A MATHEMATICAL ANALYSIS OF THE STABILITY OF TWO WHEELED VEHICLES

BY

ROBERT NEIL COLLINS

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To Professors: Easton

Nelson

Mikol

This thesis having been approved in respect to form and mechanical execution is referred to you for judgment upon its substantial merit.

J. E. Willard
Dean

Approved as satisfying in substance the doctoral thesis requirement of the University of Wisconsin.

A. H. Easton
Major Professor

D. W. Nelson

Edward P. Mikol

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SUPPLEMENTARY LIST OF SYMBOLS

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x, y, z	body fixed orthogonal axes
x', y', z'	space fixed orthogonal axes
$\bar{r}$	position vector
c.g.	center of mass
δm	particle of mass
$\bar{\omega}$	angular velocity vector
$\bar{V}$	velocity vector
$\bar{h}$	angular momentum vector
i, j, k	unit vectors along the x, y, z axes
P	product of inertia
I	moment of inertia

Chapter II

Subscript 1	front end of vehicle
Subscript 2	rear end of vehicle
ω	angular velocity (general)
Ω	angular velocity of rotating parts
ϕ	lean angle (see page 12)
$\dot{\phi}$	$\frac{d\phi}{dt}$

$\ddot{\phi}$	$\frac{d^2\phi}{dt^2}$
θ	front wheel angle (see page 12)
I'	inertia of rotating part
R	distance between ground contact points of vehicle wheels
ψ	steering angle as measured in the ground plane
r_1	effective radius of vehicle front wheel
r_2	effective radius of vehicle rear wheel
m	mass of system
g_c	local gravitational constant

Chapter III

F_{g4}	driving force tangent to the rear wheel
M_3	moment on the steering axis as applied by the rider or damping
c_1	constant of proportionality relating the drag force on the front end to the square of the speed
c_2	constant of proportionality relating the drag force on the rear end to the speed
c_3	constant of proportionality relating the force tangent to the front wheel to the speed
c_4	constant designating the driving force required for a fixed speed $= F_{g4}$

c_5	constant of proportionality relating the damping moment on the steering axis to the rate of change of the front wheel angle

Chapter IV

K_1 ---- K_{32}	constants derived in the algebraic reduction of the basic equations of motion, reduced forms shown in Chapter V

Chapter V

A_1 ---- A_4	temporary terms composing part of the polynomial coefficients, defined for convenience and eliminated almost immediately
E_0 ---- E_4	final coefficients of the polynomial equation for use in the computer solution
D	operator denoting the rate of change with respect to time
C_1 ---- C_4	integration constants for the solution of the final equation

INTRODUCTION

The "handling" characteristics for a two wheeled vehicle of the basic motorcycle geometry are evaluated primarily in terms of (1) its stability near the upright equilibrium position and (2) its cornering response at higher speeds. The analysis in this paper is concerned only with stability near upright equilibrium but it should be noted that many of the same design features influence both stability and cornering.

Motorcycle stability stems from the action of the front wheel. Deflection of the front wheel from the straight ahead position sets up reaction forces at the base of both wheels which not only accelerate the machine toward its instantaneous center, but also apply a moment tending to change the lean angle. Control of the front wheel is thus the key to stability.

At low speeds the rate of change of the lean angle is slow enough to permit the rider to exercise control of the front wheel. At the higher rates of change which occur in normal operation, the rider reaction time is too long hence the stability must be designed into the machine itself.

An extensive literature search reveals very little analytical work in the field of motorcycle stability. This is the result of the complexity of the problem and the fact that it has been handled with a great deal of success by trial and error methods.

The work of Dohring[1] is the most recent analytical study in this field. Dohring used the bicycle equations of Klein and Sommerfield[2] to study the stability of three industrial models. Prior to this were analytical papers by Pearsall[3] and Bower[4].

In addition to these investigations a paper by Wallace[5] included an important study of motorcycle geometry and the more recent works of Irving[6] and Wilson-Jones[7] provide excellent general background along with some limited experimental results.

(1) E. Dohring, "Stability of Single-Track Vehicles", Forschung Ing. - Wes. 21, no. 2,50-62 (1955); Translated by J. Lotsof for Cornell Aeronautical Laboratory Inc.

(2) F. Klein and A. Sommerfield, The Theory of the Gyroscope, Vol. IV, Technical Applications, Berlin and Leipzig (1910).

(3) R. H. Pearsall, "The Stability of the Bicycle", Proc. Inst. Automobile Eng., Vol. XVII, p. 395, (1922).

(4) George S. Bower, "Steering and Stability of Single Track Vehicles", The Automobile Engineer, Vol. V, p. 280-283, (1915).

(5) John Wallace, "The Super Sports Motorcycle," The Institute of Automobile Engineers Proceedings, XXIV, 161-231, (1929).

(6) P. E. Irving, <u>Motorcycle Engineering</u>, Temple Press, London (1961).

(7) R. A. Wilson-Jones, "Steering and Stability of Single-Track Vehicles", Instn. of Mech. Engrs.-Proc.(Automobile Div.) pt. 4, p. 191-199, (1951).

The analysis here is based on the following assumptions:

(1) the vehicle is riderless

(2) the wheels are thin discs which remain in contact with the ground without slipping

(3) the operation is on a level surface

(4) the vehicle has no suspension system.

The general approach is to break the vehicle into two systems with System 1 as the front wheel and steering forks and System 2 as the rear frame, rear wheel, and engine. Then the equations of motion are written using Eulers equations for a set of body fixed axes selected at the center of gravity of each system. This procedure yields twelve equations in terms of the three basic variables (lean angle, front wheel angle, and rear wheel speed), nine reaction forces, and three reaction moments. The three remaining equations necessary for a solution come from the following conditions:

(1) an equations relating the moment on the front wheel resulting from the tangential ground forces to the moment imposed on the front frame at the wheel bearing.

(2) an equation describing the driving force on the rear wheel as a function of the square of the rear wheel speed.

(3) an equation describing the moment about the

steering fork axis as coulomb friction.

Then by algebraically eliminating the reaction terms and restricting the analysis to small angles, the equations are reduced to a system of two second order linear differential equations with constant coefficients.

In turn these equations are reduced to the two fourth order differential equations which are the basis for the stability study.

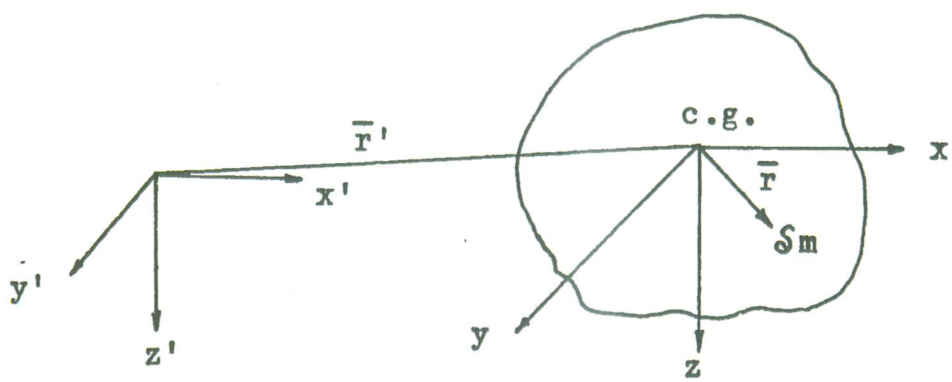
CHAPTER I

DERIVATION OF THE MOMENTUM AND ACCELERATION EQUATIONS

The material in this section is primarily a derivation of the Euler Equations which deal with the angular momentum and acceleration referred to body fixed axes. These general equations are to be applied to the motorcycle in the following chapter.

Angular Momentum of a Rigid Body

Consider a rigid body free to rotate and translate in space. Let the right handed coordinate systems x, y, z and x', y', z' be respectively (1) a set of body fixed axes with origin at the center of gravity and (2) a set of space fixed axes as shown in Figure (1.1).



Rigid Body in Space

Figure (1.1)

First writing an expression for the velocity of the mass particle δm with respect to the space fixed axes

$$(1.1) \quad \bar{V} = \bar{V}_{c.g.} + \bar{\omega} \times \bar{r}$$

where $\bar{\omega}$ is the angular velocity vector of the rigid body.

Using this equation together with the position vector, $\bar{r}$, the angular momentum, $\delta \bar{h}$, becomes

$$(1.2) \quad \delta \bar{h} = (\bar{r}' + \bar{r}) \times \bar{V} \delta m$$

$$= (\bar{r}' + \bar{r}) \times (\bar{V}_{c.g.} + \bar{\omega} \times \bar{r}) \delta m$$

Since the choice is arbitrary, let the space fixed axes be made to coincide instantaneously with the body fixed axes at the center of gravity. With this choice, Equation (1.2) simplifies to

$$(1.3) \quad \delta \bar{h} = \bar{r} \times (\bar{V}_{c.g.} + \bar{\omega} \times \bar{r}) \delta m$$

$$(1.4) \quad \delta \bar{h} = \left[\bar{r} \times \bar{V}_{c.g.} + \bar{r} \times (\bar{\omega} \times \bar{r}) \right] \delta m$$

The second term of Equation (1.4) can be expanded by an identity for the vector triple product.

$$(1.5) \quad \delta \bar{h} = \left[\bar{r} \times V_{c.g.} + \bar{r} \cdot \bar{r} (\bar{\omega}) - \bar{r} \cdot \bar{\omega} (\bar{r}) \right] \delta m$$

Summing over the entire mass yields the angular momentum for the body with respect to the space fixed axes.

$$(1.6) \quad \bar{h} = \sum (\bar{r} \times \bar{V}_{c.g.}) \delta m + \sum r^2 \bar{\omega} \delta m - \sum \bar{r} \cdot \bar{\omega} (\bar{r}) \delta m$$

The first term in Equation (1.6) becomes zero by virtue of the location of the origin. That is, for the origin at the center of gravity $\sum \bar{r}\, \delta m = 0$. Then expanding the remaining terms of Equation (1.6)

$$(1.7) \quad \bar{h} = \sum (x^2 + y^2 + z^2)(\omega_x i + \omega_y j + \omega_z k)\, \delta m - \sum (x\omega_x + y\omega_y + z\omega_z)(xi + yj + zk)\, \delta m$$

where i, j, and k are unit vectors along the x, y, and z axes.

Breaking these terms into scalar components and collecting terms yields

$$(1.8) \quad (a) \quad h_{x'} = \sum \left[(y^2 + z^2)\omega_x - xy\,\omega_y - xz\,\omega_z \right] \delta m$$

$$(b) \quad h_{y'} = \sum \left[(x^2 + z^2)\omega_y - xy\,\omega_x - yz\,\omega_z \right] \delta m$$

$$(c) \quad h_{z'} = \sum \left[(x^2 + y^2)\omega_z - xz\,\omega_x - yz\,\omega_y \right] \delta m.$$

The summations contained in Equation (1.8) can be recognized as the various moments and products of inertia. Rewriting Equation (1.8) with the usual inertia symbols and dropping the prime notation because of the coincidence of the axes gives

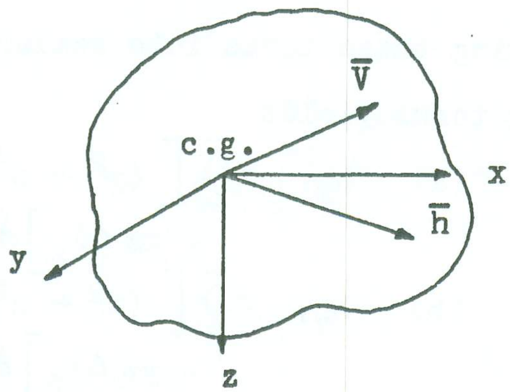
$$(1.9) \quad (a) \quad h_x = I_x \omega_x - P_{xy}\omega_y - P_{xz}\omega_z$$

$$(b) \quad h_y = I_y \omega_y - P_{xy}\omega_x - P_{yz}\omega_z$$

$$(c) \quad h_z = I_z \omega_z - P_{xz}\omega_x - P_{yz}\omega_y.$$

Momentum Rate Referred to Body Axes

Here, if the time rate of change of angular momentum is established with the space fixed axes as a reference, the inertia terms are variables because of the body rotation. However, the inertia terms can be made constants by using the body fixed axes as a reference.



Body Fixed Axes

Figure (1.2)

With this as the object consider the rigid body in Figure (1.2). The body fixed axes rotate and translate with the body and as shown previously Equation (1.9) represents the angular momentum with respect to these axes.

$$(1.10) \qquad \frac{d\bar{h}}{dt} = \dot{h}_x i + \dot{h}_y j + \dot{h}_z k + \bar{\omega} X \bar{h}$$

In Equation (1.10) the first three terms represent the rate of change of the magnitude of the momentum components and the last term represents the rate of change caused by the rotation of the axes of reference. Expanding the cross product of Equation (1.10) yields

$$(1.11)\quad \frac{d\bar{h}}{dt} = \dot{h}_x i + \dot{h}_y j + \dot{h}_z k + (\omega_y h_z - \omega_z h_y) i + (\omega_z h_x - \omega_x h_z) j + (\omega_x h_y - \omega_y h_x) k.$$

The components of Equation (1.11) can then be equated to the sum of the external moments about each of the body fixed axes.

$$(1.12)\quad \text{(a)}\ M_x = \dot{h}_x + \omega_y h_z - \omega_z h_y$$
$$\text{(b)}\ M_y = \dot{h}_y + \omega_z h_x - \omega_x h_z$$
$$\text{(c)}\ M_z = \dot{h}_z + \omega_x h_y - \omega_y h_x$$

Angular Momentum of Rotating Parts

Since the motorcycle has rotating parts, the angular momentum contribution of these parts must be added to Equation (1.12). From the derivation it follows that Equations (1.9) and (1.12) can be applied to a rotating part since it alone can be regarded as a rigid body. Using a prime notation to designate the rotating part and Ω to indicate its angular velocity, the final moment equation becomes

$$(1.13)\quad \text{(a)}\ M_x = \dot{h}_x + \omega_y h_z - \omega_z h_y + \dot{h}_x' + \Omega_y h_z' - \Omega_z h_y'$$
$$\text{(b)}\ M_y = \dot{h}_y + \omega_z h_x - \omega_x h_z + \dot{h}_y' + \Omega_z h_x' - \Omega_x h_z'$$
$$\text{(c)}\ M_z = \dot{h}_z + \omega_x h_y - \omega_y h_x + \dot{h}_z'$$

$$+\Omega_x h_{y'} - \Omega_y h_{x'}.$$

Acceleration Along Body Axes

In addition to this momentum equation an expression for the acceleration of the center of gravity along the body fixed axes is necessary. As in the case of the angular momentum vector the rate of change of the velocity vector is due to both the change of the magnitude and the rotation of the axes.

$$(1.14) \quad \dot{\bar{V}} = \dot{V}_x i + \dot{V}_y j + \dot{V}_z k + \bar{\omega} \times \bar{V}$$

Again expanding the cross product and breaking Equation (1.14) into its scalar components the acceleration becomes

$$(1.15) \quad (a) \quad a_x = \dot{V}_x + \omega_y V_z - \omega_z V_y$$
$$(b) \quad a_y = \dot{V}_y + \omega_z V_x - \omega_x V_z$$
$$(c) \quad a_z = \dot{V}_z + \omega_x V_y - \omega_y V_x$$

and since these expressions represent the acceleration of the center of gravity of the body the external force becomes

$$(1.16) \quad (a) \quad F_x = m(\dot{V}_x + \omega_y V_z - \omega_z V_y)$$
$$(b) \quad F_y = m(\dot{V}_y + \omega_z V_x - \omega_x V_z)$$
$$(c) \quad F_z = m(\dot{V}_z + \omega_x V_y - \omega_y V_x).$$

Equations (1.13) and (1.16) are the relationships to be applied to the motorcycle in the following chapter.

CHAPTER II

APPLICATION OF THE MOMENTUM AND ACCELERATION EQUATIONS

In this chapter Equation (1.12) and (1.16) are to be applied to each of the two systems shown in Figure (2.1). Henceforth the front end of the motorcycle will be referred to as System 1 and the rear end as System 2. The subscripts 1 and 2 will correspond to these systems.

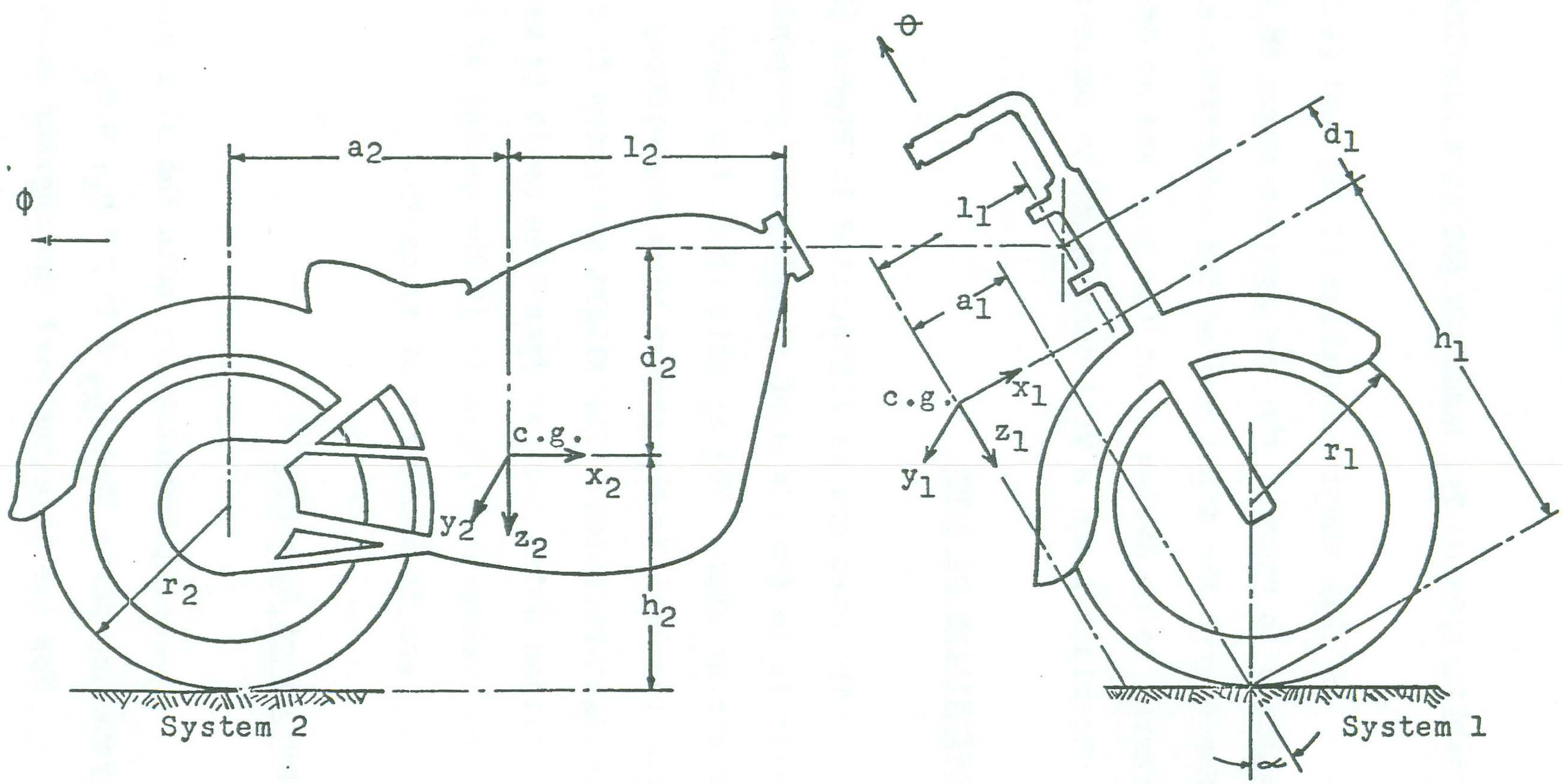
Definition of Axes

The axes are as illustrated in Figure (2.1). The x_1 axis is in the plane of symmetry and perpendicular to the steering axis. The x_2 axis is in the plane of symmetry and parallel to the ground when the motorcycle is in the upright position. The slight deviation from this parallel position which results from lean angle is neglected. In each system the origin is at the center of gravity and the x-z plane is the plane of symmetry.

