdot}17 \pm {\cdot}716i.$$

15. The natural period of oscillation of the machine is comparable with the period of revolution of the pedals. If the gear be $2f$, the angular velocity of the cranks is $\frac{V}{f}$. The ratio, period of oscillation : period of revolution of cranks

$$= V/f : \text{imaginary part of } \lambda,$$

$$= 1 : (f/b) \times \text{imaginary part of } \zeta.$$

The ratio is approximately unity, when the gear $= 2f = 3b$.

The forced oscillations due to the lateral motion of the rider in pedalling have the same period as the revolution of the pedals, but the oscillations do not tend to become excessive, as there is a large damping effect at velocities greater than V_2.

16. In the case under consideration the coefficient of ϵ is negative, and it is owing to this fact that there is an upper limit to the velocity consistent with stable steady motion. The coefficient is small, and it is advisable to investigate whether it is necessarily negative for all machines.

$\epsilon_1 = -$ coefficient of ϵ : coefficient of ϵ^2 in XXVIII.

$= -$ coefficient of gV^2 : coefficient of g^2b in XXIII.

$$= \cos\theta \left[\frac{m'r'}{W'\gamma' + \mu'\Pi} - \frac{mr}{W\gamma - \mu\Pi} \right] \text{ by XXIV. and XXV.}$$

This gives the same value of ϵ_1, and therefore of V_1 as was obtained by writing $\delta = 0$ in XXI.

The velocity V_1 was previously shewn, to be one, for which the steady motion is neutral for some disturbances. This investigation shews that for velocities greater than V_1 the steady motion is unstable. By making the back wheel very heavy ϵ_1 might be made negative, and then there would be no superior limit to the velocity. Practically it is sufficient to make ϵ_1 small by increasing $\mu'\Pi$. It is interesting to notice that by leaning forward the rider increases κ, and therefore Π, so that the limiting velocity of steady motion is increased when he adopts a stoop.

The instability introduced is not very great. Thus, in the numerical example of § 13 the positive root tends when ϵ is small to $\zeta = \frac{1 \cdot 12}{3 \cdot 4}\epsilon$.

If a disturbance of the type corresponding to this root be set up, the time in which ϕ is multiplied by e bears to the period of revolution of the pedals the ratio

$$\frac{1}{\lambda} : \frac{2\pi f}{V} = 1 : 2\pi\zeta f/b.$$

Thus when $f = b$ and $\epsilon = \cdot 1$, so that the velocity is approximately 36 km./hr, ϕ is multiplied by e after 20 revolutions of the pedals. A tendency to fall of this character can be overcome by a very small motion of the rider's body relative to the machine.

The use of handles.

17. When the handles are in use, the rider exerts a couple on the front-frame. The alteration of position of the rider's body and arms is very slight, so that rider and frame may still be considered one rigid body. The only modification in the system of equations is due to $C_1 \neq 0$.

The most interesting problem in this connection is to discover whether the steady motion can be rendered stable by the introduction of a couple C_1, whose value depends only on the angle between the planes of the two wheels. To a first approximation such a couple may be written

$$C_1 = Zg(\phi - \phi')\,wb.$$

The sign is so chosen that, when Z is positive, the couple tends to reduce $(\phi - \phi')$.

When this value of C_1 is substituted in XV. the coefficient of ϕ on the left of that equation is increased by $-\dfrac{b\cos\theta}{\mu\mu'}.Zgwb.$

Let $\Delta + \Delta_1$ be the determinant derived by elimination of ϕ, &c., from XIV., III., and the modified form of XV.

$$\Delta_1 = \begin{vmatrix} [(J_2\sin\theta + I\cos\theta)\lambda^2 + (mr + \omega h)\cos\theta\nu\lambda + (\mu\pi - \omega\gamma)g], & [(J_2'\sin\theta + I'\cos\theta)\lambda^2 + (m'r' + \omega'h')\cos\theta\nu\lambda - (\mu'\pi + \omega'\gamma')g], & [(I+I')\lambda + \nu(mr + \omega h + m'r' + \omega'h')] \\ -Z\dfrac{wgb^2\cos\theta}{\mu\mu'}, & Z\dfrac{wgb^2\cos\theta}{\mu\mu'}, & 0 \\ \mu\lambda + \nu\cos\theta, & -(\mu'\lambda + \nu\cos\theta), & b \end{vmatrix}$$

$$= \frac{wZgb^3\cos\theta}{\mu\mu'}[\lambda^2(J_2 + J_2') - (\omega h + \omega'h')g]\sin\theta \quad\text{.........................XXIX.,}$$

where J_2 is the moment of inertia of back-frame and rider about the axis in direction B_2 through T.

In the case of the typical machine

$$J_2 + J_2' = 85{\cdot}4wb^3, \qquad Wh + W'h' = 80{\cdot}8wb^2,$$

$$\frac{wZgb^3\cos\theta\sin\theta}{\mu\mu'}\left(\frac{V}{b}\right)^2(J_2+J_2') = 7{\cdot}5Z\epsilon\left[264{\cdot}7\frac{\cos^2\theta\sin\theta}{\mu}w^2V^4b\right],$$

$$\frac{wZgb^3\cos\theta\sin\theta}{\mu\mu'}g(Wh+W'h') = 7{\cdot}1Z\epsilon^2\left[264{\cdot}7\frac{\cos^2\theta\sin\theta}{\mu}w^2V^4b\right],$$

therefore

$$\Delta + \Delta_1 \equiv 86{\cdot}9w^2V^4b.[\zeta^4 + 4{\cdot}05\zeta^3 + (3{\cdot}73 - 4{\cdot}6\epsilon - 7{\cdot}5Z\epsilon)\zeta^2 + (3{\cdot}4 - 3{\cdot}41\epsilon)\zeta - (1{\cdot}12\epsilon - 3{\cdot}09\epsilon^2 + 7{\cdot}1Z\epsilon^2)] = 0 \text{ ...XXX.}$$

In the first place, this equation shews that, if Z exceed $\dfrac{3{\cdot}09}{7{\cdot}1} \equiv Z_0$, XXX. has a positive root for every value of ϵ. Therefore, if the constraining couple exceed Z_0wbg, the motion is unstable. This is what we should expect, for under such circumstances the machine approximates to one with a locked head, which certainly cannot be ridden.