Effect of Symmetry

Assuming symmetry for both the x, z plane and the rotating parts then $P_{xy} = P_{yz} = h_{x'} = h_{z'} = \dot{h}_{x'} = \dot{h}_{z'} = 0$.

For the rotating part the angular momentum term can



System Breakdown for Analysis

Figure 2.1

then be expressed as

$$(2.1) \quad h_y' = -I'\Omega$$

where Ω is the angular velocity of the wheel. Note that in the case of System 2, I' is the effective inertia of the rear wheel and engine combined.

Taking advantage of these simplifications and substituting Equation (1.9) into Equation (1.13) yields

$$(2.2) \quad (a) \quad M_x = I_x\dot{\omega}_x - P_{xz}\dot{\omega}_z + I_z\omega_z\omega_y - P_{xz}\omega_x\omega_y - I_y\omega_y\omega_z + I'\Omega\omega_z$$

$$(b) \quad M_y = I_y\dot{\omega}_y + I_x\omega_x\omega_z - P_{xz}\omega_z^2 - I_z\omega_x\omega_z + P_{xz}\omega_x^2 - I'\dot{\Omega}$$

$$(c) \quad M_z = I_z\dot{\omega}_z - P_{xz}\dot{\omega}_x + I_y\omega_x\omega_y - I_x\omega_x\omega_y + P_{xz}\omega_y\omega_z - I'\Omega\omega_x$$

Definition of Terms

The variables in Equations (1.16) and (2.2) must now be evaluated in terms of the basic variables, θ, ϕ, and Ω. These variables are respectively:

(1) the angle of lean, ϕ, as indicated by a vector along the x axis of System 2, with the upright position as zero, and with counter clockwise rotation as viewed from the rear as positive.

(2) the angle of front wheel rotation, θ, as indicated by a vector along the steering axis, with

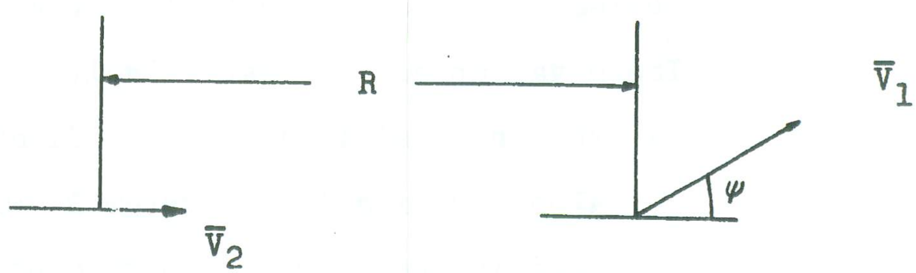
the straight ahead position as zero, and with counter clockwise rotation as viewed from the top as positive.

(3) the angular velocity, Ω, of the wheel in question.

In addition to these, several other machine dimensions are necessary. They will be defined by the various figures as the evaluation of the terms proceeds.

Evaluation of Angular Velocity Terms

The vectors $\overline{V}_1$ and $\overline{V}_2$ in Figure (2.2) are in the ground plane and represent the velocity of the front and rear wheels respectively. The quantity R, which can be treated as a constant for ordinary values of θ, is the distance between the points where the wheels make contact with the ground.



Wheel Velocities in the Ground Plane

Figure (2.2)

With Ω as the angular velocity, r as the wheel radius, and $\varphi^{(1)} = \tan^{-1}\left[\frac{\cos\alpha}{\cos\phi\cot\theta - \sin\phi\sin\alpha}\right]$ as the angle between the two velocity vectors, the following relationships hold:

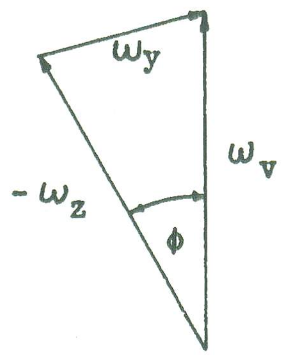
$$(2.3) \quad V_2 = V_1 \cos\varphi = r_2\Omega_2$$

$$(2.4) \quad V_1 = \frac{r_2\Omega_2}{\cos\varphi} = r_1\Omega_1$$

$$(2.5) \quad \Omega_1 = \frac{r_2}{r_1}\frac{\Omega_2}{\cos\varphi} .$$

These equations are based on the assumption that the tires do not slip and remain in contact with the ground.

Using these relationships the angular velocity components in Figure (2.3) can be evaluated. The term ω_v is the angular velocity of System 2 about a line perpendicular to the ground.



Rear View of Left Hand Turn

Figure (2.3)

$$(2.6) \quad \omega_v = \frac{V_1 \sin\varphi}{R} = \frac{r_2\Omega_2}{\cos\varphi}\,\frac{\sin\varphi}{R} = \frac{r_2\Omega_2}{R}\tan\varphi$$

(1) Wallace, p. 187.

The components of ω_v are

(2.7) $\omega_y = \frac{r_2}{R}\Omega_2 \tan\varphi \sin\phi$

(2.8) $\omega_z = -\frac{r_2}{R}\Omega_2 \tan\varphi \cos\phi$.

Thus for System 2

(2.9) (a) $\omega_{x2} = -\dot{\phi}$

(b) $\omega_{y2} = \frac{r_2}{R}\Omega_2 \sin\phi \tan\varphi$

(c) $\omega_{z2} = -\frac{r_2}{R}\Omega_2 \cos\phi \tan\varphi$.

By differentiation

(2.10) (a) $\dot{\omega}_{x2} = -\ddot{\phi}$

(b) $\dot{\omega}_{y2} = \frac{r_2}{R}\left[\Omega_2(\sin\phi \sec^2\varphi\dot{\varphi} + \cos\phi \tan\varphi\dot{\phi}) + \dot{\Omega}_2 \sin\phi \tan\varphi\right]$

(c) $\dot{\omega}_{z2} = -\frac{r_2}{R}\left[\Omega_2(\cos\phi \sec^2\varphi\dot{\varphi} - \sin\phi \tan\varphi\dot{\phi}) + \dot{\Omega}_2 \cos\phi \tan\varphi\right]$

These expressions for System 2 are used in calculating the angular velocities of System 1.

The components of the angular velocities of System 2 which contribute to the angular velocities of System 1 are as follows:

the contributions to ω_{x1}, ω_{y1}, and ω_{z1} are respectively

(a) By ω_{x2}

$\omega_{x2} \cos\alpha \cos\theta$

$\omega_{x2} \cos\alpha \sin\theta$

$\omega_{x2} \sin\alpha$

(b) By ω_{y2}

$-\omega_{y2} \sin\theta$

$+\omega_{y2} \cos\theta$

zero

(c) By ω_{z2}

$-\omega_{z2} \sin\alpha \cos\theta$

$-\omega_{z2} \sin\alpha \sin\theta$

$+\omega_{z2} \cos\alpha$.

Substituting the expressions for ω_{x2}, ω_{y2}, and ω_{z2} into these terms and adding $\dot{\theta}$ to the z component gives the angular velocities for System 1.

(2.11) (a) $$\omega_{x1} = -\dot{\phi}\cos\alpha\cos\theta - \frac{r_2}{R}\Omega_2 \sin\phi\tan\psi\sin\theta + \frac{r_2}{R}\Omega_2 \cos\phi\tan\psi\sin\alpha\cos\theta$$

(b) $$\omega_{y1} = \frac{r_2}{R}\Omega_2 \cos\phi\tan\psi\sin\alpha\sin\theta + \frac{r_2}{R}\Omega_2 \sin\phi\tan\psi\cos\theta - \dot{\phi}\cos\alpha\sin\theta$$

(c) $$\omega_{z1} = -\dot{\phi}\sin\alpha - \frac{r_2}{R}\Omega_2 \cos\phi\tan\psi\cos\alpha - \dot{\theta}$$

By differentiation

(2.12) (a) $$\dot{\omega}_{x1} = -\cos\alpha\left[-\dot{\phi}\sin\theta\dot{\theta} + \ddot{\phi}\cos\theta\right]$$

$$- \frac{r_2}{R}\Big[\Omega_2 \sin\phi(\tan\psi \cos\theta\dot{\theta}$$

$$+ \sec^2\psi\dot{\psi} \quad \sin\theta)$$

$$+ \tan\psi \sin\theta(\Omega_2 \cos\phi\dot{\phi}$$

$$+ \dot{\Omega}_2 \sin\phi)\Big]$$

$$+ \frac{r_2}{R} \sin\alpha\Big[\Omega_2 \cos\phi(-\tan\psi \sin\theta\dot{\theta}$$

$$+ \cos\theta\sec^2\psi\dot{\psi})$$

$$+ \tan\psi\cos\theta(-\Omega_2 \sin\phi\dot{\phi}$$

$$+ \dot{\Omega}_2 \cos\phi)\Big]$$

$$\text{(b)}\ \dot{\omega}_{y1} = \frac{r_2}{R} \sin\alpha\Big[\Omega_2 \cos\phi(\tan\psi \cos\theta\dot{\theta}$$

$$+ \sin\theta\sec^2\psi\dot{\psi})$$

$$+ \tan\psi \sin\theta(-\Omega_2 \sin\phi\dot{\phi}$$

$$+ \Omega_2 \cos\phi)\Big]$$

$$+ \frac{r_2}{R}\Big[\Omega_2 \sin\phi(-\tan\psi \sin\theta\dot{\theta}$$

$$+ \cos\theta\sec^2\psi\dot{\psi})$$

$$+ \tan\psi\cos\theta(\Omega_2 \cos\phi\dot{\phi}$$

$$+ \dot{\Omega}_2 \sin\phi)\Big]$$

$$- \cos\alpha(\dot{\phi}\cos\theta\dot{\theta} + \ddot{\phi}\sin\theta)$$

$$\text{(c)}\ \dot{\omega}_{z1} = -\ddot{\phi}\sin\alpha - \frac{r_2}{R}\cos\alpha\Big[\Omega_2 \cos\phi\sec^2\psi\dot{\psi}$$

$$+ \tan\psi(-\Omega_2 \sin\phi\dot{\phi} + \dot{\Omega}_2 \cos\phi)\Big]$$

$$- \ddot{\theta}$$

This accounts for all of the terms in Equation (2.2). The velocity terms in Equation (1.16) must now be evaluated.

Evaluation of Velocity Terms

The terms in Equation (1.16) are the scalar components of velocity along axes which are space fixed coincident with the body fixed axes.

Since the point of contact of the rear wheel has no side component of velocity, the velocity components can be expressed as

(2.13) (a) $V_{x2} = r_2\Omega_2$

(b) $V_{y2} = -h_2\dot{\phi} - a_2\omega_v \cos\phi$

$= -h_2\dot{\phi} - \frac{a_2 r_2 \Omega_2}{R}\tan\psi\cos\phi$

(c) $V_{z2} = -a_2\omega_v \sin\phi$

$= -\frac{a_2 r_2 \Omega_2}{R}\tan\psi\sin\phi$

By differentiation

(2.14) (a) $\dot{V}_{x2} = r_2\ddot{\Omega}_2$

(b) $\dot{V}_{y2} = -h_2\ddot{\phi} - \frac{a_2 r_2}{R}\Big[\Omega_2(\cos\phi\sec^2\psi\dot{\psi}$

$- \tan\psi\sin\phi\dot{\phi}) + \dot{\Omega}_2\cos\phi\tan\psi\Big]$

(c) $\dot{V}_{z2} = -\frac{a_2 r_2}{R}\Big[\Omega_2(\sin\phi\sec^2\psi\dot{\psi}$

$$+ \cos\phi\dot{\phi}\tan\psi) + \dot{\Omega}_2 \sin\phi\tan\psi \Big]$$

Treating a_1 and h_1 as constants and neglecting the contributions of $a_1\omega_{y1}$ and $h_1\omega_{y1}$ because of the small values of ω_{y1}, the velocities of System 1 become

$$(2.15)\quad (a)\quad V_{x1} = \frac{r_2\Omega_2}{\cos\psi}\cos\alpha$$

$$(b)\quad V_{y1} = -a_1\omega_{z1} + h_1\omega_{x1}$$

$$(c)\quad V_{z1} = \frac{r_2\Omega_2 \sin\alpha}{\cos\psi}$$

Substituting for ω_{x1} and ω_{z1} from Equation (2.11)

$$(2.16)\quad (a)\quad V_{x1} = \frac{r_2\Omega_2 \cos\alpha}{\cos\psi}$$

$$(b)\quad V_{y1} = -a_1\Big[-\dot{\phi}\sin\alpha - \frac{r_2\Omega_2}{R}\cos\phi\tan\psi\cos\alpha - \dot{\theta}\Big] + h_1\Big[-\dot{\phi}\cos\alpha\cos\theta - \frac{r_2\Omega_2}{R}\sin\phi\tan\psi\sin\theta + \frac{r_2\Omega_2}{R}\cos\phi\tan\psi\sin\alpha\cos\theta\Big]$$

$$(c)\quad V_{z1} = \frac{r_2\Omega_2}{\cos\psi}\sin\alpha$$

By differentiation

$$(2.17)\quad (a)\quad \dot{V}_{x1} = r_2\cos\alpha\Big[\frac{\Omega_2\sin\psi\dot{\psi}}{\cos^2\psi} + \frac{\dot{\Omega}}{\cos\psi}\Big]$$

$$(b)\quad \dot{V}_{y1} = -a_1\Big[-\ddot{\phi}\sin\alpha$$

$$- \frac{r_2}{R} \cos\alpha \Big\{ \Omega_2 (\cos\phi \sec^2\psi\dot{\psi}$$

$$- \sin\phi \tan\psi\dot{\phi}) + \cos\phi \tan\psi\dot{\Omega}_2 \Big\}$$

$$- \ddot{\theta} \Big]$$

$$+ h_1 \Big[- \cos\alpha(-\dot{\phi} \sin\theta\dot{\theta}$$

$$+ \ddot{\phi} \cos\theta)$$

$$- \frac{r_2}{R} \Big\{ \Omega_2 \sin\phi(\tan\psi \cos\theta\dot{\theta}$$

$$+ \sin\theta \sec^2\psi\dot{\psi})$$

$$+ \tan\psi \sin\theta(\Omega_2 \phi \dot{\phi} + \sin\phi\dot{\Omega}_2) \Big\}$$

$$+ \frac{r_2}{R} \sin\alpha \Big\{ \Omega_2 \cos\phi (-\tan\psi \sin\theta\dot{\theta}$$

$$+ \cos\theta \sec^2\psi\dot{\psi})$$

$$+ \tan\psi \cos\theta(-\Omega_2 \sin\phi\dot{\phi}$$

$$+ \cos\phi\dot{\Omega}_2) \Big\} \Big]$$

(c) $$\dot{V}_{z1} = r_2 \sin\alpha \Big[\Omega_2 \frac{\sin\psi}{\cos^2\psi} \dot{\psi} + \frac{\dot{\Omega}_2}{\cos\psi} \Big]$$

This completes the evaluation of all of the velocity terms to be used in Equation (1.16). Equations (2.9) thru (2.17) can now be substituted directly into Equations (1.16) and (2.2). This step will be taken up in the next chapter.

The remainder of this chapter will be concerned with developing the expressions for the external forces and moments for the two systems.

Summing External Forces

Figure (2.4) is a free body diagram showing the external forces and moments acting on System 1. The numerical subscripts 1, 2, and 3 denote the forces exerted by System 2 on System 1. The "a" subscript denotes the drag force which is assumed to be in the x, z plane and proportional to the square of the front wheel speed. The "g" subscripts denote the ground reaction forces acting on the point of contact. Note that these ground reaction forces are defined to always be parallel to the previously chosen body-fixed axes.