For velocities below the limit V_2, when $Z = 0$, two of the roots of XXX. are complex with their real part positive. If it is posible by choice of Z to make the real part negative, there must be a limiting value of Z for which the two roots are purely imaginary. In that case $[4{\cdot}05\zeta^2 + (3{\cdot}4 - 3{\cdot}41\epsilon)]$ is a factor of $\Delta + \Delta_1$, and $\zeta = \pm{\cdot}91\sqrt{(\epsilon_4 - \epsilon)}\,i$ gives the corresponding roots, if

$$\epsilon_4 = 3{\cdot}40/3{\cdot}41 = {\cdot}997.$$

The two factors of Δ are

$$[\zeta^2 + {\cdot}84(1-\epsilon)][\zeta^2 + 4{\cdot}05\zeta + (2{\cdot}89 - 3{\cdot}76\epsilon + 7{\cdot}5Z\epsilon)],$$

if the absolute term agrees on comparison with XXX.

This condition leads to

$${\cdot}84(1-\epsilon)(2{\cdot}89 - 3{\cdot}76\epsilon + 7{\cdot}5Z\epsilon) = 3{\cdot}09\epsilon^2 - 1{\cdot}12\epsilon - 7{\cdot}1Z\epsilon^2,$$

or

$$(6{\cdot}3\epsilon + {\cdot}8\epsilon^2)Z = -2{\cdot}43 + 4{\cdot}47\epsilon - {\cdot}07\epsilon^2.$$

Denote the solution of this equation by Z_1,

Z_1 is positive when $\epsilon_4 > \epsilon > \epsilon_5$,

Let Z_2 be the root of $7{\cdot}1Z\epsilon^2 = 3{\cdot}09\epsilon^2 - 1{\cdot}12\epsilon$.

$Z_1 < Z_2$, when $\epsilon = \epsilon_5$, and it can be shewn that this inequality holds for all values of $\epsilon < \epsilon_1$. For, if the inequality could be reversed for any value of ϵ, there would be an intermediate value of ϵ, for which $Z_1 = Z_2$. That value would be given by

$$\left.\begin{array}{l} 3{\cdot}09\epsilon^2 - 1{\cdot}12\epsilon - 7{\cdot}1Z\epsilon^2 = 0 \\ 3{\cdot}76\epsilon - 2{\cdot}89 - 7{\cdot}5Z\epsilon = 0 \end{array}\right\},$$

$$\epsilon = 3{\cdot}5.$$

This value of $\epsilon \not< \epsilon_4$, and therefore for all values of $\epsilon < \epsilon_4$, $Z_1 < Z_2$.

Further, when

$$\epsilon_4 > \epsilon > \epsilon_2, \qquad 0 < Z_1 < Z_2,$$
$$\epsilon_2 > \epsilon > \epsilon_1, \qquad Z_1 < 0 < Z_2,$$
$$\epsilon_1 > \epsilon > 0, \qquad Z_1 < Z_2 < 0.$$

In each of these cases XXX. has roots with their real parts negative, as long as Z lies between Z_1 and Z_2.

When $\epsilon_4 > \epsilon > \epsilon_2$, Z must be positive, and the constraining couple tends to restore the handles to their mean position. To obtain an idea of the magnitude of the couple write $\epsilon = {\cdot}8$,

$$Z_1 = {\cdot}2, \qquad Z_2 = {\cdot}45.$$

Let the distance between the handles be $2y$. When the angle between the front- and back-frames is $(\phi' - \phi)$ the displacement of either handle relative to the back-frame has a component $(\phi' - \phi)\,y$ in the direction H_2. If the rider exert the couple $Z(\phi' - \phi)\,bgw$ by means of equal and opposite forces in the direction M_2 on the handles, the magnitude of either force is $wgZ(\phi' - \phi).b/2y$, therefore the ratio, force on handles: displacement $= Zwgb/2y^2$. Taking $y = {\cdot}22b$ the restraining force on the handle must be between 20 and 45 grammes weight for every centimetre the handle is displaced.

When $\epsilon_2 > \epsilon > \epsilon_1$, no constraint is necessary to insure stability, as was shewn in the investigation of § 14. A small couple increasing or decreasing the displacement of the handles will not introduce instability.

When $\epsilon_1 > \epsilon > 0$, Z must be negative, and the rider must exert a couple tending to increase the angle between the two wheels.

Thus, when $\epsilon = {\cdot}1$, $V = {\cdot}36$ kilom./hr about,

$$Z_1 = -2{\cdot}8, \qquad Z_2 = -1{\cdot}1.$$

With the same assumptions as before it appears that the force on either handle must be about 2 kilogramme weight for every centimetre the handles are displaced.

For small values of ϵ, $Z_2 \propto -\dfrac{1}{\epsilon}$.

Therefore the difficulty of balancing by the method under investigation increases with the velocity. The value of ϵ_4 and V_4 can be found in the general case.

$$\epsilon_4 = -\text{coefficient of } \zeta : \text{coefficient of } \epsilon\zeta \text{ in XXVIII.}$$

$$= -\text{coefficient of } \lambda V^3 : \text{coefficient of } \lambda b V g \text{ in } \Delta$$

$$= \frac{\left(\dfrac{mr}{\mu} + \dfrac{m'r'}{\mu'}\right)(mr + Wh + m'r' + W'h')}{\left[\dfrac{-(W'\gamma' + \mu'\Pi)\,I + (W\gamma - \mu\Pi)\,I'}{\mu\mu'}\right] + \left(\dfrac{J_1}{\mu} + \dfrac{J_1'}{\mu'}\right)\dfrac{Wh + W'h'}{b} + \left(\dfrac{h\gamma'}{\mu} - \dfrac{h'\gamma}{\mu'}\right)\dfrac{WW'}{\cos\theta}}$$

by the results XXVI. and XXVII.

Balance.

18. When riding with hands off the handles, the rider steers the machine by a lateral movement of his body. In the above analysis it was assumed that c_3 and therefore δ were constants. When variations of δ are considered it is found that additional terms must be added to the various equations.

$M\ddot{c}_3$ or $W\ddot{\delta}$ to the right-hand side of V. (3),

$(c_3 + a\sin\theta)\,W\ddot{\delta}$ " " VII.,

$-(c_1 + a\cos\theta)\,W\ddot{\delta}$ " " VIII.,

$-(c_1\cos\theta + c_2\sin\theta + a)\,W\ddot{\delta}$ to the left-hand side of XIV.,

$-\dfrac{c_2 + a\sin\theta}{\mu}W\ddot{\delta}$ " " XV.

Neglecting $m:M$, the terms in δ and $\ddot{\delta}$ on the left of XIV. and XV. are respectively

$$[-h\ddot{\delta}+g\delta]\,W \text{ and } [-\gamma\ddot{\delta}+g\delta\sin\theta]\frac{W}{\mu},$$

If the rider sways his body automatically, the movement will probably depend on that of the back-frame, with which he is physically connected. In this case $\delta = A\phi$ may be taken as the relation between δ and ϕ. On this assumption the eliminant of XIV., XV., XIII. becomes $\Delta+\Delta'=0$, where

$$\Delta' = W.A. \begin{vmatrix} (-h\lambda^2+g) & [(\Gamma_1'\sin\theta+\Gamma'_{12}\cos\theta)\lambda^2 + (m'r'+W'h')\cos\theta V\lambda - (\mu'\Pi+W'\gamma')g], & [(I+I')\lambda + V(mr+Wh+m'r'+W'h')] \\ \dfrac{-\gamma\lambda^2+g\sin\theta}{\mu}, & \dfrac{1}{\mu'}[\Gamma_1'\lambda^2+W'\gamma'\cos\theta V\lambda - (\mu'\Pi+W'\gamma')g\sin\theta], & \left[\left(\dfrac{\Gamma_1\cos\theta-\Gamma_{12}\sin\theta}{\mu} + \dfrac{\Gamma_1'\cos\theta-\Gamma'_{12}\sin\theta}{\mu'}\right)\lambda + \left(\dfrac{mr\sin\theta+W\gamma}{\mu} + \dfrac{m'r'\sin\theta+W'\gamma'}{\mu'}\right)V\right] \\ 0 & -(\mu'\lambda+V\cos\theta), & b \end{vmatrix}.$$

This expression cannot be simplified in the general case. In the case of the machine previously discussed

$$\Delta' = -AW\frac{\cos\theta\sin\theta}{\mu}wb.\begin{vmatrix} \lambda^2 b-g, & \cdot158\lambda^2 b+1\cdot3\lambda V-1\cdot32g, & 20\lambda b+81\cdot8V \\ 1\cdot625\lambda^2 b-g, & 7\cdot25\lambda^2 b+16\cdot8\lambda V-27\cdot7g, & 48\cdot94\lambda b+157\cdot8V \\ 0, & -(\cdot05\lambda b+V), & b \end{vmatrix}$$

$$= -AW\frac{\cos\theta\sin\theta}{\mu}wb.[7\cdot82\lambda^4b^3+32\cdot3\lambda^3b^2V+25\lambda^2bV^2-34\cdot1\lambda^2b^2g-48\cdot2\lambda bVg-76gV^2+26\cdot4g^2b]$$

$$= -600\frac{A\sin\theta}{b^2}w^2V^4b[\zeta^4+4\cdot13\zeta^3+3\cdot2\zeta^2-4\cdot3\epsilon\zeta^2-6\cdot2\epsilon\zeta-9\cdot7\epsilon+3\cdot4\epsilon^2].$$

The leading terms in this expression are nearly the same as those in Δ. Write

$$\frac{600A\sin\theta}{86\cdot9b-600A\sin\theta}=\chi.$$

$$\frac{\Delta+\Delta'}{86\cdot9b^2-600A\sin\theta} \equiv \frac{w^2V^4}{b}[\{\zeta^4+4\cdot05\zeta^3+(3\cdot73-4\cdot6\epsilon)\zeta^2+(3.40-3.41\epsilon)\zeta-(1\cdot12\epsilon-3\cdot09\epsilon^2)\}$$

$$+\chi\{-\cdot08\zeta^3+(\cdot5-\cdot3\epsilon)\zeta^2+(3\cdot4+2\cdot8\epsilon)\zeta+(8\cdot6\epsilon-\cdot3\epsilon^2)\}\ldots\text{XXXII.}$$

When the motion is stable, the absolute term in this expression must be positive. Therefore $\chi>\chi_1$, where

$$(8\cdot6-\cdot3\epsilon)\chi_1=(1\cdot12-3\cdot09\epsilon)\ \ldots\ldots\ldots\ldots\ldots\ldots\text{XXXIII.}$$

The other limit to the value of χ is that at which the real part of the complex roots changes sign.