By summing forces on the free body for System 1

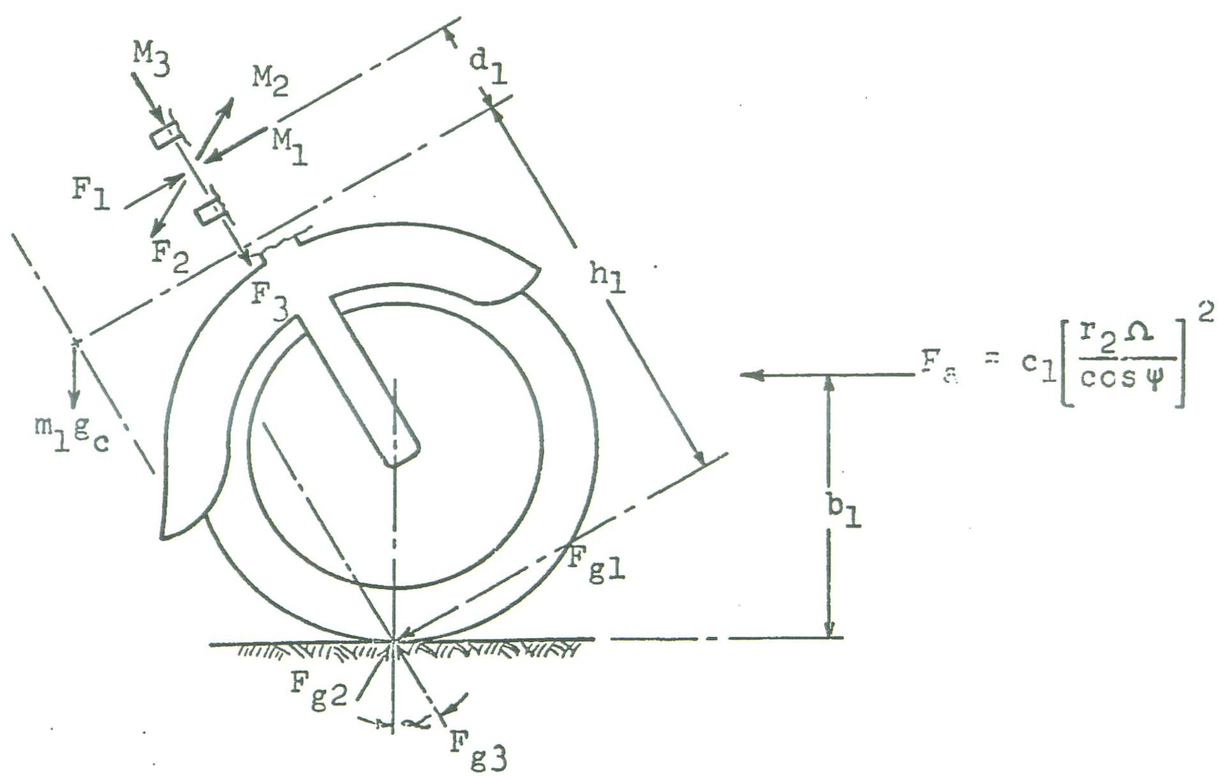
(2.18) (a) $$F_{x1} = F_1 - F_{g1} - c_1\left(\frac{r_2\Omega_2}{\cos\psi}\right)^2 \cos\alpha - m_1 g_c \cos\beta \sin\alpha$$

(b) $$F_{y1} = F_2 - F_{g2} - m_1 g_c \sin\beta$$

(c) $$F_{z1} = F_3 - F_{g3} - c_1\left(\frac{r_2\Omega_2}{\cos\psi}\right)^2 \sin\alpha + m_1 g_c \cos\beta \cos\alpha$$

The variable β is the angle between the vertical and the plane of the front wheel and mathematically is equal [2] to $\sin^{-1}\left[\sin\theta\cos\phi\sin\alpha + \sin\phi\cos\theta\right]$.

(2) Wallace, p. 188.



Free Body Diagram of System 1

Figure 2.4

From the free body diagram in Figure (2.5)

$$(2.19)\ (a)\quad F_{x2} = F_{g4} - F_4 - c_2(r_2\Omega_2)^2$$

$$(b)\quad F_{y2} = -\left[F_{g5} + F_5 + m_2 g_c \sin\phi\right]$$

$$(c)\quad F_{z2} = m_2 g_c \cos\phi - F_{g6} - F_6$$

where $c_2(r_2\Omega_2)^2$ is the drag force on System 2.

But F_4, F_5, and F_6 can be expressed in terms of F_1, F_2, and F_3 as follows:

$$(2.20)\ (a)\quad F_4 = F_3 \sin\alpha + F_1 \cos\theta\cos\alpha + F_2 \sin\theta\cos\alpha$$

$$(b)\quad F_5 = -F_1 \sin\theta + F_2 \cos\theta$$

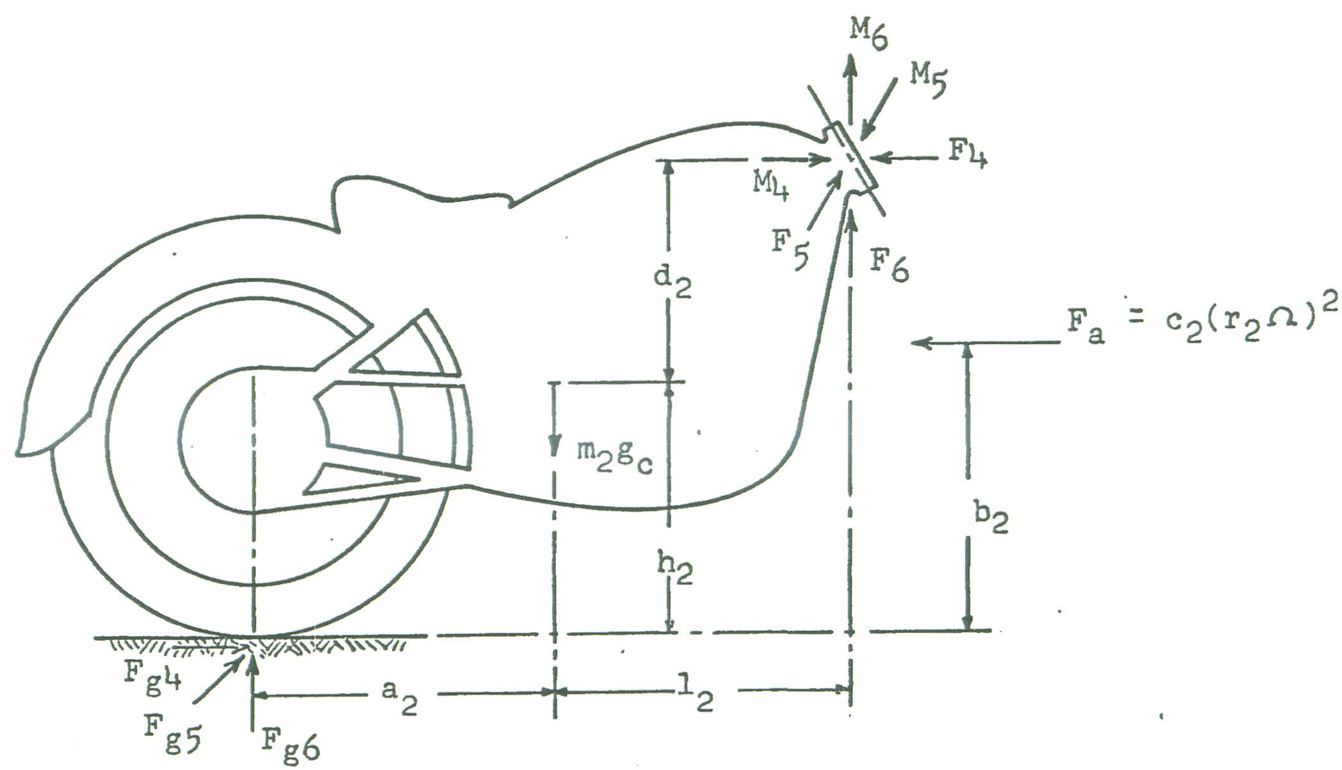
$$(c)\quad F_6 = -F_1 \cos\theta \sin\alpha - F_2 \sin\theta\sin\alpha + F_3 \cos\alpha$$

These expressions can be substituted into Equation (2.19) to give

$$(2.21)\ (a)\quad F_{x2} = F_{g4} - F_3 \sin\alpha - F_1 \cos\theta\cos\alpha - F_2 \sin\theta\cos\alpha - c_2(r_2\Omega_2)^2$$

$$(b)\quad F_{y2} = -F_{g5} + F_1 \sin\theta - F_2 \cos\theta - m_2 g_c \sin\phi$$

$$(c)\quad F_{z2} = m_2 g_c \cos\phi - F_{g6} + F_1 \cos\theta\sin\alpha + F_2 \sin\theta\sin\alpha - F_3 \cos\alpha$$



Free Body Diagram of System 2

Figure 2.5

The expressions for the external forces given in Equations (2.18) and (2.21) are to be equated to the expanded forms of Equation (1.16). This will be taken up in Chapter III.

Shift of Front Wheel Contact Point

Before evaluating the external moments the variation of the position of the front wheel point of contact as a function of ϕ and θ should be mentioned. In general the shift of the contact point is small and will be neglected for reasonable values of ϕ and θ. However, it should be pointed out that either large values of ϕ, θ, and α or low values of "$l_1 - a_1$" (see Figure (2.1)) will make this shift important.

The x, y, and z components of the shift of the point of contact, as derived in the previously cited work of Wallace, are given in the foot notes X_1, Y_1, and Z_1.[(3)]

(3) From Wallace, p. 186.

$$X_1 = \left[\sin\alpha(1 - \cos\theta)\right]\left[r_1 \cos(\alpha+\gamma) - e\right]$$

$$Y_1 = \left[\sin\theta - \cos\alpha\sin\phi(1 - \cos\theta) - \sin\theta(1 - \cos\phi)\right]\left[r_1 \cos(\alpha+\gamma) - e\right]$$

$$Z_1 = \left[\cos\alpha\cos\phi(1 - \cos\theta) + \sin\theta\sin\phi\right]\left[r_1 \cos(\alpha+\gamma) - e\right]$$

where e = offset of fork from the hub and

$$\gamma = \cot^{-1}\left[\frac{\cos\phi}{\sin\alpha[\cos\alpha(1 - \cos\theta) + \sin\theta\tan\phi]} - \cot\alpha\right]$$

<u>Summing External Moments</u>

Referring again to Figure (2.4) the external moments for System 1 can be written

(2.22) (a) $$M_{x1} = F_{g2}h_1 - M_1 + F_2 d_1$$

(b) $$M_{y1} = F_{g3}a_1 - F_{g1}h_1 - M_2 - F_1 d_1 - F_3 l_1 + c_1\left(\frac{r_2 \Omega_2}{\cos \psi}\right)^2 (b_1 - h_1 \cos\alpha)$$

(c) $$M_{z1} = - F_{g2}a_1 + M_3 + F_2 l_1$$

Using Figure (2.5) for System 2

(2.23) (a) $$M_{x2} = F_{g5}h_2 - F_5 d_2 + M_4$$

(b) $$M_{y2} = F_{g4} + c_2 (r_2 \Omega_2)^2 (b_2 - h_2) - F_{g6}a_2 + F_6 l_2 + F_4 d_2 + M_5$$

(c) $$M_{z2} = F_{g5}a_2 - M_6 - F_5 l_2$$

But M_4, M_5, and M_6 can be expressed in terms of M_1, M_2, and M_3 as follows

(2.24) (a) $$M_4 = - M_3 \sin\alpha + M_1 \cos\theta \cos\alpha + M_2 \sin\theta \cos\alpha$$

(b) $$M_5 = - M_1 \sin\theta + M_2 \cos\theta$$

(c) $$M_6 = M_1 \cos\theta \sin\alpha + M_2 \sin\theta \sin\alpha + M_3 \cos\alpha$$

Now substituting Equations (2.20) and (2.24) into Equation (2.23) gives

$$(2.25)\ (a)\quad M_{x2} = F_{g5}h_2 + (F_1 \sin\theta - F_2 \cos\theta)d_2 - (M_3 \sin\alpha - M_1 \cos\theta\cos\alpha - M_2 \sin\theta\cos\alpha)$$

$$(b)\quad M_{y2} = F_{g4}h_2 + c_2(b_2 - h_2)(r_2\Omega_2)^2 - F_{g6}a_2 - (F_1 \cos\theta\sin\alpha + F_2 \sin\theta\sin\alpha - F_3 \cos\alpha)l_2 + (F_2 \sin\theta\cos\alpha + F_3 \sin\alpha + F_1 \cos\theta\cos\alpha)d_2 - M_1 \sin\theta + M_2 \cos\theta$$

$$(c)\quad M_{z2} = F_{g5}a_2 - (M_1 \cos\theta\sin\alpha + M_2 \sin\theta\sin\alpha + M_3 \cos\alpha) + (F_1 \sin\theta - F_2 \cos\theta)l_2$$

This completes the expressions for the external moments.

Using the expressions developed in this chapter, Equations (1.16) and (2.2) can be expanded for each system to yield twelve independent equations in terms of the fifteen time varying quantities; F_1, F_2, F_3, F_{g1},

F_{g2}, F_{g3}, F_{g4}, F_{g5}, F_{g6}, M_1, M_2, M_3, Θ, Φ, and Ω.

Chapter III deals with the linearization and expansion of these equations as well as forming the three additional equations necessary for a solution.

CHAPTER III

THE LINEARIZED EQUATIONS OF MOTION

A considerable reduction in algebraic manipulation can be effected by linearizing the velocity and angular velocity expressions before actually substituting them into Equations (1.16) and (2.2). Consequently the material in this chapter is in order;

(1) the linearization of the velocity and angular velocity expressions developed in Chapter II,

(2) the substitution of these linearized expressions into Equations (1.16) and (2.2) to form the basic 12 equations in terms of 15 variables

(3) the formation of the 3 additional equations necessary for a solution.

Approximations Used for Linearizing Equations

The following approximations are used in linearizing the equations:

$$\phi << 1, \theta << 1, \dot{\Omega} = 0, \sin\phi = \phi, \sin\theta = \theta,$$

$$\cos\phi = 1, \cos\theta = 1, \tan\psi = \psi = \theta\cos\alpha,$$

$$\dot{\psi} = \dot{\theta}\cos\alpha, \frac{r_1}{r_2}\Omega_1 = \Omega_2 = \Omega, \sin\beta = \theta\sin\alpha + \phi,$$

$$\beta = \theta\sin\alpha + \phi, \cos\beta = 1$$

$$\phi^2 = \theta^2 = \phi\theta = \dot{\phi}\theta = \phi\dot{\theta} = \dot{\phi}^2 = \dot{\theta}^2 = 0$$

In view of these restrictions, some discussion concerning the practical significance of the solutions is in order. However, since this discussion bears directly on the results, it will be delayed until Chapter VI. Suffice it to say here, that the use of these approximations restricts the solutions to constant vehicle speeds in the neighborhood of the upright equilibrium position.

Linearized Velocity and Angular Velocity Expressions

Using approximations defined above, the terms derived in Chapter II can be simplified to the following:

(3.1) (a) $\omega_{x2} \doteq -\dot{\phi}$

(b) $\omega_{y2} \doteq 0$

(c) $\omega_{z2} \doteq -\frac{r_2}{R} \cos\alpha\, \Omega\theta$

(3.2) (a) $\dot{\omega}_{x2} \doteq -\ddot{\phi}$

(b) $\dot{\omega}_{y2} \doteq 0$

(c) $\dot{\omega}_{z2} \doteq -\frac{r_2}{R}\Omega\dot{\theta}\cos\alpha$

(3.3) (a) $\omega_{x1} \doteq -\dot{\phi}\cos\alpha + \frac{r_2}{R}\Omega\theta\sin\alpha\cos\alpha$

(b) $\omega_{y1} \doteq 0$

(c) $\omega_{z1} \doteq -\dot{\phi}\sin\alpha - \frac{r_2}{R}\Omega\theta\cos^2\alpha - \dot{\theta}$

(3.4) (a) $\dot{\omega}_{x1} \doteq -\ddot{\phi}\cos\alpha + \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha\cos\alpha$

(b) $\dot{\omega}_{y1} \doteq 0$

(c) $\dot{\omega}_{z1} = -\ddot{\phi}\sin\alpha - \frac{r_2}{R}\Omega\dot{\theta}\cos^2\alpha - \ddot{\theta}$

(3.5) (a) $V_{x2} = r_2\Omega$

(b) $V_{y2} = -h_2\dot{\phi} - \frac{a_2 r_2}{R}\Omega\theta\cos\alpha$

(c) $V_{z2} = 0$

(3.6) (a) $\dot{V}_{x2} = 0$

(b) $\dot{V}_{y2} = -h_2\ddot{\phi} - \frac{a_2 r_2}{R}\Omega\dot{\theta}\cos\alpha$

(c) $\dot{V}_{z2} = 0$

(3.7) (a) $V_{x1} = r_2\Omega\cos\alpha$

(b) $V_{y1} = a_1\left[\dot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\theta\cos^2\alpha + \dot{\theta}\right] - h_1\cos\alpha\left[\dot{\phi} - \frac{r_2}{R}\Omega\theta\sin\alpha\right]$

(c) $V_{z1} = r_2\Omega\sin\alpha$

(3.8) (a) $\dot{V}_{x1} = 0$

(b) $\dot{V}_{y1} = a_1\left[\ddot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\dot{\theta}\cos^2\alpha + \ddot{\theta}\right] - h_1\cos\alpha\left[\ddot{\phi} - \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha\right]$

All products of the angular velocity combinations which appear in Equation (2.2) are secondary terms hence

(3.9) (a) $\omega_{x1}\omega_{y1} = \omega_{y1}\omega_{z1} = \omega_{x1}\omega_{z1} = \omega_{x1}^2 = \omega_{y1}^2 = \omega_{z1}^2 = 0$

$$(b)\ \omega_{x2}\omega_{y2} \doteq \omega_{y2}\omega_{z2} \doteq \omega_{x2}\omega_{z2}$$
$$\doteq \omega_{x2}^2 \doteq \omega_{y2}^2 \doteq \omega_{z2}^2 \doteq 0$$

The following products of velocity and angular velocity which appear in Equation (1.16) are secondary terms hence

$$(3.10)\ (a)\ \omega_{y1} V_{z1} \doteq \omega_{z1} V_{y1} \doteq \omega_{x1} V_{y1}$$
$$\doteq \omega_{y1} V_{x1} \doteq 0$$

$$(b)\ \omega_{y2} V_{z2} \doteq \omega_{z2} V_{y2} \doteq \omega_{x2} V_{y2}$$
$$\doteq \omega_{y2} V_{x2} \doteq \omega_{x2} V_{z2} \doteq 0$$

Using the expressions derived in this chapter, Equations (1.16) and (2.2) can now be expanded for both systems.

$$(3.11)\ (a)\ F_{x1} \doteq 0$$

$$(b)\ F_{y1} \doteq m_1 \Big[a_1(\ddot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\dot{\theta}\cos\alpha + \ddot{\theta})$$
$$- h_1 \cos\alpha(\ddot{\phi} - \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha)$$
$$- (\dot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\theta\cos^2\alpha$$
$$+ \dot{\theta})(r_2\Omega\cos\alpha)$$
$$+ (\dot{\phi}\cos\alpha$$
$$- \frac{r_2}{R}\Omega\theta\sin\alpha\cos\alpha)(r_2\Omega\sin\alpha) \Big]$$

$$= m_1 \Big[a_1(\ddot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\dot{\theta}\cos\alpha$$

$$+ \ddot{\theta}) - h_1 \cos\alpha \, (\ddot{\phi}$$

$$- \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha \,)$$

$$- \frac{(r_2 \,\Omega\,)^2}{R}\theta\cos\alpha - r_2\Omega\dot{\theta}\cos\alpha \Big]$$