For that value $\Delta + \Delta'$ has factors

$$[\zeta^2 + \{\cdot 84 (1 - \epsilon) + (\cdot 84 + \cdot 69\epsilon)\chi\}]$$
$$\times [\zeta^2 + 4\cdot 1\zeta + \{(2\cdot 89 - 3\cdot 76\epsilon) - (\cdot 3 + \epsilon)\chi\}],$$

and the value of χ is given by the condition that the absolute terms in this expression and XXXII. agree.

$$(\cdot 86\epsilon - \cdot 3\epsilon^2)\chi - (1\cdot 12\epsilon - 3\cdot 09\epsilon^2)$$
$$= [\cdot 84 (1 - \epsilon) + (\cdot 84 + \cdot 69\epsilon)\chi][(2\cdot 89 - 3\cdot 76\epsilon) - (\cdot 3 + \epsilon)\chi],$$
$$\chi^2(\cdot 69\epsilon^2 + 1\cdot 05\epsilon + \cdot 25) + \chi(2\cdot 4\epsilon^2 + 10\cdot 4\epsilon - 2\cdot 2)$$
$$+ (-\cdot 07\epsilon^2 + 4\cdot 47\epsilon - 2\cdot 43) = 0 \ldots \text{XXXIV.}$$

When $\epsilon < \epsilon_2$ the absolute term in this equation is negative, and in that case the equation has a positive root χ_1.

When $\epsilon = \epsilon_2$, one of the roots of XXXIV. vanishes and is greater than χ_2. As ϵ changes, the root χ_1 remains greater than χ_2, until the value of ϵ is reached, for which $\chi_1 = \chi_2$. In that case, the same value of χ is given by the two equations

$$(8\cdot 6\epsilon - \cdot 3\epsilon^2)\chi - (1\cdot 12\epsilon - 3\cdot 09\epsilon^2) = 0,$$

and
$$(\cdot 84\epsilon + \cdot 69\epsilon)\chi + (\cdot 84 - \cdot 84\epsilon) = 0.$$

The result of eliminating χ from these two equations is

$$2\cdot 23\epsilon^2 + 11\cdot 1\epsilon - 9\cdot 7 = 0.$$

The positive root of this equation is $\cdot 75 = \epsilon_3$. To insure stability of motion χ must lie between χ_1 and χ_2.

Now, if

$$\epsilon_3 > \epsilon > \epsilon_2, \qquad 0 > \chi_1 > \chi_2,$$
$$\epsilon_2 > \epsilon > \epsilon_1, \qquad \chi_1 > 0 > \chi_2,$$
$$\epsilon_1 > \epsilon > 0, \qquad \chi_1 > \chi_2 > 0;$$

When $\epsilon_3 > \epsilon > \epsilon_2$, χ is negative; therefore δ/ϕ is negative, and the rider moves his body in the same direction as he is falling.

The value of χ_1 when $\epsilon = \cdot 6$, is given by XXXIV.,

$$1\cdot 13\chi^2 + 5\chi + \cdot 23,$$
$$\chi_1 = -\cdot 046, \quad A\phi = \delta = -\cdot 046b\phi \sin\theta.$$

For the same value of ϵ, XXXIII. leads to

$$\chi_2 = -\cdot 087, \quad A\phi = \delta = -\cdot 084b\phi \sin\theta.$$

Now the displacement of the C.G. of the rider's body due to the swaying of the frame is $h\phi \sin\theta$. Therefore the ratio of the rider's movement relative to the machine to his lateral movement with the machine is for this velocity (about 16 kilom./hr) between ·046 and ·084.

When $\epsilon_2 > \epsilon > \epsilon_1$, the rider need not move his body, but a small movement is consistent with stability.

When $\epsilon_1 > \epsilon > 0$, the rider must move his body in the opposite direction to that in which the frame is falling.

Thus, when $\epsilon = \cdot 1$, XXXIV. reads

$$\cdot 46\chi^2 - 1\cdot 14\chi - 1\cdot 98 = 0.$$

The solution of this equation is

$$\chi_1 = 4\cdot 46 \text{ corresponding to } \delta = \cdot 86b\phi \sin\theta;$$

XXXIII. leads to

$$\chi_2 = \cdot 095 \quad \text{,,} \quad \text{,,} \quad \delta = \cdot 09b\phi \sin\theta.$$

This indicates that for the velocity given by $\epsilon = \cdot 1$ [about 36 kilom./hr.] the rider can keep his balance by moving his C.G. in such a way that its displacement from the vertical plane through the wheel base is one-tenth of what it would be if he were fixed rigidly to the machine.

19. The foregoing investigation has shewn that there are four critical velocities for a bicycle. For velocities greater than V_1 the motion is unstable, but may be rendered stable by a rider who turns the front wheel towards the side on which he is falling, or who moves his body away from that side. The force he has to exert in the former operation is comparatively great, whereas the distance he has to move his body in the latter case is small.

For velocities betwen V_1 and V_2 the motion is stable, even when the rider does not move his body and makes no use of the handles. For velocities less than V_2 the motion without hands is unstable, but between V_2 and V_3 it is stable for a rider who moves his body through a very small distance in the same direction as the fall is carrying him. This distance is about 1/20 of the distance he is moved by the swaying of the machine. For velocities between V_3 and V_4 the motion is stable for a rider who keeps the motion of the handles as small as possible.

For velocities below V_4 a rider who combines the two methods, and uses both his weight and his hands, may be successful. The balance at such low velocities is not automatic, but is a feat which requires conscious attention.

The values of V_1 and V_4 can be found from the linear equations obtained in §§ 16, 17. V_2 and V_3 are found by solving quadratic equations with very complicated coefficients. For similar machines the values of these V's vary as the square root of the dimensions. For the machine discussed, the values obtained by taking

$$g = 9{\cdot}81 \text{ m./sec}^2, \quad b = 1{\cdot}1m$$

are

$\epsilon_1 = {\cdot}363$, $V_1 = 5{\cdot}45$ m./sec. = 19·6 km./hr. = 12·2 miles/hr.,
$\epsilon_2 = {\cdot}505$, $V_2 = 4{\cdot}62$ m./sec. = 16·6 „ = 10·4 „
$\epsilon_3 = {\cdot}75$, $V_3 = 3{\cdot}79$ m./sec. = 13·6 „ = 8·5 „
$\epsilon_4 = {\cdot}997$, $V_4 = 3{\cdot}29$ m./sec. = 11·8 „ = 7·4 „

In practice it is found to be quite easy to ride with hands off at all velocities above a limit, which is fairly definite for each machine. The ease of balance does not increase when the velocity increases beyond that limit.

The rider does not find it necessary to give his body a continuous lateral movement. He occasionally finds that the machine, as a whole, is gradually bearing to one side, and that this motion is independent of the oscillations of the handles. A small movement of his body is sufficient to restore the normal position.

Comparison with theory suggests that V_2 is the one limiting velocity, of which the existence is indicated by practice. At velocities less than V_2 the oscillations of the front-frame about the head tend to increase in amplitude. To overcome this tendency by a motion of his body, the rider would be compelled to obey the elastic law of § 18. In practice this would be very difficult. It appears that V_2 is the most important of the limiting velocities. Unfortunately the calculation of V_2 for any given machine is by no means easy.

The fact that corners can be turned by a rider who does not use his hands shews that the oscillations of the front-frame about the head are rapidly damped. Whilst the bicycle is describing the curve the handles are at a considerable distance from the symmetrical position. When the rider resumes his erect posture the handle-bar returns to the symmetrical position and shews no tendency to oscillate violently about that position. These observations are in accordance with theory, which shews that when the velocity exceeds V_2 the logarithmic decrement for oscillations of this type is large.

Spinning Friction.

20. Spinning friction tends to prevent the wheel turning about the normal through the point of contact with the ground. The simplest hypothesis is that the tyre possesses lateral rigidity, but that it is so far compressible that a small area A is in contact with the ground. The angular velocity of every element of A about T is the same. The friction is limiting, and therefore the couple about the normal at T is

$$U = \nu N g \varpi,$$

where ν is the coefficient of friction, Ng is the normal reaction, ϖ is the mean distance from T of the points of A.

The magnitude of this couple is independent of the velocity and spin of the wheel, and its sense is always opposed to that of the spin. When the spin is liable to a change of sign the motion can not be investigated by the methods of this paper. The only case of interest of which this is not true is that of the steady motion. Consider the steady motion with $C_1 = 0$, $\delta = 0$. VII., which is the equation of moments about an axis in the direction H_1 through T, is

$$\mu P_3 + U\cos\theta = Wg\gamma\phi \sin\theta - V(mr\sin\theta + W\gamma)\tau.$$

With the corresponding equation this leads to the modified form of XVIII.

$$\left(\frac{U}{\mu} + \frac{U'}{\mu'}\right)\cos\theta + \left(\frac{mr\sin\theta + W\gamma}{\mu} + \frac{m'r'\sin\theta + W'\gamma'}{\mu'}\right)\frac{V^2}{\rho}$$
$$= g\sin\theta\left(\frac{W\gamma - \mu\Pi}{\mu}\phi + \frac{W'\gamma' + \mu'\Pi}{\mu'}\phi'\right).$$

The form of XVII. is unaltered.

$$[mr\sin\theta + W\gamma + m'r'\sin\theta + W'\gamma' - \Pi b\cos\theta]\frac{V^2}{\rho}$$
$$= g\sin\theta[(W\gamma - \mu\Pi)\phi + (W'\gamma' + \mu'\Pi)\phi'].$$

Solve these equations for ϕ and ϕ'.

$$\left(\frac{U}{\mu} + \frac{U'}{\mu'}\right)\frac{\mu\mu'}{b} + (m'r' + W'\gamma' + \mu'\Pi)\frac{V^2}{\rho} = g\sin\theta(W'\gamma' + \mu'\Pi)\phi',$$

$$-\left(\frac{U}{\mu} + \frac{U'}{\mu'}\right)\frac{\mu\mu'}{b} + (mr + W\gamma - \mu\Pi)\frac{V^2}{\rho} = g\sin\theta(W\gamma - \mu\Pi)\phi,$$

and, since $\phi' - \phi = \dfrac{b}{\rho \cos\theta}$,

$$\left(\frac{m'r'}{W'\gamma' + \mu'\Pi} - \frac{mr}{W\gamma - \mu\Pi}\right)\frac{V^2}{g\rho}$$

$$= \frac{b}{\rho\cos\theta} - \left(\frac{U}{\mu} + \frac{U'}{\mu'}\right)\frac{\mu\mu'}{gb}\left(\frac{1}{W'\gamma' + \mu'\Pi} + \frac{1}{W\gamma - \mu\Pi}\right),$$

This equation determines the amount by which the critical velocity is reduced by spinning friction.

21. This theory is probably correct when the spin is comparable with the roll Ω of the wheel. When the spin is small, the fact that the lateral rigidity of the tyre is finite is of importance. This suggests the hypothesis that there is no slipping, and that the so-called spinning friction is due entirely to the lateral rigidity of the tyre.