(c) $F_{z1} = 0$

(3.12) (a) $F_{x2} = 0$

(b) $F_{y2} = m_2 \Big[- h_2\ddot{\phi} - \frac{a_2 r_2}{R}\Omega\dot{\theta}\cos\alpha$

$$+ (- \frac{r_2}{R}\cos\alpha\,\Omega\theta)(r_2\Omega) \Big]$$

(c) $F_{z2} = 0$

(3.13) (a) $M_{x1} = I_{x1}(-\ddot{\phi}\cos\alpha + \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha\cos\alpha)$

$$+ P_{xz1}(\ddot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\dot{\theta}\cos^2\alpha$$

$$+ \ddot{\theta}) - I_1'\Omega(\dot{\phi}\sin\alpha$$

$$+ \frac{r_2}{R}\Omega\theta\cos^2\alpha + \dot{\theta})$$

(b) $M_{y1} = 0$

(c) $M_{z1} = - I_{z1}(\ddot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\theta\cos^2\alpha$

$$+ \ddot{\theta}) + P_{xz1}(\ddot{\phi}\cos\alpha$$

$$- \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha\cos\alpha\,)$$

$$+ I_1'\Omega(\dot{\phi}\cos\alpha - \frac{r_2}{R}\Omega\theta\sin\alpha\cos\alpha)$$

$$(3.14)\ (a)\quad M_{x2} = -I_{x2}\ddot{\phi} + P_{xz2}\frac{r_2}{R}\Omega\dot{\theta}\cos\alpha - I_2'\Omega\frac{r_2}{R}\Omega\theta\cos\alpha$$

$$(b)\quad M_{y2} = 0$$

$$(c)\quad M_{z2} = -I_{z2}\frac{r_2}{R}\Omega\dot{\theta}\cos\alpha + P_{xz2}\ddot{\phi} + I_2'\Omega\dot{\phi}$$

The expressions for the external forces and moments derived in Chapter II can now be linearized and equated to Equations (3.11) thru (3.14) as follows:

$$(3.15)\quad F_{x1} = F_1 - F_{g1} - c_1(r_2\Omega)^2\cos\alpha - m_1 g_c\sin\alpha = 0$$

$$(3.16)\quad F_{y1} = F_2 - F_{g2} - m_1 g_c(\theta\sin\alpha + \phi) = m_1\Big[a_1(\ddot{\phi}\sin\alpha + \frac{r_2}{R}\Omega\dot{\theta}\cos^2\alpha + \ddot{\theta}) - h_1\cos\alpha(\ddot{\phi} - \frac{r_2}{R}\Omega\dot{\theta}\sin\alpha) - \frac{(r_2\Omega)^2}{R}\theta\cos\alpha - r_2\Omega\dot{\theta}\cos\alpha\Big]$$

$$(3.17)\quad F_{z1} = F_3 - F_{g3} - c_1(r_2\Omega)^2\sin\alpha + m_1 g_c\cos\alpha = 0$$

$$(3.18)\quad F_{x2} = F_{g4} - F_3\sin\alpha - F_1\cos\alpha - F_2\theta\cos\alpha - c_2(r_2\Omega)^2 = 0$$

$$(3.19)\quad F_{y2} = -F_{g5} + F_1\theta - F_2 - m_2 g_c\phi$$

$$= - m_2 \left[h_2 \ddot{\Phi} + \frac{a_2 r_2}{R} \Omega \dot{\Theta} \cos\alpha + \frac{(r_2 \Omega)^2}{R} \Theta \cos\alpha \right]$$

$$(3.20) \quad F_{z2} = m_2 g_c - F_{g6} + F_1 \sin\alpha + F_2 \Theta \sin\alpha - F_3 \cos\alpha = 0$$

$$(3.21) \quad M_{x1} = F_{g2} h_1 - M_1 + F_2 d_1 = I_{x1}(-\ddot{\Phi} \cos\alpha + \frac{r_2}{R} \Omega \dot{\Theta} \sin\alpha \cos\alpha) + P_{xz1}(\ddot{\Phi} \sin\alpha + \frac{r_2}{R} \Omega \dot{\Theta} \cos^2\alpha + \ddot{\Theta}) - I_1' \frac{r_2}{r_1} \Omega (\dot{\Phi} \sin\alpha + \frac{r_2}{R} \Omega \Theta \cos^2\alpha + \dot{\Theta})$$

$$(3.22) \quad M_{y1} = F_{g3} a_1 - F_{g1} h_1 - M_2 - F_1 d_1 - F_3 l_1 + c_1 (r_2 \Omega)^2 (b_1 - h_1 \cos\alpha) = 0$$

$$(3.23) \quad M_{z1} = - F_{g2} a_1 + M_3 + F_2 l_1 = - I_{z1}(\ddot{\Phi} \sin\alpha + \frac{r_2}{R} \Omega \dot{\Theta} \cos^2\alpha + \ddot{\Theta}) + P_{xz1}(\ddot{\Phi} \cos\alpha - \frac{r_2}{R} \Omega \dot{\Theta} \sin\alpha \cos\alpha) + I_1' \frac{r_2}{r_1} \Omega (\dot{\Phi} \cos\alpha - \frac{r_2}{R} \Omega \Theta \sin\alpha \cos\alpha)$$

$$(3.24) \quad M_{x2} = F_{g5} h_2 + F_1 d_2 \Theta - F_2 d_2 - M_3 \sin\alpha + M_1 \cos\alpha + M_2 \Theta \cos\alpha = - I_{x2} \ddot{\Phi}$$

$$+ P_{xz2}\frac{r_2}{R}\Omega\dot{\theta}\cos\alpha$$

$$- I_2' \frac{r_2}{R}\Omega^2\theta\cos\alpha$$

$$(3.25)\quad M_{y2} = F_{g4}h_2 + c_2(b_2 - h_2)(r_2\Omega)^2 - F_{g6}a_2$$

$$- (F_1 \sin\alpha + F_2\theta\sin\alpha - F_3\cos\alpha)l_2$$

$$+ (F_2\theta\cos\alpha + F_3\sin\alpha + F_1\cos\alpha)d_2$$

$$- M_1\theta + M_2 = 0$$

$$(3.26)\quad M_{z2} = F_{g5}a_2 - M_1 \sin\alpha - M_2\theta\sin\alpha$$

$$- M_3\cos\alpha + (F_1\theta - F_2)l_2$$

$$= - I_{z2}\frac{r_2}{R}\Omega\dot{\theta}\cos\alpha + P_{xz2}\ddot{\phi} + I_2'\Omega\dot{\phi}$$

Equations (3.15) thru (3.26) now form a system of twelve independent equations in terms of the fifteen time varying quantities F_1, F_2, F_3, F_{g1}, F_{g2}, F_{g3}, F_{g4}, F_{g5}, M_1, M_2, M_3, ϕ, θ, and Ω. Conditions relating F_{g1}, F_{g3}, F_{g4}, and Ω must be imposed now to set up the three additional equations necessary for solution.

<u>Three Additional Equations</u>

The first condition is based on a description of the driving force applied at the rear wheel. This force ordinarily oscillates slightly about a mean value because of the nature of the torque output of reciprocating engines. However, since the variation is small and the frequency is high, this force will be treated as a constant for the constant speed solutions. By summing the external forces on the motorcycle

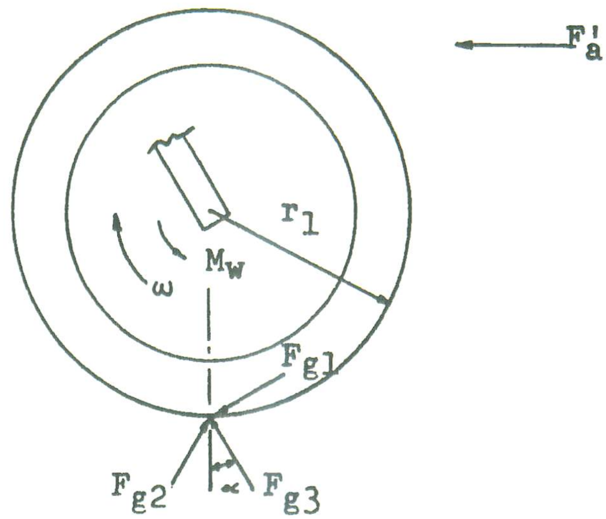
$$(3.27)\quad F_{g4} = c_4 = F_{g3}\sin\alpha + F_{g1}\cos\alpha + c_1(r_2\Omega)^2 + c_2(r_2\Omega)^2$$

The second condition relates the first two terms of Equation (3.27) to the ground forces tangent to the front wheel. In the absence of acceleration the moment generated by these forces about the front wheel hub must be in equilibrium with the moment caused by the fractional part of the total drag force acting on the front wheel alone (F_a') and the moment (M_w) transmitted to the wheel through the front wheel bearing. In practice the sum "$F_a' r_1 + M_w$" will generally be small. These two terms are lumped together in this analysis and assumed to be directly proportional to the product of the front wheel radius and the motorcycle speed. Using c_3 as this constant of proportionality the moment equation about the front wheel hub can be written by reference to Figure (3.1) as

$$(3.28)\quad F_{g3}r_1\sin\alpha + F_{g1}r_1\cos\alpha = c_3 r_1(r_2\Omega)$$

Cancelling r_1

$$(3.29)\quad F_{g3}\sin\alpha + F_{g1}\cos\alpha = c_3 r_2\Omega$$



Moment About the Front Wheel Hub

Figure (3.1)

Substituting Equation (3.29) into Equation (3.27) gives

(3.30) $F_{g4} = c_4 = c_3 r_2 \Omega + (c_1 + c_2)(r_2 \Omega)^2$

Equation (3.30) combines both Equation (3.27) and Equation (3.29) thus it eliminates both F_{g4} and Ω as time variables.

The third condition is a description of the moment M_3 imposed by System 2 on System 1. Note that here a rider or servo input could be included, but for the solutions formed here M_3 is assumed to be entirely a result of coulomb damping.

Using c_5 as the constant of proportionality

(3.31) $M_3 = c_5 \dot{\theta}$

Equations (3.15) thru (3.26) together with Equations (3.30) and (3.31) constitute the complete equations of motion. All of the time varying quantities except ϕ and θ can be eliminated by substitution. The algebra of this process is shown in Chapter IV.

CHAPTER IV

ALGEBRA OF REDUCTION TO TWO VARIABLES

The material in this Chapter consists entirely of the algebraic elimination of F_1, F_2, F_3, F_{g1}, F_{g2}, F_{g3}, F_{g5}, F_{g6}, M_1 and M_2. The two terms F_{g4} and M_3 are carried forward to Chapter V to make it easier to change the nature of the road load and to allow for the possibility of a rider input in later solutions. In a few instances secondary terms appear and are immediately cancelled, otherwise the procedure is one of straightforward substitution.

Equations (3.15) thru (3.26) can be rewritten in order as

$$(4.1) \quad F_{g1} = F_1 - c_1(r_2\Omega)^2\cos\alpha - m_1 g_c \sin\alpha$$

$$(4.2) \quad F_{g2} = F_2 - m_1 g_c \sin\alpha\,\theta - m_1 g_c\,\phi - (a_1 \sin\alpha - h_1 \cos\alpha)m_1\ddot{\phi} - \left[\frac{a_1 \cos\alpha}{R} + \frac{h_1 \sin\alpha}{R} - 1\right] m_1 r_2 \cos\alpha\,\Omega\,\dot{\theta} - m_1 a_1 \ddot{\theta} + \frac{m_1(r_2\Omega)^2\cos\alpha\,\theta}{R}$$

$$(4.3) \quad F_{g3} = F_3 - c_1(r_2\Omega)^2\sin\alpha + m_1 g_c \cos\alpha$$

$$(4.4) \quad F_3 \sin\alpha = F_{g4} - F_1 \cos\alpha - F_2 \cos\alpha\,\theta - c_2(r_2\Omega)^2$$

$$(4.5) \quad m_2h_2\ddot{\phi} = F_{g5} - F_1\theta + F_2 + m_2g_c\phi - \frac{m_2a_2r_2\Omega\cos\alpha\dot{\theta}}{R} - \frac{m_2(r_2\Omega)^2\cos\alpha\theta}{R}$$

$$(4.6) \quad F_{g6} = m_2g_c + F_1\sin\alpha + F_2\sin\alpha\theta - F_3\cos\alpha$$

$$(4.7) \quad M_1 = F_{g2}h_1 + F_2d_1 + (I_{x1}\cos\alpha - P_{xz1}\sin\alpha)\ddot{\phi} - I_{x1}\left[(I_{x1}\sin\alpha + P_{xz1}\cos\alpha)\frac{\cos\alpha}{R} - \frac{I_1'}{r_1}\right]r_2\Omega\dot{\theta} - P_{xz1}\ddot{\theta} + \frac{I_1'r_2\Omega\sin\alpha\dot{\phi}}{r_1} + \frac{I_1'(r_2\Omega)^2\theta}{r_1R}$$

$$(4.8) \quad F_1d_1 = F_{g5}a_1 - F_{g1}h_1 - M_2 - F_3l_1 + c_1(b_1 - h_1\cos\alpha)(r_2\Omega)^2$$

$$(4.9) \quad I_{z1}\ddot{\theta} = F_{g2}a_1 - M_3 - (I_{z1}\sin\alpha - P_{xz1}\cos\alpha)\ddot{\phi} - (I_{z1}\cos\alpha + P_{xz1}\sin\alpha)\frac{r_2\Omega\cos\alpha\dot{\theta}}{R} + \frac{I_1'r_2\Omega\cos\alpha\dot{\phi}}{R} - \frac{I_1'(r_2\Omega)^2\sin\alpha\cos\alpha\theta}{r_1R} - F_2l_1$$

$$(4.10) \quad F_{g5}h_2 = M_3\sin\alpha - M_1\cos\alpha - M_2\cos\alpha\theta + F_2d_2 - I_{x2}\ddot{\phi} - F_1d_2\theta + \frac{P_{xz2}r_2\Omega\cos\alpha\dot{\theta}}{R} - \frac{I_2'r_2\Omega^2\cos\alpha\theta}{R}$$

$$(4.11)\quad M_2 = -F_{g4}h_2 - c_2(b_2 - h_2)(r_2\Omega)^2 + F_{g6}a_2$$
$$+ F_1 l_2 \sin\alpha + F_2 l_2 \sin\alpha\,\theta - F_3 l_2 \cos\alpha$$
$$- F_2 d_2 \cos\alpha\,\theta - F_3 d_2 \sin\alpha - F_1 d_2 \cos\alpha$$
$$+ M_1\theta$$

$$(4.12)\quad F_2 l_2 = -P_{xz2}\ddot{\phi} - I_2'\Omega\dot{\phi} + F_{g5}a_2 - M_1 \sin\alpha$$
$$- M_2 \sin\alpha\,\theta - M_3 \cos\alpha + F_1 l_2\,\theta$$
$$+ \frac{I_{z2} r_2 \Omega \cos\alpha\,\dot{\theta}}{R}$$

Eliminating Equation (4.6) by substituting it into Equation (4.11)

$$(4.11\ a)\quad M_2 = -F_{g4}h_2 - c_2(b_2 - h_2)(r_2\Omega)^2$$
$$+ m_2 g_c a_2 + (a_2 \sin\alpha + l_2 \sin\alpha$$
$$- d_2 \cos\alpha)(F_1 + F_2\theta) - (a_2 \cos\alpha$$
$$+ l_2 \cos\alpha + d_2 \sin\alpha)F_3 + M_1\theta$$

Eliminating Equation (4.10) by substituting it into Equations (4.5) and (4.12)

$$(4.5\ a)\quad m_2 h_2 \ddot{\phi} = \frac{1}{h_2}(M_3 \sin\alpha - M_1 \cos\alpha$$
$$- M_2 \cos\alpha\,\theta - I_{x2}\ddot{\phi} + \frac{P_{xz2} r_2 \Omega \cos\alpha\,\dot{\theta}}{R}$$
$$- \frac{I_2' r_2 \Omega^2 \cos\alpha\,\theta}{R}) + (1 + \frac{d_2}{h_2})(F_2 - F_1\theta)$$

$$m_2 g_c \Phi - \frac{m_2 a_2 r_2 \Omega \cos\alpha \dot{\theta}}{R}$$

$$- \frac{m_2 (r_2 \Omega) \cos\alpha \, \theta}{R}$$

(4.12 a) $$F_2 l_2 = - P_{xz2} \ddot{\Phi} - I_2' \Omega \dot{\Phi} + \left(\frac{a_2 \sin\alpha}{h_2} - \cos\alpha\right) M_3 - \left(\frac{a_2 \cos\alpha}{h_2} + \sin\alpha\right)(M_1 + M_2 \theta) + \frac{a_2 d_2}{h_2} F_2 - \frac{a_2 I_{x2} \ddot{\Phi}}{h_2} + \left(l_2 - \frac{a_2 d_2}{h_2}\right) F_1 \theta + \frac{P_{xz2} a_2 r_2 \Omega \cos\alpha \dot{\theta}}{h_2 R} - \frac{I_2' a_2 r_2 \Omega^2 \cos\alpha \, \theta}{h_2 R} + \frac{I_{z2} r_2 \Omega \cos\alpha \dot{\theta}}{R}$$

Eliminating Equation (4.3) by substituting it into Equation (4.8)

(4.8 a) $$F_1 d_1 = (a_1 - l_1) F_3 + a_1 (-c_1 (r_2 \Omega)^2 \sin\alpha + m_1 g_c \cos\alpha) - F_{g1} h_1 - M_2 + c_1 (r_2 \Omega)^2 (b_1 - h_1 \cos\alpha)$$

Eliminating Equation (4.1) by substituting it into Equation (4.8 a)

(4.8 b) $$F_1 d_1 = (a_1 - l_1) F_3 + a_1 \left[-c_1 (r_2 \Omega)^2 \sin\alpha + m_1 g_c \cos\alpha \right] - h_1 \left[F_1 - c_1 (r_2 \Omega)^2 \cos\alpha - m_1 g_c \sin\alpha \right] - M_2 + c_1 (r_2 \Omega)^2 (b_1 - h_1 \cos\alpha)$$