Consider the elements of the tyre, which naturally lies on a circle S, drawn on the surface at a given distance from the central line (fig. 4). Denote by β the position which one of these elements would occupy if the tyre were not strained. Let B be the position it actually occupies. Let M be the point with which β coincides when nearest to T. M is the point of contact with the ground of the circle S.

The tangents at M to the locus of M and to the circle S coincide. Let B_1 be the point of the tyre, lying on the circle S, which has just come into contact with the ground. The tyre is not appreciably strained at B_1; therefore B_1 lies on the tangent at M. The locus of B is determined by the condition:—The tangent from B to the locus of M is constant. If the tyre be perfectly resilient, the elastic force on the element of the tyre at B is proportional to $B\beta$. When the curvature of the path of M is uniform, the locus of B is symmetrical about the normal at M. In that case the moment about the vertical through T of the elastic forces vanishes. When the curvature of the locus of M varies, the couple is approximately proportional to the variation of curvature in the time during which an element of tyre is on the ground.

The curvature of the locus of M is $\dfrac{\tau + \dot\phi\cos\theta}{V}$, and, on the hypothesis under consideration, the couple of spinning friction is small and proportional to $\dfrac{\dot\tau + \ddot\phi\cos\theta}{V^2}$.

The value of this couple is proportional to the fifth power of the linear dimensions of the area of contact. Assuming that the coefficient of elasticity is the same, there is much less spinning friction in the case of a well-inflated tyre than with a flat one. When the velocity of the machine is greater than V_2, any force which prevents the front wheel answering to the inclination of the back-frame is disadvantageous. It follows that in practice a well-inflated tyre is conducive to stability.

Rolling Friction, &c.

22. In obtaining the above results, rolling friction and wind resistance, which are necessarily accompanied by pedalling, have been neglected. Apart from the lateral motion in pedalling, which sets up transverse oscillations of the type hitherto considered, the additional features of the system are

(1) A couple $(-L_3)$ applied to the back wheel, and reacting on the rider and frame.

(2) A couple about an axis through T' in the direction B_3.

(3) Additional forces acting in the direction B_3 on all parts of the system. The centres of pressure will not in general coincide with the C.G.'s of the several portions of the system.

The value of $(-L_3)$ is easily obtained from the equation of work. When the motion of translation is steady, the only effect of the new circumstances on the transverse oscillations is due to the small alterations in C_2 and Π, and is therefore almost negligible.

When the rider does not pedal evenly and continuously $(-L_3)$ varies, and therefore V varies. In this case XIV. and XV. are true, but the assumption $\phi \propto e^{\lambda t}$ is not justifiable. If $V = V_0 + A\sin\left(\dfrac{V_0}{f}t\right)$ be taken as the approximate value of V, the problem resembles that of the oscillations of a string under variable tension. In that case Melde found that if the tension vary in a period half one of the natural periods the oscillation becomes excessive. The bicycle has no pure natural periods, and therefore the character of the oscillations will not be much affected.

The motion of a tricycle.

23. The only type of tricycle at present constructed is known as the cripper. In this machine the back-frame is supported by two equal wheels symmetrically placed on opposite sides of the mean plane. The front-frame and wheel are of the same pattern as those in the safety bicycle, and the steering arrangements are also identical. Fig. 3 represents the tricycle when the signification of the letters T and O is modified. T represents the point mid-way between the points of contact of the back wheels. O is the point in which the common axle of the two back wheels cuts the mean plane.

The effect of the introduction of the two back wheels is to keep the mean plane of the back-frame vertical. Therefore $\phi = 0$.

Let V be the velocity of TO, 2ϖ the width of the tricycle between the centres of the back wheels. The velocity of the centre of the right wheel is $V - \varpi\tau$.

Therefore the spin about the axle is $\dfrac{V-\varpi\tau}{a}$.

The force exerted by the ground on this wheel has a component in the direction B_2,

$$-\frac{1}{a}\cdot\frac{d}{dt}\cdot B_3\cdot\frac{V-\varpi\tau}{a} = \frac{B_3\varpi}{a^2}\cdot\dot{\tau}.$$

Similarly, the force on the other wheel is $\dfrac{B_3\varpi}{a^2}\dot{\tau}$ in the opposite direction.

The moment about the vertical through T of the two forces is

$$\left[\frac{2B_3\varpi^2}{a^2}\right]\dot{\tau}.\equiv K\dot{\tau}.$$

The equation of moments about the vertical through T differs from the corresponding equations for the bicycle by the introduction of this term.

This equation is obtained by multiplying VII. by $\cos\theta$, VIII. by $\sin\theta$, and adding, noticing that

$$\Gamma_1\cos^2\theta + \Gamma_2\sin^2\theta - 2\Gamma_{12}\sin\theta\cos\theta = J_1$$

is the moment of inertia of the back-frame wheels and rider about the vertical through T.

$$K\dot{\tau} + C_1\cos\theta + C_2\sin\theta + p\cos\theta P_3 = -J_1\dot{\tau} - W\kappa V\tau \text{...XXXV.}$$

The equations of motion of the front wheel are unaltered. Multiply the equations corresponding to VII. and VIII. by $\cos\theta$, $\sin\theta$ respectively, and add

$$C_1'\cos\theta + C_2'\sin\theta + p'\cos\theta P_3' = \phi' W'\kappa' g\sin\theta - J_1'\dot{\tau} - W'\kappa' V\tau$$
$$-(\Gamma_1'\cos\theta - \Gamma'_{12}\sin\theta)\ddot{\phi}' - W'\kappa' V\dot{\phi}'\cos\theta \text{......XXXV'.}$$

$$C_1' + \mu' P_3' = W' g\gamma'\phi'\sin\theta - \Gamma_1'(\ddot{\phi}' + \dot{\tau}\cos\theta) + \Gamma'_{12}\dot{\tau}\sin\theta$$
$$- V[m'r'\tau\sin\theta + W'\gamma'(\tau + \dot{\phi}'\cos\theta)] \text{...VII'.}$$