Eliminating Equation (4.2) by substituting it into Equations (4.7) and (4.9)

$$(4.7\text{ a})\quad M_1 = (h_1 + d_1)F_2 + h_1\Big[- m_1 g_c \sin\alpha\,\theta - m_1 g_c\,\phi - (a_1 \sin\alpha - h_1 \cos\alpha)m_1\ddot{\phi} - \Big(\frac{a_1 \cos\alpha}{R} + \frac{h_1 \sin\alpha}{R} - 1\Big)m_1 r_2 \Omega \cos\alpha\,\dot{\theta} - m_1 a_1 \ddot{\theta} + \frac{m_1 \cos\alpha\,(r_2\Omega)^2\theta}{R}\Big] + (I_{x1} \cos\alpha - P_{xz1} \sin\alpha)\ddot{\phi} - \Big[(I_{x1} \sin\alpha + P_{xz1} \cos\alpha)\frac{\cos\alpha}{R} - \frac{I_1'}{r_1}\Big] r_2 \Omega \dot{\theta} - P_{xz1}\ddot{\theta} + \frac{I_1' r_2 \Omega \sin\alpha\,\dot{\phi}}{r_1} + \frac{I_1'(r_2\Omega)^2\cos^2\alpha\,\theta}{r_1 R}$$

$$(4.9\text{ a})\quad I_{z1}\ddot{\theta} = (a_1 - l_1)F_2 + a_1\Big[- m_1 g_c \sin\alpha\,\theta - m_1 g_c\,\phi - (a_1 \sin\alpha - h_1 \cos\alpha)m_1\ddot{\phi} - \Big(\frac{a_1 \cos\alpha}{R} + \frac{h_1 \sin\alpha}{R} - 1\Big)m_1 r_2 \Omega \cos\alpha\,\dot{\theta} - m_1 a_1 \ddot{\theta} + \frac{m_1 \cos\alpha\,(r_2\Omega)^2\theta}{R}\Big] - M_3 - (I_{z1} \sin\alpha - P_{xz1} \cos\alpha)\ddot{\phi} - (I_{z1} \cos\alpha + P_{xz1} \sin\alpha)\frac{r_2 \Omega \cos\alpha\,\dot{\theta}}{R} + \frac{I_1' r_2 \Omega \cos\alpha\,\dot{\phi}}{r_1} - \frac{I_1'(r_2\Omega)^2 \sin\alpha \cos\alpha\,\theta}{r_1 R}$$

Eliminating Equation (4.7 a) by substituting it into

Equations (4.11 a), (4.12 a) and (4.5 a)

$$(4.11\text{ b})\quad M_2 = -F_{g4}h_2 - c_2(r_2\Omega)^2(b_2 - h_2)\Omega + m_2a_2g_c + (a_2\sin\alpha + l_2\sin\alpha - d_2\cos\alpha)F_1 + (a_2\sin\alpha + l_2\sin\alpha - d_2\cos\alpha + h_1 + d_1)F_2\theta - (a_2\cos\alpha + l_2\cos\alpha + d_2\sin\alpha)F_3$$

$$(4.12\text{ b})\quad F_2l_2 = -P_{xz2}\ddot{\Phi} - I_2'\Omega\dot{\Phi} + \left(\frac{a_2\sin\alpha}{h_2} - \cos\alpha\right)M_3 + \frac{a_2d_2}{h_2} - \left(\frac{a_2\cos\alpha}{h_2} + \sin\alpha\right)(h_1+d_1)F_2 - \left(\frac{a_2\cos\alpha}{h_2} + \sin\alpha\right)\Big\{h_1\Big[-m_1g_c\sin\alpha\,\theta - m_1g_c\Phi - (a_1\sin\alpha - h_1\cos\alpha)m_1\ddot{\Phi} - \left(\frac{a_1\cos\alpha}{R} + \frac{h_1\sin\alpha}{R} - 1\right)m_1r_2\Omega\cos\alpha\,\dot{\theta} - m_1a_1\ddot{\theta} + \frac{m_1(r_2\Omega)^2\cos\alpha\,\theta}{R}\Big] + (I_{x1}\cos\alpha - P_{xz1}\sin\alpha)\ddot{\Phi} - \Big[(I_{x1}\sin\alpha + P_{xz1}\cos\alpha)\frac{\cos\alpha}{R} - \frac{I_1'}{r_1}\Big]r_2\Omega\dot{\theta} - P_{xz1}\ddot{\theta} + \frac{I_1'r_2\Omega\sin\alpha\,\dot{\Phi}}{r_1}$$

$$+ \frac{I_1'(r_2\Omega)^2\cos^2\alpha\,\theta}{r_1 R}\Bigg\} - \Big(\frac{a_2\cos\alpha}{h_2}$$

$$+ \sin\alpha\Big)M_2\theta - \frac{a_2 I_{x2}\ddot{\phi}}{h_2} + \Big(l_2 - \frac{a_2 d_2}{h_2}\Big)F_1\theta$$

$$+ \frac{P_{xz2}a_2 r_2\Omega\cos\alpha\,\dot{\theta}}{h_2 R} - I_2'\frac{a_2 r_2\Omega^2\cos\alpha\,\theta}{h_2 R}$$

$$+ \frac{I_{z2}r_2\Omega\cos\alpha\,\dot{\theta}}{R}$$

(4.5 b) $$m_2 h_2\ddot{\phi} = \frac{M_3\sin\alpha}{h_2} - \Bigg[(h_1 + d_1)\frac{\cos\alpha}{h_2}$$

$$- \Big(1 + \frac{d_2}{h_2}\Big)\Bigg]F_2 - \frac{h_1\cos\alpha}{h_2}\Bigg\{- m_1 g_c\sin\alpha\,\theta$$

$$- m_1 g_c\phi - (a_1\sin\alpha - h_1\cos\alpha)m_1\ddot{\phi}$$

$$- \Big(\frac{a_1\cos\alpha}{R} + \frac{h_1\sin\alpha}{R} - 1\Big)m_1 r_2\Omega\cos\alpha\,\dot{\theta}$$

$$- m_1 a_1\ddot{\theta} + \frac{m_1(r_2\Omega)^2\cos\alpha\,\theta}{R}\Bigg\}$$

$$- \frac{\cos\alpha}{h_2}\Bigg\{(I_{x1}\cos\alpha - P_{xz1}\sin\alpha)\ddot{\phi}$$

$$- \Bigg[(I_{x1}\sin\alpha + P_{xz1}\cos\alpha)\frac{\cos\alpha}{R}$$

$$- \frac{I_1'}{R}\Bigg]r_2\Omega\dot{\theta} - P_{xz1}\ddot{\theta} + \frac{I_1' r_2\Omega\sin\alpha\,\dot{\phi}}{r_1}$$

$$+ \frac{I_1'(r_2\Omega)^2\cos^2\alpha\,\theta}{r_1 R}\Bigg\} + \frac{1}{h_2}(- M_2\cos\alpha\,\theta$$

$$- I_{x2}\ddot{\phi} + \frac{P_{xz2}r_2\Omega\cos\alpha\,\dot{\theta}}{R} - \frac{I_2'r_2\Omega^2\cos\alpha\,\theta}{R})$$

$$- (1 + \frac{d_2}{h_2})F_1\theta + m_2 g_c\phi - \frac{m_2 a_2 r_2\Omega\cos\alpha\,\dot{\theta}}{R}$$

$$- \frac{m_2(r_2\,\Omega)^2\cos\alpha\,\theta}{R}$$

Eliminating Equation (4.11 b) by substituting it into Equations (4.5 b), (4.8 b), and (4.12 b)

$$(4.5\ c)\quad m_2h_2\ddot{\phi} = \frac{M_3\sin\alpha}{h_2} - \left[\frac{(h_1 + d_1)\cos\alpha}{h_2}\right.$$

$$\left. - 1 - \frac{d_2}{h_2}\right] F_2 + \frac{m_1h_1g_c\sin\alpha\cos\alpha\,\theta}{h_2}$$

$$+ \frac{m_1h_1g_c\cos\alpha\,\phi}{h_2} + \frac{m_1h_1a_1\sin\alpha\cos\alpha\,\ddot{\phi}}{h_2}$$

$$- \frac{m_1h_1{}^2\cos^2\alpha\,\ddot{\phi}}{h_2} + \frac{m_1h_1a_1r_2\Omega\cos^3\alpha\,\dot{\theta}}{h_2R}$$

$$+ \frac{m_1h_1{}^2r_2\,\Omega\sin\alpha\cos^2\alpha\,\dot{\theta}}{h_2R}$$

$$- \frac{m_1h_1r_2\,\Omega\cos^2\alpha\,\dot{\theta}}{h_2} + \frac{m_1a_1h_1\cos\alpha\,\ddot{\theta}}{h_2}$$

$$- \frac{m_1h_1(r_2\,\Omega)^2\cos^2\alpha\,\theta}{h_2R}$$

$$- \frac{\cos\alpha}{h_2}\left[\, I_{x1}\cos\alpha\,\ddot{\phi} - P_{xz1}\sin\alpha\,\ddot{\phi}\right.$$

$$- \frac{I_{x1}r_2\Omega\sin\alpha\cos\alpha\,\dot{\theta}}{R}$$

$$- \frac{P_{xz1} r_2 \Omega \cos^2\alpha\, \dot{\theta}}{R} + \frac{I_1' r_2 \Omega \dot{\theta}}{r_1}$$

$$- P_{xz1} \ddot{\theta} + \frac{I_1' r_2 \Omega \sin\alpha\, \dot{\phi}}{r_1}$$

$$+ \left. \frac{I_1' (r_2 \Omega)^2 \cos^2\alpha\, \theta}{r_1 R} \right] + \frac{1}{h_2} \Big[F_{g4} h_2 \cos\alpha\, \theta$$

$$+ c_2 (r_2 \Omega)^2 (b_2 - h_2) \cos\alpha\, \theta$$

$$- m_2 g_c a_2 \cos\alpha\, \theta - I_{x2} \ddot{\phi} + \frac{P_{xz2} r_2 \Omega \cos\alpha\, \dot{\theta}}{R}$$

$$- \left. \frac{I_2' r_2 \Omega^2 \cos\alpha\, \theta}{R} \right] + m_2 g_c \phi$$

$$- \frac{m_2 a_2 r_2 \Omega \cos\alpha\, \dot{\theta}}{R} - \frac{m_2 (r_2 \Omega)^2 \cos\alpha\, \theta}{R}$$

$$- \Big[(a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha) \frac{\cos\alpha}{h_2}$$

$$+ 1 + \left. \frac{d_2}{h_2} \right] F_1 \theta + (a_2 \cos\alpha + l_2 \cos\alpha$$

$$+ d_2 \sin\alpha) \frac{\cos\alpha}{h_2} F_3 \theta$$

(4.8 c) $$F_1 d_1 = (a_1 - l_1) F_3 + a_1 (- c_1 (r_2 \Omega)^2 \sin\alpha$$

$$+ m_1 g_c \cos\alpha) - h_1 (F_1 - c_1 (r_2 \Omega)^2 \cos\alpha$$

$$- m_1 g_c \sin\alpha) + F_{g4} h_2 + c_2 (r_2 \Omega)^2 (b_2 - h_2)$$

$$- m_2 a_2 g_c - (a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha) F_1$$

$$- (a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha + h_1$$

$$+d_1)F_2\theta + (a_2 \cos\alpha + l_2 \cos\alpha$$

$$+d_2 \sin\alpha)F_3 + c_2(r_2\Omega)^2(b_1 - h_1 \cos\alpha)$$

$$\text{(4.12 c)} \quad F_2 l_2 = - P_{xz2}\ddot{\phi} - I_2'\Omega\dot{\phi} + (\frac{a_2 \sin\alpha}{h_2}$$

$$- \cos\alpha)M_3 + \Bigg[\frac{a_2 d_2}{h_2} - (\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha)(h_1 + d_1)\Bigg] F_2 - (\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha \Bigg\{ h_1 \Bigg[- m_1 g_c \sin\alpha\, \theta - m_1 g_c \phi$$

$$- (a_1 \sin\alpha - h_1 \cos\alpha) m_1 \ddot{\phi} - (\frac{a_1 \cos\alpha}{R}$$

$$+ \frac{h_1 \sin\alpha}{R} - 1) m_1 r_2 \Omega \cos\alpha\, \dot{\theta} - m_1 a_1 \ddot{\theta}$$

$$+ \frac{m_1 (r_2\Omega)^2 \cos\alpha\, \theta}{R} \Bigg] + (I_{x1} \cos\alpha$$

$$- P_{xz1} \sin\alpha)\ddot{\phi} - \Bigg[(I_{x1} \sin\alpha$$

$$+ P_{xz1} \cos\alpha)\frac{\cos\alpha}{R} - \frac{I_1'}{r_1} \Bigg] r_2\Omega\dot{\theta}$$

$$- P_{xz1}\ddot{\theta} + \frac{I_1' r_2 \Omega \sin\alpha\, \dot{\phi}}{r_1}$$

$$+ \frac{I_1'(r_2\Omega)^2 \cos^2\alpha\, \theta}{r_1 R} \Bigg\} - (\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha) \Bigg[- F_{g4} h_2 \theta - c_2 (r_2\Omega)^2 (b_2 - h_2)\theta$$

$$+ m_2 g_c a_2 \theta + (a_2 \sin\alpha + l_2 \sin\alpha$$

$$- d_2 \cos \alpha)F_1\theta - (a_2 \cos\alpha + l_2 \cos\alpha$$

$$+ d_2 \sin \alpha)F_3\theta\Big] - \frac{a_2\, I_{x2}\, \ddot{\phi}}{h_2} + (l_2$$

$$- \frac{a_2 d_2}{h_2})F_1\theta + \frac{P_{xz2}a_2 r_2\, \Omega \cos \alpha \dot{\theta}}{h_2 R}$$

$$- \frac{I_2' a_2 r_2\, \Omega \cos \alpha\, \theta}{h_2 R} + \frac{I_{z2} r_2\, \Omega \cos \alpha\, \dot{\theta}}{R}$$

Factoring F_1 from Equation (4.8 c) and F_2 from Equation (4.12 c) gives

(4.8 d) $$F_1(a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha + h_1$$

$$+ d_1) = (a_1 - l_1 + a_2 \cos\alpha + l_2 \cos\alpha$$

$$+ d_2 \sin \alpha)F_3 - (a_2 \sin\alpha + l_2 \sin\alpha$$

$$- d_2 \cos\alpha + h_1 + d_1)F_2\theta - a_1 c_1(r_2\Omega)^2 \sin\alpha$$

$$+ m_1 g_c a_1 \cos\alpha + c_1 h_1 (r_2\Omega)^2 \cos\alpha$$

$$+ m_1 g_c h_1 \sin\alpha + F_{g4} h_2 + c_2(r_2\Omega)^2(b_2 - h_2)$$

$$- m_2 g_c a_2 + c_1(r_2\Omega)^2(b_1 - h_1 \cos\alpha)$$

(4.12 d) $$F_2\Big\{l_2\Big[- \frac{a_2 d_2}{h_2} - (\frac{a_2 \cos\alpha}{h_2} + \sin\alpha)(h_1$$

$$+ d_1)\Big]\Big\} = - P_{xz2}\ddot{\phi} - I_2'\Omega\dot{\phi} + (\frac{a_2 \sin\alpha}{h_2}$$

$$- \cos\alpha)M_3 - (\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha)\Big[- m_1 g_c h_1 \sin\alpha\, \theta - m_1 g_c h_1 \phi$$

$$- m_1 a_1 h_1 \sin\alpha\,\ddot{\phi} + m_1 h_1^2 \cos\alpha\,\ddot{\phi} - \Big(\frac{a_1 \cos\alpha}{R}$$

$$+ \frac{h_1 \sin\alpha}{R} - 1\Big) m_1 h_1 r_2 \Omega \cos\alpha\,\dot{\theta} - m_1 a_1 h_1 \ddot{\theta}$$

$$+ \frac{m_1 h_1 \cos\alpha\,(r_2\Omega)^2\theta}{R} + I_{x1} \cos\alpha\,\ddot{\phi}$$

$$- P_{xz1} \sin\alpha\,\ddot{\phi} - \frac{I_{x1} r_2 \Omega \sin\alpha \cos\alpha\,\dot{\theta}}{R}$$

$$- \frac{P_{xz1} r_2 \Omega \cos^2\alpha\,\dot{\theta}}{R} + \frac{I'_1 r_2 \Omega\,\dot{\theta}}{r_1} - P_{xz1}\ddot{\theta}$$

$$+ \frac{I'_1 r_2 \Omega \sin\alpha\,\dot{\phi}}{r_1} + \frac{I'_1 (r_2\Omega)^2 \cos^2\alpha\,\theta}{r_1 R}\Bigg]$$

$$+ \Bigg[l_2 - \frac{a_2 d_2}{h_2} - \Big(\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha\Big)(a_2 \sin\alpha + l_2 \sin\alpha$$

$$- d_2 \cos\alpha)\Bigg] F_1 \theta + \Big(\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha\Big)(a_2 \cos\alpha + l_2 \cos\alpha$$

$$+ d_2 \sin\alpha) F_3 \theta - \Big(\frac{a_2 \cos\alpha}{h_2}$$

$$+ \sin\alpha\Big) \Big[- F_{g4} h_2 \theta - c_2 (r_2\Omega)^2 (b_2 - h_2)\theta$$

$$+ m_2 g_c a_2 \theta \Big] + \frac{P_{xz2} a_2 r_2 \Omega \cos\alpha\,\dot{\theta}}{h_2 R}$$

$$- \frac{I'_2 a_2 r_2 \Omega^2 \cos\alpha\,\theta}{h_2 R} + \frac{I_{z2} r_2 \Omega \cos\alpha\,\dot{\theta}}{R}$$

$$- \frac{a_2 I_{x2} \ddot{\phi}}{h_2}$$

For convenience in writing, some definitions are made at this point. In a few instances the constants defined here and on the following pages can be conveniently modified or else eliminated entirely; hence the final forms used for the computer solution will be given at the beginning of Chapter V.

$$K_1 \equiv l_2 - \left[\frac{a_2 d_2}{h_2} - \left(\frac{a_2 \cos\alpha}{h_2} + \sin\alpha\right)(h_1 + d_1) \right]$$

$$K_2 \equiv a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha + h_1 + d_1$$

$$K_3 \equiv a_1 - l_1 + a_2 \cos\alpha + l_2 \cos\alpha + d_2 \sin\alpha$$

$$K_4 \equiv K_2 + K_3 \frac{\cos\alpha}{\sin\alpha}$$

$$K_5 \equiv K_2 - d_1$$

$$K_6 \equiv \frac{a_2}{h_2} \cos\alpha + \sin\alpha$$

$$K_7 \equiv K_4 + K_3 \frac{\cos\alpha}{\sin\alpha}$$

K_8 not defined

$$K_9 \equiv \left\{ \left[l_2 - \frac{a_2 d_2}{h_2} - K_6(a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha) \right] - K_6 \left[a_2 \cos\alpha + l_2 \cos\alpha + d_2 \sin\alpha \right] \frac{\cos\alpha}{\sin\alpha} \right\}$$

$$K_{10} \equiv (a_2 \cos\alpha + l_2 \cos\alpha + d_2 \sin\alpha)$$

$$K_{11} \equiv (a_2 \sin\alpha + l_2 \sin\alpha - d_2 \cos\alpha)$$

$$K_{12} \equiv \left[\frac{(h_1 + d_1)\cos\alpha}{h_2} - 1 - \frac{d_2}{h_2} \right]$$