The following equations also hold:

$$R_3 = P_3 = -(P_3' + \phi'\Pi g\sin\theta) \text{ ....XI.}$$

$$\left.\begin{aligned} C_2 + C_2' + \phi' C_3 - qP_3 &= 0,\\ C_1 + C_1' &= 0, \end{aligned}\right\} \text{............XII.}$$

$$C_3' + p'\Pi\cos\theta = -W' g\kappa' \text{ ....IX.}$$

Add XXXV. and XXXV.', and reduce by IX.

$$bP_3' = (J_1 + K + J_1')\dot{\tau} + \Pi b V\tau + (\Gamma_1'\cos\theta - \Gamma'_{12}\sin\theta)\ddot{\phi}'$$
$$+ W'\kappa' V\dot{\phi}'\cos\theta - \Pi b g\sin\theta\phi' \text{...XXXVI.}$$

Eliminate P_3' from VII.' and XXXVI. The result is given, when the dashes are dropped in front-frame symbols, and a bar is placed over back-frame symbols, by

$$-C_1 = \{\Gamma_1\cos\theta - \Gamma_{12}\sin\theta + \frac{\mu}{b}(\bar{J}_1 + \bar{K} + J_1)\}\dot{\tau}$$
$$+ \{\mu\Pi + W\gamma + mr\sin\theta\}V\tau + \{\Gamma_1 + \frac{\mu}{b}(\Gamma_1\cos\theta - \Gamma_{12}\sin\theta)\}\ddot{\phi}$$
$$+ \left(\frac{\mu\kappa + \gamma}{b}\right) WV\cos\theta\dot{\phi} - (\mu\Pi + W\gamma)g\sin\theta\phi.$$

XIII. takes the simple form

$$\tau b = V\cos\theta\phi + \mu\dot{\phi}.$$

therefore

$$-C_1=\left[\Gamma_1+\frac{2\mu}{b}(\Gamma_1\cos\theta-\Gamma_{12}\sin\theta)+\left(\frac{\mu}{b}\right)^2(\overline{J}_1+\overline{K}+J_1)\right]\ddot{\phi}$$

$$+\left[(\Gamma_1\cos\theta-\Gamma_{12}\sin\theta)+\frac{\mu}{b}(\overline{J}_1+\overline{K}+J_1)\right.$$

$$\left.+(\mu\Pi+W\gamma+mr\sin\theta)\frac{\mu}{b}+\left(\frac{\mu}{b}\kappa+\gamma\right)W\right]V\cos\theta\,\dot{\phi}$$

$$+\left[(\mu\Pi+W\gamma+mr\sin\theta)\frac{V^2\cos\theta}{b}-(\mu\Pi+W\gamma)g\sin\theta\right]\phi,$$

where the letters without dashes refer to the front wheel.

The only critical velocity for the tricycle is that for which the coefficient of ϕ above vanishes.

In that case

$$\epsilon'\equiv\frac{bg}{V'^2}=\cot\theta+\frac{mr\cos\theta}{\mu\Pi+W\gamma}.$$

For steady motion when $V<V'$, C_1 is positive,

$$V>V',\ C_1 \text{ is negative},$$

i.e. when $V<V'$ the rider has to apply a couple in the sense of a rotation decreasing the angle between the planes of the wheels, when $V>V'$ the couple applied is in the sense of a rotation increasing that angle.

The effect of a disturbance of the steady motion when the rider is not using the handles is dissipated when $V>V'$.

N.B.—The velocity V' was obtained in § 11, as that at which there is a possible steady motion of the corresponding bicycle with hands off and back-frame vertical. In that case the motion was unstable. In this case V' is the limiting velocity of stable steady motion.

ON THE RESIDUE WITH RESPECT TO p^{n+1} OF A BINOMIAL-THEOREM COEFFICIENT DIVISIBLE BY p^n.

By J. W. L. GLAISHER.

§ 1. IN a paper in the present volume of the *Quarterly Journal*,* it was shown that if

$$n=kp+q,$$

$$r=gp+s,$$

q and s being less than a prime p, then the binomial-theorem coefficient $(n)_r$, *i.e.* the number of combinations of n things taken r together, is equal to $(k)_g\times$ a quantity $\equiv(q)_s$, mod. p, if $s<$ or $=q$. I proceed now to obtain the corresponding result for the case when $s>q$.

§ 2. From §§ 3–5 of the preceding paper (taking case II. of § 5, p. 152) we see that, if $s>q$, then $(n)_r$ is divisible by p, and we have

$$(n)_r=\frac{k!}{g!\,(k-g-1)!}\,p\times\text{a quantity}\equiv-\frac{q!}{s!\,(p+q-s)!},\ \text{mod}\,p.$$

Now, if m be any number $<p$,

$$(p-m)!=\frac{1.2.3\ldots(p-1)!}{(p-m+1)\,(p-m+2)\ldots(p-1)!}.$$

The numerator $\equiv-1$, mod. p, and, m being $<p$, the denominator

$$\equiv(-1)^{m-1}\,1.2.3\ldots(m-1),\quad\text{mod. }p.$$

Thus $$(p-m)!\equiv\frac{(-1)^m}{(m-1)!},\quad\text{mod. }p;$$

and therefore, taking $m=s-q$,

$$\frac{q!}{s!\,(p+q-s)!}\equiv(-1)^{s-q}\frac{q!\,(s-q)!}{s!},\quad\text{mod. }p,$$