Eliminating Equation (4.4) by substituting it into Equations (4.5 c), (4.8 d), and (4.12 d) gives

$$(4.5\ d) \quad m_2 h_2 \ddot{\phi} = \frac{M_3 \sin\alpha}{h_2} + \frac{m_1 g_c h_1 \sin\alpha \cos\alpha\, \theta}{h_2}$$

$$+ \frac{m_1 g_c h_1 \cos\alpha\, \phi}{h_2} + \frac{m_1 a_1 h_1 \sin\alpha \cos\alpha\, \ddot{\phi}}{h_2}$$

$$- \frac{m_1 h_1^2 \cos^2\alpha\, \ddot{\phi}}{h_2} + \frac{m_1 h_1 a_1 r_2 \Omega \cos^3\alpha\, \dot{\theta}}{h_2 R}$$

$$+ \frac{m_1 h_1 r_2 \Omega \sin\alpha \cos^2\alpha\, \dot{\theta}}{h_2 R} - \frac{m_1 h_1 r_2 \Omega \cos^2\alpha\, \dot{\theta}}{h_2}$$

$$+ \frac{m_1 a_1 h_1 \cos\alpha\, \ddot{\theta}}{h_2} - \frac{m_1 h_1 (r_2 \Omega)^2 \cos^2\alpha\, \theta}{h_2 R}$$

$$- \frac{\cos\alpha}{h_2} \left[I_{x1} \cos\alpha\, \ddot{\phi} - P_{xz1} \sin\alpha\, \ddot{\phi} \right.$$

$$- \frac{I_{x1} r_2 \Omega \sin\alpha \cos\alpha\, \dot{\theta}}{R} - \frac{P_{xz1} r_2 \Omega \cos^2\alpha\, \dot{\theta}}{R}$$

$$+ \frac{I_1' r_2 \Omega \dot{\theta}}{r_1} - P_{xz1} \ddot{\theta} - \frac{I_1' r_2 \Omega \sin\alpha\, \dot{\phi}}{r_1}$$

$$\left. + \frac{I_1' (r_2 \Omega)^2 \cos^2\alpha\, \theta}{r_1 R} \right] + \frac{1}{h_2} \left[F_{g4} h_2 \cos\alpha\, \theta \right.$$

$$+ c_2 (r_2 \Omega)^2 (b_2 - h_2) \cos\alpha\, \theta - m_2 g_c a_2 \cos\alpha\, \theta$$

$$\left. - I_{x2} \ddot{\phi} + \frac{P_{xz2} r_2 \Omega \cos\alpha\, \dot{\theta}}{R} - \frac{I_2' r_2 \Omega^2 \cos\alpha\, \theta}{R} \right]$$

$$+ m_2 g_c \phi - \frac{m_2 a_2 r_2 \Omega \cos\alpha\, \dot{\theta}}{R}$$

$$- \frac{m_2 (r_2\Omega)^2 \cos\alpha\, \theta}{R} - \Big[(a_2 \sin\alpha$$

$$+ l_2 \sin\alpha - d_2 \cos\alpha) \frac{\cos\alpha}{h_2} + 1 + \frac{d_2}{h_2}$$

$$+ \frac{\cos^2\alpha}{h_2 \sin\alpha} (a_2 \cos\alpha + l_2 \cos\alpha$$

$$+ d_2 \sin\alpha) \Big] F_1 \theta + (a_2 \cos\alpha + l_2 \cos\alpha$$

$$+ d_2 \sin\alpha) \frac{\cos\alpha\, \theta}{h_2 \sin\alpha} \Big[F_{g4} - c_2 (r_2\Omega)^2 \Big]$$

$$(4.8\ e) \quad F_1 \Big(K_2 + \frac{K_3 \cos\alpha}{\sin\alpha}\Big) = - \Big(K_2 + \frac{K_3 \cos\alpha}{\sin\alpha}\Big) F_2 \theta$$

$$+ \frac{K_3}{\sin\alpha} \Big[F_{g4} - c_2 (r_2\Omega)^2 \Big] - c_1 a_1 (r_2\Omega)^2 \sin\alpha$$

$$+ m_1 g_c a_1 \cos\alpha + c_1 h_1 (r_2 \Omega)^2 \cos\alpha$$

$$+ m_1 g_c h_1 \sin\alpha + F_{g4} h_2 + c_2 (r_2\Omega)^2 (b_2 - h_2)$$

$$- m_2 g_c a_2 + c_1 (r_2\Omega)^2 (b_1 - h_1 \cos\alpha)$$

$$(4.12\ e) \quad F_2 K_1 = \Big[l_2 - \frac{a_2 d_2}{h_2} - K_6 (a_2 \sin\alpha$$

$$+ l_2 \sin\alpha - d_2 \cos\alpha) - K_6 (a_2 \cos\alpha$$

$$+ l_2 \cos\alpha + d_2 \sin\alpha) \frac{\cos\alpha}{\sin\alpha} \Big] F_1 \theta$$

$$- P_{xz2} \ddot{\phi} - I_2' \Omega \dot{\phi} + \Big(\frac{a_2 \sin\alpha}{h_2}$$

$$- \cos\alpha)M_3 - K_6\Big[- m_1 g_c h_1 \sin\alpha\,\theta$$

$$- m_1 g_c h_1 \phi - m_1 a_1 h_1 \sin\alpha\,\ddot{\phi}$$

$$+ m_1 h_1^2 \cos\alpha\,\ddot{\phi} - (\frac{a_1 \cos\alpha}{R} + \frac{h_1 \sin\alpha}{R}$$

$$- 1) m_1 h_1 r_2 \Omega \cos\alpha\,\dot{\theta} - m_1 a_1 h_1 \ddot{\theta}$$

$$+ \frac{m_1 h_1 (r_2\Omega)^2 \cos\alpha\,\theta}{R} + I_{x1} \cos\alpha\,\ddot{\phi}$$

$$- P_{xz1} \sin\alpha\,\ddot{\phi} - \frac{I_{x1} r_2 \Omega \sin\alpha \cos\alpha\,\dot{\theta}}{R}$$

$$- \frac{P_{xz1} r_2 \Omega \cos^2\alpha\,\dot{\theta}}{R} + \frac{I_1' r_2 \Omega\,\dot{\theta}}{r_1} - P_{xz1}\ddot{\theta}$$

$$+ \frac{I_1' r_2 \Omega \sin\alpha\,\dot{\phi}}{r_1} + \frac{I_1' (r_2\Omega)^2 \cos^2\alpha\,\theta}{r_1 R}\Big]$$

$$+ \frac{K_6 K_{10}}{\sin\alpha}\Big[F_{g4} - c_2 (r_2\Omega)^2\Big]\theta + K_6\Big[F_{g4} h_2$$

$$+ c_2 (r_2\Omega)^2 (b_2 - h_2) - m_2 g_c a_2\Big]\theta$$

$$+ \frac{P_{xz2} a_2 r_2 \Omega \cos\alpha\,\dot{\theta}}{h_2 R} - \frac{I_2' a_2 r_2 \Omega^2 \cos\alpha\,\theta}{h_2 R}$$

$$+ \frac{I_{z2} r_2 \Omega \cos\alpha\,\dot{\theta}}{R} - \frac{a_2 I_{x2} \ddot{\phi}}{h_2}$$

Cancelling the higher order terms, Equations (4.8 e) and (4.12 e) can be substituted into Equations (4.5 d) and (4.9 a) to give

$$(4.5\ e)\quad m_2 h_2 \ddot{\phi} = \frac{M_3 \sin\alpha}{h_2} - \frac{K_{12}}{K_1}\left\{ \frac{K_9}{K_4} \left[\frac{K_3 F_{g4} \theta}{\sin\alpha} \right.\right.$$

$$- \frac{K_3 c_1 (r_2 \Omega)^2 \theta}{\sin\alpha} - c_1 a_1 (r_2\Omega)^2 \sin\alpha\, \theta$$

$$+ m_1 g_c a_1 \cos\alpha\, \theta + c_1 h_1 (r_2\Omega)^2 \cos\alpha\, \theta$$

$$+ m_1 g_c h_1 \sin\alpha\, \theta + F_{g4} h_2 \theta$$

$$+ c_2 (r_2\Omega)^2 (b_2 - h_2)\theta - m_2 g_c a_2 \theta$$

$$\left. + c_1 (r_2\Omega)^2 (b_1 - h_1 \cos\alpha)\theta \right] - I_2' \Omega \dot{\phi}$$

$$+ \frac{a_2 \sin\alpha\, M_3}{h_2} - \cos\alpha\, M_3$$

$$+ K_6 \Big[m_1 g_c h_1 \sin\alpha\, \theta + m_1 g_c h_1 \phi$$

$$+ a_1 h_1 m_1 \sin\alpha\, \ddot{\phi} - m_1 h_1^2 \cos\alpha\, \ddot{\phi}$$

$$+ \frac{m_1 a_1 h_1 r_2 \Omega \cos^2\alpha\, \dot{\theta}}{R} + \frac{m_1 h_1^2 r_2 \Omega \sin\alpha \cos\alpha\, \dot{\theta}}{R}$$

$$- m_1 h_1 r_2 \Omega \cos\alpha\, \dot{\theta} + m_1 a_1 h_1 \ddot{\theta}$$

$$- \frac{m_1 h_1 (r_2\Omega)^2 \cos\alpha\, \theta}{R} - I_{x1} \cos\alpha\, \ddot{\phi}$$

$$+ P_{xz1} \sin\alpha\, \ddot{\phi} + \frac{I_{x1} r_2 \Omega \sin\alpha \cos\alpha\, \dot{\theta}}{R}$$

$$+ \frac{P_{xz1} r_2 \Omega \cos^2\alpha\, \dot{\theta}}{R} - \frac{I_1' r_2 \Omega \dot{\theta}}{r_1} + P_{xz1} \ddot{\theta}$$

$$- \frac{I_1' r_2 \Omega \sin\alpha \, \dot{\phi}}{r_1} - \frac{I_1' (r_2 \Omega)^2 \cos^2\alpha \, \theta}{r_1 R}$$

$$+ \frac{K_{10} F_{g4} \, \theta}{\sin\alpha} - \frac{K_{10} c_2 (r_2 \Omega)^2 \theta}{\sin\alpha} + h_2 F_{g4} \, \theta$$

$$\left. + c_2 (r_2 \Omega)^2 (b_2 - h_2) - m_2 g_c a_2 \, \theta \right]$$

$$+ \frac{P_{xz2} a_2 r_2 \Omega \cos\alpha \, \dot{\theta}}{h_2 R} - \frac{I_2' a_2 r_2 \Omega^2 \cos\alpha \, \theta}{h_2 R}$$

$$\left. + \frac{I_{z2} r_2 \Omega \cos\alpha \, \dot{\theta}}{R} - \frac{a_2 I_{x2} \, \ddot{\phi}}{h_2} \right\}$$

$$+ \frac{m_1 g_c h_1 \sin\alpha \cos\alpha \, \theta}{h_2} + \frac{m_1 g_c h_1 \cos\alpha \, \phi}{h_2}$$

$$+ \frac{m_1 a_1 h_1 \sin\alpha \cos\alpha \, \ddot{\phi}}{h_2} - \frac{m_1 h_1^2 \cos\alpha \, \ddot{\phi}}{h_2}$$

$$+ \frac{m_1 a_1 h_1 r_2 \Omega \cos^3\alpha \, \dot{\theta}}{h_2 R}$$

$$+ \frac{m_1 h_1^2 r_2 \Omega \sin\alpha \cos^2\alpha \, \dot{\theta}}{h_2 R}$$

$$- \frac{m_1 h_1 r_2 \Omega \cos^2\alpha \, \dot{\theta}}{h_2} + \frac{m_1 a_1 h_1 \cos\alpha \, \ddot{\theta}}{h_2}$$

$$- \frac{m_1 h_1 (r_2 \Omega)^2 \cos^2\alpha \, \theta}{h_2 R} - \frac{I_{x1} \cos^2\alpha \, \ddot{\phi}}{h_2}$$

$$+ \frac{P_{xz1} \sin\alpha \cos\alpha \, \ddot{\phi}}{h_2}$$

$$+ \frac{I_{x1} r_2 \Omega \sin\alpha \cos^2\alpha \, \dot{\theta}}{h_2 R}$$

$$+ \frac{P_{xz1} r_2 \Omega \cos^3\alpha\, \dot{\theta}}{h_2 R} - \frac{I_1' r_2 \Omega \dot{\theta}}{h_2 r_1}$$

$$+ \frac{P_{xz1} \cos\alpha\, \ddot{\theta}}{h_2} + \frac{I_1' r_2 \Omega \sin\alpha \cos\alpha\, \dot{\phi}}{h_2 r_1}$$

$$- \frac{I_1'(r_2\Omega)^2 \cos^3\alpha\, \theta}{h_2 r_1 R} + F_{g4} \cos\alpha\, \theta$$

$$+ \frac{c_2 (r_2\Omega)^2 (b_2 - h_2) \cos\alpha\, \theta}{h_2}$$

$$- \frac{m_2 g_c a_2 \cos\alpha\, \theta}{h_2} - \frac{I_{x2}\, \ddot{\phi}}{h_2}$$

$$+ \frac{P_{xz2} r_2 \Omega \cos\alpha\, \dot{\theta}}{h_2 R} - \frac{I_2' r_2 \Omega^2 \cos\alpha\, \theta}{h_2 R}$$

$$+ m_2 g_c \phi - \frac{m_2 a_2 r_2 \Omega \cos\alpha\, \dot{\theta}}{R}$$

$$- \frac{m_2 (r_2\Omega)^2 \cos\alpha\, \theta}{R} - \left[\frac{K_{11} \cos}{h_2} + 1 \right.$$

$$\left. + \frac{d_2}{h_2} + \frac{K_{10} \cos^2\alpha}{h_2 \sin\alpha} \right] \frac{1}{K_4} \left[\frac{K_3 F_{g4} \theta}{\sin\alpha} \right.$$

$$- \frac{K_3 c_1 (r_2\Omega)^2 \theta}{\sin\alpha} - c_1 a_1 (r_2\Omega)^2 \sin\alpha\, \theta$$

$$+ m_1 g_c a_1 \cos\alpha\, \theta + c_1 h_1 (r_2\Omega)^2 \cos\alpha\, \theta$$

$$+ m_1 g_c h_1 \sin\alpha\, \theta + F_{g4} h_2 \theta$$

$$+ c_2 (r_2\Omega)^2 (b_2 - h_2) \theta - m_2 g_c a_2 \theta$$

$$+ c_1(r_2\Omega)^2(b_1 - h_1 \cos \alpha)\theta \Big]$$

$$+ \frac{K_{10}\cos \alpha F_{g4}\,\theta}{h_2 \sin \alpha} - \frac{K_{10}\cos \alpha \; c_2(r_2\Omega)^2\theta}{h_2 \sin \alpha}$$

$$(4.9\ b) \quad I_{z1}\ddot{\theta} = \frac{a_1 - l_1}{K_1}\Big\{ \frac{K_9}{K_4}\Big[\frac{F_{g4}K_3\theta}{\sin \alpha}$$

$$- \frac{K_3c_1(r_2\Omega)^2}{\sin \alpha} - c_1a_1(r_2\Omega)^2\sin \alpha\,\theta$$

$$+ m_1g_ca_1 \cos \alpha\,\theta + c_1h_1(r_2\Omega)^2\cos \alpha\,\theta$$

$$+ m_1g_ch_1 \sin \alpha\,\theta + F_{g4}h_2\theta + c_2b_2(r_2\Omega)^2\theta$$

$$- c_2h_2(r_2\Omega)^2\theta - m_2g_ca_2\theta + c_1b_1(r_2\Omega)^2\theta$$

$$- c_1h_1(r_2\Omega)^2\cos \alpha\,\theta \Big] - P_{xz2}\ddot{\phi} - I_2'\Omega\dot{\phi}$$

$$+ \left(\frac{a_2 \sin \alpha}{h_2} - \cos \alpha\right)M_3$$

$$+ K6 \Big[m_1g_ch_1 \sin \alpha\,\theta + m_1g_ch_1\phi$$

$$+ m_1a_1h_1 \sin \alpha\,\ddot{\phi} - m_1h_1^2 \cos \alpha\,\ddot{\phi}$$

$$+ \frac{m_1a_1h_1r_2\Omega \cos^2 \alpha\,\dot{\theta}}{R}$$

$$+ \frac{m_1h_1^2\, r_2\Omega \sin \alpha \cos \alpha\,\dot{\theta}}{R} - m_1h_1r_2\Omega \cos \alpha\,\dot{\theta}$$

$$+ m_1a_1h_1\ddot{\theta} - \frac{m_1h_1(r_2\Omega)^2\cos \alpha\,\theta}{R}$$

$$- I_{x1} \cos \alpha\,\ddot{\phi} + P_{xz1} \sin \alpha\,\ddot{\phi}$$

$$+ \frac{I_{x1} r_2 \Omega \sin\alpha \cos\alpha \dot{\theta}}{R} + \frac{P_{xz1} r_2 \Omega \cos^2\alpha \dot{\theta}}{R}$$

$$- \frac{I_1' r_2 \Omega \dot{\theta}}{r_1} + P_{xz1} \ddot{\theta} - \frac{I_1' r_2 \Omega \sin\alpha \dot{\phi}}{r_1}$$

$$- \frac{I_1' (r_2 \Omega)^2 \cos^2\alpha \theta}{r_1 R} \Bigg] + \frac{K_6 \theta}{\sin\alpha} \Bigg[a_2 \cos\alpha F_{g4}$$

$$- c_2 a_2 (r_2 \Omega)^2 \cos\alpha + l_2 \cos\alpha F_{g4}$$

$$- c_2 l_2 (r_2 \Omega)^2 \cos\alpha + d_2 \sin\alpha F_{g4}$$

$$- c_1 d_2 (r_2 \Omega)^2 \sin\alpha \Bigg] + K_6 \theta \Bigg[h_2 F_{g4}$$

$$+ c_2 b_2 (r_2 \Omega)^2 - c_2 h_2 (r_2 \Omega)^2 - m_2 g_c a_2 \Bigg]$$

$$+ \frac{P_{xz2} a_2 r_2 \Omega \cos\alpha \dot{\theta}}{h_2 R} - \frac{I_2' a_2 r_2 \Omega^2 \cos\alpha \theta}{h_2 R}$$

$$+ \frac{I_{z2} r_2 \Omega \cos\alpha \dot{\theta}}{R} - \frac{a_2 I_{x2} \ddot{\phi}}{h_2} \Bigg\} - m_1 g_c a_1 \sin\alpha \theta$$

$$- m_1 g_c a_1 \phi - m_1 a_1^2 \sin\alpha \ddot{\phi} + m_1 a_1 h_1 \cos\alpha \ddot{\phi}$$

$$- \frac{m_1 a_1^2 r_2 \Omega \cos^2\alpha \dot{\theta}}{R} - \frac{m_1 a_1 h_1 r_2 \Omega \sin\alpha \cos\alpha \dot{\theta}}{}$$

$$+ m_1 a_1 r_2 \Omega \cos\alpha \dot{\theta} - m_1 a_1^2 \ddot{\theta}$$

$$- \frac{m_1 a_1 (r_2 \Omega)^2 \cos\alpha \theta}{R} - M_3 - I_{z1} \sin\alpha \ddot{\phi}$$

$$+ P_{xz1} \cos\alpha \ddot{\phi} - \frac{I_{z1} r_2 \Omega \cos^2\alpha \dot{\theta}}{R}$$

$$- \frac{P_{xz1} r_2 \Omega \sin\alpha \cos\alpha \Omega \dot{\theta}}{R}$$

$$+ \frac{I_1' r_2 \Omega \cos\alpha \dot{\phi}}{r_1} - \frac{I_1' (r_2 \Omega)^2 \sin\alpha \cos\alpha \theta}{r_1 R}$$