* "On the residue of a binomial-theorem coefficient with respect to a prime modulus," pp. 150—156.

will contain T_{1,τ^2} and therefore $T_{1,\alpha}$, where α is an arbitrary mark in the $GF[2^n]$. Further, $R_{1,2,\lambda}$ transforms $T_{1,\alpha}$ into $R_{1,2,\lambda(1+\alpha)}T_{1,\alpha}$. Hence, if $n > 1$, so that the $GF[2^n]$ contains a mark α neither zero nor unity, the group I contains a substitution $R_{1,2,\lambda(1+\alpha)}$ not the identity. But M_1M_2 transforms $R_{1,2,1}$ into $N_{1,2,1}$. With $N_{1,2,1}$, I contains M_i, $L_{i,1}$ $(i=1, 2)$, since it contained their products two at a time. Transforming $L_{i,1}$ and $N_{i,j,1}$ by $T_{i,\tau}$, we obtain L_{i,τ^2} and $N_{i,j,\tau}$ respectively. Hence I contains every $L_{i,\alpha}$ and $N_{i,j,\alpha}$. Finally I contains

$$M_iL_{i,\alpha}M_iL_{i,\alpha^{-1}}M_iL_{i,\alpha} = T_{1,\alpha}.$$

The invariant subgroup I therefore coincides with H, which is thus simple.

The simple group H has the order $2^{4n}(2^{4n}-1)(2^{2n}-1)$, which for $n=2$ and 3 is respectively

$$2^8.3^2.5^2.17 = 979,200, \quad 2^{12}.3^4.7^2.5.13 = 1,173,836,560.$$

ERRATA IN THE PAPER, pp. 1–16.

p. 6, l. 19, *for* $Q_{2,3,\gamma_3}$ *read* $Q_{2,3.\alpha_3'}$.

p. 14, l. 1, *omit* -1 *after* γ_2''.

p. 16, l. 3 of Errata, *for* γ_2' *read* γ_1'.

University of California,
January 6, 1899.

END OF VOL. XXX.

Plate to Vol. XXX, No. 120.

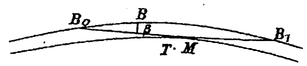


Fig. 4.

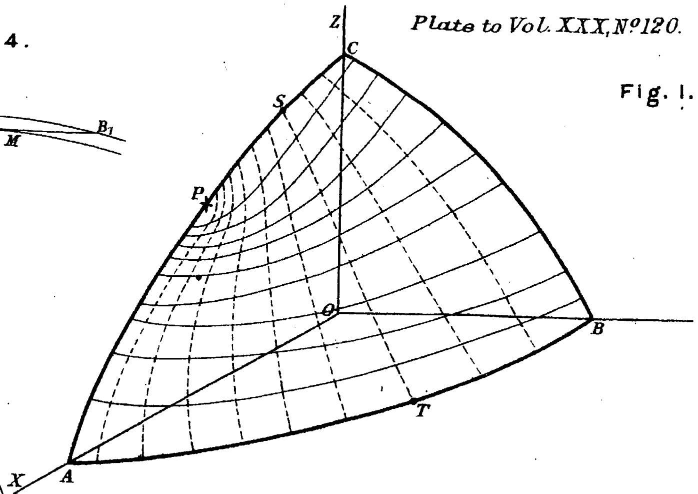


Fig. 1.

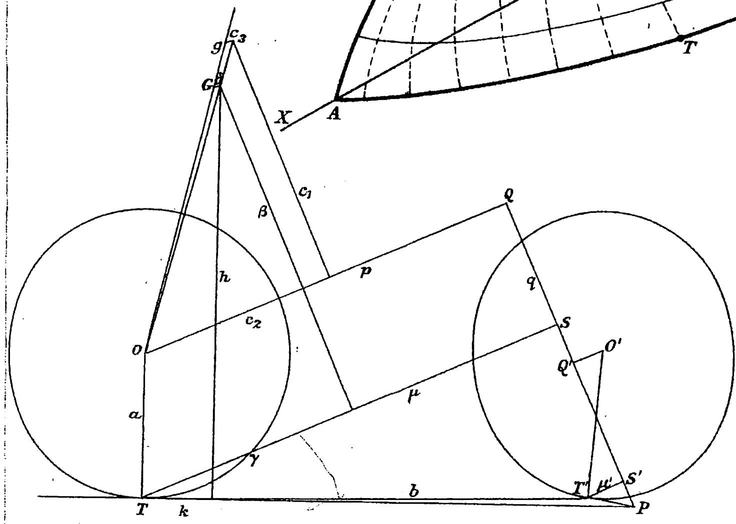


Fig. 2.

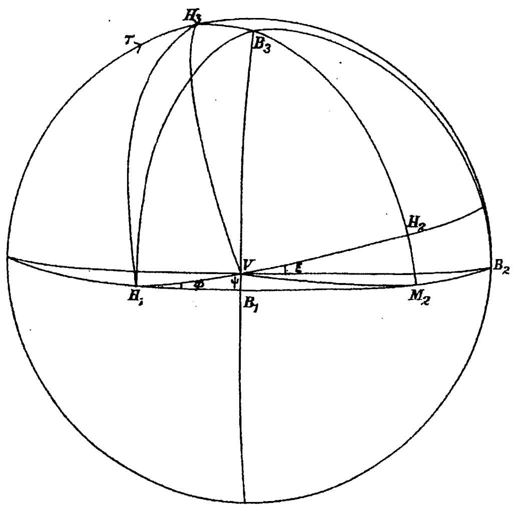


Fig. 3.