Collecting terms in Equations (4.5 e) and (4.9 b)

$$(4.5\ f) \quad \ddot{\phi} \Bigg\{ m_2 h_2 - \frac{K_{12} P_{xz2}}{K_1} + \frac{K_{12} K_6}{K_1} \Big[m_1 a_1 h_1 \sin\alpha - m_1 h_1^2 \cos\alpha - I_{x1} \cos\alpha + P_{xz1} \sin\alpha \Big]$$

$$- \frac{K_{12}}{K_1} \frac{a_2 I_{x2}}{h_2} - \frac{m_1 a_1 h_1 \sin\alpha \cos\alpha}{h_2}$$

$$+ \frac{m_1 h_1 \cos^2\alpha}{h_2} + \frac{I_{x1} \cos^2\alpha}{h_2} - \frac{P_{xz1} \sin\alpha \cos\alpha}{h_2}$$

$$+ \frac{I_{x2}}{h_2} \Bigg\} = \ddot{\theta} \Bigg\{ - \frac{K_{12} K_6 m_1 a_1 h_1}{K_1} - \frac{K_{12} K_6 P_{xz1}}{K_1}$$

$$+ \frac{m_1 a_1 h_1 \cos\alpha}{h_2} + \frac{P_{xz1} \cos\alpha}{h_2} \Bigg\} + M_3 \Bigg\{ \frac{\sin\alpha}{h_2}$$

$$- \frac{K_{12}}{K_1} \frac{a_2 \sin\alpha}{h_2} + \frac{K_{12} \cos\alpha}{K_1} \Bigg\}$$

$$+ F_{g4} \theta \Bigg\{ \frac{K_{12}}{K_1} \Bigg[- \frac{K_9 K_3}{K_4 \sin\alpha} - \frac{K_9 h_2}{K_4} - \frac{K_{10} K_6}{\sin\alpha}$$

$$- K_6 h_2 \Bigg] + \cos\alpha + \frac{K_{10} \cos\alpha}{h_2 \sin\alpha} - \frac{1}{K_4} \Bigg[\frac{K_3}{\sin\alpha}$$

$$+ h_2 + \frac{K_3 d_2}{h_2 \sin\alpha} + d_2 + \frac{K_3 K_{10} \cos^2\alpha}{h_2 \sin^2\alpha}$$

$$+ \frac{K_{10} \cos^2\alpha}{\sin\alpha} + \frac{K_3K_{11} \cos\alpha}{h_2 \sin\alpha} + K_{11}\cos\alpha \Big]\Big\}$$

$$+ \Theta (r\Omega)^2 \Big\{ \frac{K_{12}K_9}{K_1K_4} \Big[\frac{K_3c_1}{\sin\alpha} + c_1a_1 \sin\alpha$$

$$- c_1h_1 \cos\alpha - c_2(b_2 - h_2)$$

$$- c_1(b_1 - h_1 \cos\alpha) \Big] + \frac{K_{12}K_{10}K_6c_2}{K_1 \sin\alpha}$$

$$- \frac{K_{12}K_6c_2(b_2 - h_2)}{K_1} + \frac{c_1}{K_4} \Big[\frac{K_{11}K_3 \cos\alpha}{h_2 \sin\alpha}$$

$$+ \frac{K_{11}a_1 \sin\alpha \cos\alpha}{h_2} - \frac{K_{11}h_1 \cos^2\alpha}{h_2}$$

$$- \frac{K_{11}(b_1 - h_1 \cos\alpha)\cos\alpha}{h_2} + \frac{K_3}{\sin\alpha}$$

$$+ a_1 \sin\alpha - h_1 \cos\alpha - (b_1 - h_1 \cos\alpha)$$

$$+ \frac{K_3d_2}{h_2 \sin\alpha} + \frac{a_1d_2 \sin\alpha}{h_2} - \frac{h_1d_2 \cos\alpha}{h_2}$$

$$- \frac{d_2(b_1 - h_1 \cos\alpha)}{h_2} + \frac{K_3K_{10} \cos^2\alpha}{h_2 \sin^2\alpha}$$

$$+ \frac{K_{10}a_1 \cos^2\alpha}{h_2} - \frac{K_{10}h_1 \cos^3\alpha}{h_2 \sin\alpha}$$

$$- \frac{K_{10} \cos^2\alpha (b_1 - h_1 \cos\alpha)}{h_2 \sin\alpha} \Big]$$

$$+ c_2 \Big[(b_2 - h_2)\frac{\cos\alpha}{h_2}$$

$$- \frac{K_{11} \cos\alpha (b_2 - h_2)}{K_4 h_2} - \frac{(b_2 - h_2)}{K_4}$$

$$- \frac{d_2(b_2 - h_2)}{h_2 K_4} - \frac{K_{10}(b_2 - h_2)\cos^2\alpha}{h_2 K_4 \sin\alpha}$$

$$- \frac{K_{10} \cos\alpha}{h_2 \sin\alpha} \Bigg] \Bigg\}$$

$$+ \theta g_c \Bigg\{ m_1 \Bigg[\frac{K_{12} K_9 (- a_1 \cos\alpha - h_1 \sin\alpha)}{K_1 K_4}$$

$$- \frac{K_{12} K_6 h_1 \sin\alpha}{K_1} + \frac{h_1 \sin\alpha \cos\alpha}{h_2}$$

$$+ \frac{1}{K_4} \Bigg(- \frac{K_{11} a_1 \cos^2\alpha}{h_2} - \frac{K_{11} h_1 \sin\alpha \cos\alpha}{h_2}$$

$$- a_1 \cos\alpha - h_1 \sin\alpha - \frac{a_1 d_2 \cos\alpha}{h_2}$$

$$- \frac{h_1 d_2 \sin\alpha}{h_2} - \frac{K_{10} a_1 \cos^3\alpha}{h_2 \sin\alpha}$$

$$- \frac{K_{10} h_1 \cos^2\alpha}{h_2} \Bigg) \Bigg] + m_2 a_2 \Bigg[\frac{K_{12} K_9}{K_1 K_4}$$

$$+ \frac{K_{12} K_6}{K_1} - \frac{\cos\alpha}{h_2} + \frac{K_{11} \cos\alpha}{K_4 h_2} + \frac{1}{K_4}$$

$$+ \frac{d_2}{h_2 K_4} + \frac{K_{10} \cos^2\alpha}{h_2 K_4 \sin\alpha} \Bigg] \Bigg\}$$

$$+ \dot{\phi} \Bigg\{ \frac{K_{12} I_2'}{K_1} + \frac{K_{12} K_6 I_1' r_2 \sin\alpha}{K_1 r_1}$$

$$- \frac{I_1' r_2 \sin\alpha \cos\alpha}{h_2 r_1} \Bigg\} + \phi g_c \Bigg\{ - \frac{K_{12} K_6 m_1 h_1}{K_1}$$

$$+ \frac{m_1 h_1 \cos\alpha}{h_2} + m_2 \Big\}$$

$$+ \dot{\theta}\Omega \Big\{ \frac{K_{12} K_6 r_2}{K_1} \Big[- \frac{m_1 a_1 h_1 \cos^2\alpha}{R}$$

$$- \frac{m_1 h_1^2 \sin\alpha \cos\alpha}{R} + m_1 h_1 \cos\alpha$$

$$- \frac{I_{x1} \sin\alpha \cos\alpha}{R} - \frac{P_{xz1} \cos^2\alpha}{R} + \frac{I_1'}{r_1} \Big]$$

$$- \frac{K_{12} r_2 \cos\alpha}{K_1 R} \Big[\frac{a_2 P_{xz2}}{h_2} + I_{z2} \Big]$$

$$+ r_2 \cos\alpha \Big[\frac{h_1 a_1 m_1 \cos^2\alpha}{h_2 R}$$

$$+ \frac{h_1^2 m_1 \sin\alpha \cos\alpha}{h_2 R} - \frac{h_1 m_1 \cos\alpha}{h_2}$$

$$+ \frac{I_{x1} \sin\alpha \cos\alpha}{h_2 R} + \frac{P_{xz1} \cos^2\alpha}{h_2 R}$$

$$- \frac{I_1'}{h_2 r_1} + \frac{P_{xz2}}{h_2 R} - \frac{m_2 a_2}{R} \Big] \Big\}$$

$$+ \theta\Omega^2 \Big\{ \frac{K_{12} K_6 m_1 h_1 r_2^2 \cos\alpha}{K_1 R}$$

$$+ \frac{K_{12} K_6 I_1' r_2^2 \cos^2\alpha}{K_1 r_1 R} + \frac{K_{12} I_2' a_2 r_2 \cos\alpha}{K_1 h_2 R}$$

$$- \frac{h_1 m_1 r_2^2 \cos^2\alpha}{h_2 R} - \frac{I_1' r_2^2 \cos^3\alpha}{h_2 r_1 R}$$

$$- \frac{I_2' r_2 \cos\alpha}{h_2 R} - \frac{m_2 r_2^2 \cos\alpha}{R} \Big\}$$

$$(4.9\ c)\quad \ddot{\theta}\Big[I_{z1} - \frac{a_1 - l_1}{K_1}(K_6 m_1 a_1 h_1 + K_6 P_{xz1}$$

$$+ m_1 a_1^2 \Big] = \ddot{\phi}\Big[\frac{a_1 - l_1}{K_1} (- P_{xz2}$$

$$+ K_6 a_1 h_1 m_1 \sin\alpha - K_6 m_1 h_1^2 \cos\alpha$$

$$- K_6 I_{x1} \cos\alpha + K_6 P_{xz1} \sin\alpha - \frac{a_2 I_{x2}}{h_2})$$

$$- m_1 a_1^2 \sin\alpha + m_1 a_1 h_1 \cos\alpha - I_{z1} \sin\alpha$$

$$+ P_{xz1} \cos\alpha \Big] + \frac{F_{g4}\, \theta\, a_1 - l_1}{K_1} \Big[\frac{K_9 K_3}{K_4 \sin\alpha}$$

$$+ \frac{K_9 h_2}{K_4} + \frac{K_6 a_2 \cos\alpha}{\sin\alpha} + \frac{K_6 l_2 \cos\alpha}{\sin\alpha} + K_6 d_2$$

$$+ K_6 h_2 \Big] + \theta\, (r_2 \Omega)^2\ \frac{a_1 - l_1}{K_1} \Big[\frac{K_9}{K_4} (- \frac{K_3 c_1}{\sin\alpha}$$

$$- a_1 c_1 \sin\alpha + c_1 h_1 \cos\alpha + c_2 b_2 - c_2 h_2$$

$$+ c_1 b_1 - c_1 h_1 \cos\alpha) - K_6 c_2 (\frac{a_2 \cos\alpha}{\sin\alpha}$$

$$+ \frac{l_2 \cos\alpha}{\sin\alpha} + d_2 - b_2 + h_2) \Big]$$

$$+ \theta_{gc} \Big[\frac{a_1 - l_1}{K_1} (\frac{K_9 m_1 h_1 \sin\alpha}{K_4}$$

$$+ \frac{K_9 m_1 a_1 \cos\alpha}{K_4} - \frac{K_9 m_2 a_2}{K_4} + K_6 m_1 h_1 \sin\alpha$$

$$- K_6 m_2 a_2) - m_1 a_1 \sin\alpha \Big] + \dot{\phi}\Omega \left[\frac{a_1 - l_1}{K_1} (- I_2'\right.$$

$$\left. - \frac{K_6 I_1' r_2 \sin\alpha}{r_1}) + \frac{I_1' r_2 \cos\alpha}{r_1} \right]$$

$$+ M_3 \left[\frac{a_1 - l_1}{K_1} (\frac{a_2 \sin\alpha}{h_2} - \cos\alpha) - 1 \right]$$

$$+ \phi\, m_1 g_c \left[\frac{a_1 - l_1}{K_1} \quad K_6 h_1 - a_1 \right]$$

$$+ \dot{\theta}\Omega \left[\frac{a_1 - l_1}{K_1} (\frac{K_6 a_1 \cos^2\alpha\, m_1 h_1 r_2}{R} \right.$$

$$+ \frac{K_6 h_1^2 m_1 r_2 \sin\alpha \cos\alpha}{R} - K_6 m_1 h_1 r_2 \cos\alpha$$

$$+ \frac{K_6 I_{x1} r_2 \sin\alpha \cos\alpha}{R} + \frac{K_6 r_2 P_{xz1} \cos^2\alpha}{R}$$

$$- \frac{K_6 I_1' r_2}{r_1} + \frac{P_{xz2} a_2 r_2 \cos\alpha}{h_2 R} + \frac{I_{z2} r_2 \cos\alpha}{R})$$

$$- \frac{m_1 a_1^2 r_2 \cos^2\alpha}{R} - \frac{m_1 a_1 h_1 r_2 \sin\alpha \cos\alpha}{R}$$

$$+ m_1 a_1 r_2 \cos\alpha - \frac{I_{z1} r_2 \cos^2\alpha}{R}$$

$$\left. - \frac{P_{xz1} r_2 \sin\alpha \cos\alpha}{R} \right]$$

$$+ \theta\Omega^2 \left[\frac{a_1 - l_1}{K_1} (- \frac{K_6 m_1 h_1 r_2^2}{R} - \frac{K_6 I_1' r_2^2 \cos^2\alpha}{r_1 R} \right.$$

$$- \frac{I_2' a_2 r_2 \cos\alpha}{h_2 R}) + \frac{m_1 a_1 r_2^2 \cos\alpha}{R}$$

$$- \frac{I_1' r_2^2 \sin\alpha \cos\alpha}{r_1 R} \Bigg]$$

Defining new constants for convenience in writing, Equations (4.5 f) and (4.9 c) can be written

$$\text{(4.9 d)} \quad K_{13}\ddot{\theta} = K_{14}\ddot{\phi} + K_{15}F_{g4}\theta + K_{16}\Omega^2\theta + K_{17}\theta + K_{18}\Omega\dot{\phi} + K_{19}M_3 + K_{20}\phi + K_{21}\Omega\dot{\theta} + K_{22}\Omega^2\theta$$

$$\text{(4.5 g)} \quad K_{24}\ddot{\phi} = K_{23}\ddot{\theta} + K_{25}M_3 + K_{26}F_{g4}\theta + K_{27}\Omega^2\theta + K_{28}\theta + K_{29}\Omega\dot{\phi} + K_{30}\phi + K_{31}\Omega\dot{\theta} + K_{32}\Omega^2\theta$$

where the constants are defined as follows:

$$K_{13} = I_{z1} - \frac{a_1 - l_1}{K_1}(K_6 m_1 a_1 h_1 + K_6 P_{xz1}) + m_1 a_1^2$$

$$K_{14} = \Bigg[\frac{a_1 - l_1}{K_1} (-P_{xz2} + K_6 a_1 h_1 m_1 \sin\alpha - K_6 m_1 h_1^2 \cos\alpha - K_6 I_{x1} \cos\alpha + K_6 P_{xz1} \sin\alpha - \frac{a_2 I_{x2}}{h_2}) - a_1^2 m_1 \sin\alpha + a_1 h_1 m_1 \cos\alpha - I_{z1} \sin\alpha + P_{xz1} \cos\alpha \Bigg]$$

$$K_{15} = \frac{a_1 - l_1}{K_1} \Bigg[\frac{K_9 K_3}{K_4 \sin\alpha} + \frac{K_9 h_2}{K_4} + \frac{K_6 a_2 \cos\alpha}{\sin\alpha} + \frac{K_6 l_2 \cos\alpha}{\sin\alpha} + K_6 d_2 + K_6 h_2 \Bigg]$$

$$K_{16} = \frac{a_1 - l_1}{K_1} r_2^2 \Bigg[- \frac{K_3 K_9 c_1}{K_4 \sin\alpha} - \frac{K_9 a_1 c_1 \sin\alpha}{K_4}$$

$$+ \frac{K_9 c_1 h_1 \cos\alpha}{K_4} + \frac{K_9 c_2 b_2}{K_4} - \frac{K_9 c_2 h_2}{K_4} + \frac{K_9 c_1 b_1}{K_4}$$

$$- \frac{K_9 c_1 h_1 \cos\alpha}{K_4} - \frac{K_6 a_2 \cos\alpha\, c_2}{\sin\alpha}$$

$$- \frac{K_6 l_2 \cos\alpha\, c_2}{\sin\alpha} - K_6 d_2 c_2 + K_6 c_2 b_2 - K_6 c_2 h_2 \Bigg]$$

$$K_{17} = \Bigg\{ \frac{a_1 - l_1}{K_1} \Bigg[\frac{K_9 m_1 h_1 g_c \sin\alpha}{K_4}$$

$$+ \frac{K_9 m_1 a_1 g_c \cos\alpha}{K_4} - \frac{K_9 m_2 a_2 g_c}{K_4}$$

$$+ K_6 h_1 m_1 g_c \sin\alpha - K_6 m_2 a_2 g_c \Bigg] - m_1 a_1 g_c \sin\alpha \Bigg\}$$

$$K_{18} = \Bigg\{ \frac{a_1 - l_1}{K_1} \Bigg[- I_2' - \frac{K_6 I_1' r_2 \sin\alpha}{r_1} \Bigg]$$

$$+ \frac{I_1' r_2 \cos\alpha}{r_1} \Bigg\}$$

$$K_{19} = \Bigg\{ \frac{a_1 - l_1}{K_1} \Bigg[\frac{a_2 \sin\alpha}{h_2} - \cos\alpha \Bigg] -1 \Bigg\}$$

$$K_{20} = \Bigg\{ \frac{a_1 - l_1}{K_1} \Big[K_6 h_1 m_1 g_c \Big] - m_1 a_1 g_c \Bigg\}$$

$$K_{21} = \Bigg\{ \frac{a_1 - l_1}{K_1} \Bigg[\frac{K_6 a_1 \cos^2\alpha\, m_1 h_1 r_2}{R}$$

$$+ \frac{K_6 h_1^2 m_1 r_2 \sin\alpha \cos\alpha}{R} - K_6 m_1 h_1 r_2 \cos\alpha$$

$$+ \frac{K_6 I_{x1} r_2 \sin\alpha \cos\alpha}{R}$$

$$+ \frac{K_6 r_2 P_{xz1} \cos^2\alpha}{R} - \frac{K_6 I_1' r_2}{r_1}$$

$$+ \frac{P_{xz2} a_2 r_2 \cos\alpha}{h_2 R} + \frac{I_{z2} r_2 \cos\alpha}{R} \Bigg]$$

$$- \frac{a_1^2 m_1 r_2 \cos^2\alpha}{R} - \frac{a_1 h_1 m_1 r_2 \sin\alpha \cos\alpha}{R}$$

$$+ m_1 a_1 r_2 \cos\alpha - \frac{I_{z1} r_2 \cos^2\alpha}{R}$$

$$- \frac{P_{xz1} r_2 \sin\alpha \cos\alpha}{R} \Bigg\}$$

$$K_{22} = \Bigg\{ \frac{a_1 - l_1}{K_1} \Bigg[- \frac{K_6 m_1 h_1 \cos\alpha \, r_2^2}{R}$$

$$- \frac{K_6 I_1' r_2^2 \cos^2\alpha}{r_1 R} - \frac{I_2' a_2 r_2 \cos\alpha}{h_2 R} \Bigg]$$

$$+ \frac{m_1 a_1 r_2^2 \cos\alpha}{R} - \frac{I_1' r_2^2 \cos\alpha \sin\alpha}{r_1 R} \Bigg\}$$

$$K_{23} = \Bigg[- \frac{K_{12} K_6 m_1 h_1 a_1}{K_1} - \frac{K_{12} K_6 P_{xz1}}{K_1} + \frac{m_1 a_1 h_1 \cos\alpha}{h_2}$$

$$+ \frac{P_{xz1} \cos\alpha}{h_2} \Bigg]$$

$$K_{24} = \Bigg[m_2 h_2 - \frac{K_{12} P_{xz2}}{K_1} + \frac{K_{12} K_6 h_1 a_1 m_1 \sin\alpha}{K_1}$$

$$- \frac{K_{12} K_6 m_1 h_1^2 \cos\alpha}{K_1} - \frac{K_{12} K_6 I_{x1} \cos\alpha}{K_1}$$

$$+ \frac{K_{12} K_6 P_{xz1} \sin\alpha}{K_1} - \frac{K_{12} a_2 I_{x2}}{K_1 h_2}$$

$$- \frac{m_1 h_1 a_1 \sin \alpha \cos \alpha}{h_2} + \frac{m_1 h_1^2 \cos^2 \alpha}{h_2}$$

$$\left. + \frac{I_{x1} \cos^2 \alpha}{h_2} - \frac{P_{xz1} \sin \alpha \cos \alpha}{h_2} + \frac{I_{x2}}{h_2} \right]$$

$$K_{25} = \left[\frac{\sin \alpha}{h_2} - \frac{K_{12} a_2 \sin \alpha}{K_1 h_2} + \frac{K_{12} \cos \alpha}{K_1} \right]$$

$$K_{26} = \left[- \frac{K_{12} K_9 K_3}{K_1 K_4 \sin \alpha} - \frac{K_{12} K_9 h_2}{K_1 K_4} - \frac{K_{12} K_{10} K_6}{K_1 \sin \alpha} \right.$$

$$- \frac{K_{12} K_6 h_2}{K_1} + \cos \alpha - \frac{K_3}{K_4 \sin \alpha} - \frac{h_2}{K_4}$$

$$- \frac{K_3 d_2}{K_4 h_2 \sin \alpha} - \frac{d_2}{K_4} - \frac{K_3 K_{10} \cos^2 \alpha}{K_4 h_2 \sin^2 \alpha}$$

$$- \frac{K_{10} \cos^2 \alpha}{K_4 \sin \alpha} + \frac{K_{10} \cos \alpha}{h_2 \sin \alpha} - \frac{K_{11} K_3 \cos \alpha}{K_4 h_2 \sin \alpha}$$

$$\left. - \frac{K_{11} \cos \alpha}{K_4} \right]$$

$$K_{27} = \frac{K_{12} K_9 K_3 c_1 r_2}{K_1 K_4 \sin \alpha} + \frac{K_{12} K_9 a_1 c_1 r_2 \sin \alpha}{K_1 K_4}$$

$$- \frac{K_{12} K_9 c_1 h_1 r_2 \cos \alpha}{K_1 K_4} - \frac{K_{12} K_9 c_2 r_2 (b_2 - h_2)}{K_1 K_4}$$

$$- \frac{K_{12} K_9 c_1 r_2 (b_1 - h_1 \cos \alpha)}{K_1 K_4}$$

$$+ \frac{K_{12} K_{10} K_6 c_2 r_2}{K_1 \sin \alpha} - \frac{K_{12} K_6 c_2 r_2 (b_2 - h_2)}{K_1}$$

$$+ \frac{c_2 r_2 (b_2 - h_2) \cos \alpha}{h_2} + \frac{K_3 K_{11} \cos \alpha \, c_1 r_2}{K_4 h_2 \sin \alpha}$$

$$+ \frac{K_{11} \cos \alpha \, a_1 c_1 r_2 \sin \alpha}{K_4 h_2}$$

$$- \frac{K_{11} \cos^2 \alpha \, c_1 h_1 r_2}{K_4 h_2} - \frac{K_{11} \cos \alpha \, c_2 r_2 (b_2 - h_2)}{K_4 h_2}$$

$$- \frac{K_{11} \cos \alpha \, c_1 r_2 (b_1 - h_1 \cos \alpha)}{K_4 h_2} + \frac{K_3 c_1 r_2}{K_4 \sin \alpha}$$

$$+ \frac{a_1 c_1 r_2 \sin \alpha}{K_4} - \frac{c_1 h_1 r_2 \cos \alpha}{K_4}$$

$$- \frac{c_2 r_2 (b_2 - h_2)}{K_4} - \frac{c_1 r_2 (b_1 - h_1 \cos \alpha)}{K_4}$$

$$+ \frac{K_3 d_2 c_1 r_2}{K_4 h_2 \sin \alpha} + \frac{a_1 c_1 r_2 d_2 \sin \alpha}{K_4 h_2}$$

$$- \frac{d_2 c_1 h_1 r_2 \cos \alpha}{K_4 h_2} - \frac{d_2 c_2 r_2 (b_2 - h_2)}{K_4 h_2}$$

$$- \frac{d_2 c_1 r_2 (b_1 - h_1 \cos \alpha)}{K_4 h_2}$$

$$+ \frac{K_3 K_{10} \cos^2 \alpha \, c_1 r_2}{K_4 h_2 \sin^2 \alpha} + \frac{K_{10} \cos^2 \alpha \, a_1 c_1 r_2}{K_4 h_2}$$

$$- \frac{K_{10} c_1 h_1 r_2 \cos^3 \alpha}{K_4 h_2 \sin \alpha} - \frac{K_{10} c_2 r_2 (b_2 - h_2) \cos^2 \alpha}{K_4 h_2 \sin \alpha}$$

$$- \frac{K_{10} \cos^2 \alpha \, c_1 r_2 (b_1 - h_1 \cos \alpha)}{K_4 h_2 \sin \alpha}$$

$$- \frac{K_{10} \cos \alpha \, c_2 r_2}{h_2 \sin \alpha}$$

$$K_{28} = \left[- \frac{K_{12}K_9 a_1 m_1 g_c \cos\alpha}{K_1 K_4} - \frac{K_{12}K_9 m_1 h_1 g_c \sin\alpha}{K_1 K_4} \right.$$

$$+ \frac{K_{12}K_9 m_2 a_2 g_c}{K_1 K_4} - \frac{K_{12}K_6 h_1 m_1 g_c \sin\alpha}{K_1}$$

$$+ \frac{K_{12}K_6 m_2 a_2 g_c}{K_1} + \frac{m_1 h_1 g_c \sin\alpha \cos\alpha}{h_2}$$

$$- \frac{m_2 a_2 g_c \cos\alpha}{h_2} - \frac{K_{11} \cos^2\alpha \, a_1 m_1 g_c}{K_4 h_2}$$

$$- \frac{K_{11} m_1 h_1 g_c \cos\alpha \sin\alpha}{K_4 h_2}$$

$$+ \frac{K_{11} \cos\alpha \, m_2 a_2 g_c}{K_4 h_2} - \frac{a_1 m_1 g_c \cos\alpha}{K_4}$$

$$- \frac{m_1 h_1 g_c \sin\alpha}{K_4} + \frac{m_2 a_2 g_c}{K_4} - \frac{d_2 a_1 m_1 g_c \cos\alpha}{h_2 K_4}$$

$$- \frac{d_2 m_1 h_1 g_c \sin\alpha}{h_2 K_4} + \frac{d_2 m_2 a_2 g_c}{h_2 K_4}$$

$$- \frac{K_{10} a_1 m_1 g_c \cos^3\alpha}{K_4 h_2 \sin\alpha} - \frac{K_{10} m_1 h_1 g_c \cos^2\alpha}{K_4 h_2}$$

$$\left. + \frac{K_{10} \cos^2\alpha \, m_2 a_2 g_c}{K_4 h_2 \sin\alpha} \right]$$

$$K_{29} = \left[\frac{K_{12} I_2'}{K_1} + \frac{K_{12} K_6 I_1' \frac{r_2}{r_1} \sin\alpha}{K_1} \right.$$

$$\left. - \frac{I_1' r_2 \cos\alpha \sin\alpha}{h_2 r_1} \right]$$

$$K_{30} = \left[- \frac{K_{12}K_6 h_1 m_1 g_c}{K_1} + \frac{m_1 h_1 g_c \cos\alpha}{h_2} + m_2 g_c \right]$$

$$K_{31} = \Bigg[- \frac{K_{12}K_6a_1 \cos^2\alpha\, m_1h_1r_2}{K_1R}$$

$$- \frac{K_{12}K_6h_1^2m_1r_2 \sin\alpha \cos\alpha}{K_1R}$$

$$+ \frac{K_{12}K_6m_1h_1r_2 \cos\alpha}{K_1} - \frac{K_{12}K_6I_{x1} \sin\alpha \cos\alpha\, r_2}{K_1R}$$

$$- \frac{K_{12}K_6P_{xz1} \cos^2\alpha\, r_2}{K_1R} + \frac{K_{12}K_6I_1'r_2}{K_1r_1}$$

$$- \frac{K_{12}P_{xz2}a_2r_2 \cos\alpha}{K_1h_2R} - \frac{K_{12}I_{z2}r_2 \cos\alpha}{K_1R}$$

$$+ \frac{h_1a_1m_1r_2 \cos^3\alpha}{h_2R} + \frac{h_1^2 \sin\alpha \cos^2\alpha\, m_1r_2}{h_2R}$$

$$- \frac{h_1 \cos^2\alpha\, m_1r_2}{h_2} + \frac{I_{x1} \sin\alpha \cos^2\alpha\, r_2}{h_2R}$$

$$+ \frac{P_{xz1}r_2 \cos^3\alpha}{h_2R} - \frac{I_1'r_2 \cos\alpha}{h_2r_1}$$

$$+ \frac{P_{xz2}r_2 \cos\alpha}{h_2R} - \frac{m_2a_2r_2 \cos\alpha}{R} \Bigg]$$

$$K_{32} = \Bigg[\frac{K_{12}K_6m_1h_1 \cos\alpha\, r_2^2}{K_1R} + \frac{K_{12}K_6I_1'r_2^2 \cos^2\alpha}{K_1r_1R}$$

$$+ \frac{K_{12}I_2'a_2r_2 \cos\alpha}{K_1h_2R} - \frac{h_1 \cos^2\alpha\, m_1r_2^2}{h_2R}$$

$$- \frac{I_1'r_2^2 \cos^3\alpha}{r_1Rh_2} - \frac{I_2'r_2 \cos\alpha}{h_2R} - \frac{m_2r_2^2 \cos\alpha}{R} \Bigg]$$

The geometrical relationships below can be used to somewhat reduce the 32 constants defined in this Chapter.

(4.13) $R = (a_1 - l_1)\cos\alpha + a_2 + l_2 + (h_1 + d_1)\sin\alpha$

(4.14) $h_2 + d_2 = (h_1 + d_1)\cos\alpha - (a_1 - l_1)\sin\alpha$

The reduced forms are listed at the beginning of Chapter V.

CHAPTER V

SOLUTION OF THE EQUATIONS

The System of Two Second-Order Differential Equations

Equations (4.9 d) and (4.5 g) are rewritten here as Equations (5.1) and (5.2) respectively. It remains now to write the reduced forms of the constants developed in Chapter IV and to eliminate F_{g4} and M_3 from the motion equations.

$$(5.1)\quad K_{13}\ddot{\theta} = K_{14}\ddot{\phi} + K_{15}F_{g4}\theta + K_{16}\Omega^2\theta + K_{17}\theta + K_{18}\Omega\dot{\phi} + K_{19}M_3 + K_{20}\phi + K_{21}\Omega\dot{\theta} + K_{22}\Omega^2\theta$$

$$(5.2)\quad K_{24}\ddot{\phi} = K_{23}\ddot{\theta} + K_{25}M_3 + K_{26}F_{g4}\theta + K_{27}\Omega^2\theta + K_{28}\theta + K_{29}\Omega\dot{\phi} + K_{30}\phi + K_{31}\Omega\dot{\theta} + K_{32}\Omega^2\theta$$

The coefficients, as they are to be programmed in to the computer, are as follows:

$$K_1 = R + (a_1 - l_1)\left(\frac{a_2}{h_2}\sin\alpha - \cos\alpha\right)$$

$$K_2 = R\sin\alpha + h_2\cos\alpha$$

$$K_3 = R\cos\alpha - h_2\sin\alpha$$

$$K_4 = \frac{R}{\sin\alpha}$$

$$K_5 = R \sin\alpha - d_1 + h_2 \cos\alpha$$

$$K_6 = \frac{a_2}{h_2} \cos\alpha + \sin\alpha$$

$$K_7 = \frac{R(1+\cos^2\alpha)}{\sin\alpha} - h_2 \cos\alpha$$

K_8 not defined

$$K_9 = -\frac{a_2}{h_2 \sin\alpha}\left\{ R\cos\alpha - (a_1 - l_1)\right\}$$

$$K_{10} = R\cos\alpha - (a_1 - l_1) - h_2 \sin\alpha$$

$$K_{11} = R\sin\alpha - (h_1 + d_1) + h_2 \cos\alpha$$

$$K_{12} = \frac{(a_1 - l_1)\sin\alpha}{h_2}$$

$$K_{13} = I_{z1} + m_1 a_1^2 - \frac{(a_1 - l_1)}{K_1}(K_6)(m_1 a_1 h_1 + P_{xz1})$$

$$K_{14} = \frac{a_1 - l_1}{K_1}\left\{ - P_{xz2} + K_6\left[(a_1 h_1 m_1 + P_{xz1})\sin\alpha - (m_1 h_1^2 + I_{x1})\cos\alpha\right] - \frac{a_2 I_{x2}}{h_2}\right\} - (a_1^2 m_1 + I_{z1})\sin\alpha$$
$$+ (a_1 h_1 m_1 + P_{xz1})\cos\alpha$$

$$K_{15} = \frac{(a_1 - l_1)\left[R\cos\alpha - (a_1 - l_1)\right]}{K_1}$$

$$K_{16} = \frac{(a_1 - l_1)}{K_1}\left\{\frac{a_2\left[R\cos\alpha - (a_1 - l_1)\right]}{h_2 R}\left[c_1(r_2)^2\left(\frac{R\cos\alpha}{\sin\alpha} - h_2 + a_1 \sin\alpha - b_1\right) - c_2(r_2)^2(b_2 - h_2)\right]\right.$$
$$\left. - \frac{c_2(r_2)^2}{\sin\alpha}(K_6)\left[R\cos\alpha - (a_1 - l_1) - b_2 \sin\alpha\right]\right\}$$

$$K_{17} = g_c \left\{ \frac{a_1 - l_1}{K_1} \left\{ - \frac{a_2}{h_2 R} \left[R \cos\alpha - (a_1 - l_1) \right] \left[m_1(h_1 \sin\alpha + a_1 \cos\alpha) - m_2 a_2 \right] + (K_6)(m_1 h_1 \sin\alpha - m_2 a_2) \right\} - m_1 a_1 \sin\alpha \right\}$$

$$K_{18} = \left\{ \frac{a_1 - l_1}{K_1} \left[- I_2' - (K_6)(I_1' \frac{r_2}{r_1} \sin\alpha) \right] + I_1' \frac{r_2}{r_1} \cos\alpha \right\}$$

$$K_{19} = \left\{ \frac{a_1 - l_1}{K_1} \left[\frac{a_2}{h_2} \sin\alpha - \cos\alpha \right] -1 \right\}$$

$$K_{20} = m_1 g_c \left[\frac{(a_1 - l_1)}{K_1} h_1 (K_6) - a_1 \right]$$

$$K_{21} = \frac{(a_1 - l_1)\cos\alpha}{K_1 R} \left\{ r_2(K_6) \left[m_1 h_1 (a_1 \cos\alpha + h_1 \sin\alpha - R) + I_{x1} \sin\alpha + P_{xz1} \cos\alpha - \frac{I_1' R}{r_1 \cos\alpha} \right] + r_2 \left[\frac{P_{xz2} a_2}{h_2} + I_{z2} \right] \right\} - \frac{r_2 \cos\alpha}{R} \left\{ m_1 a_1 \left[a_1 \cos\alpha + h_1 \sin\alpha - R \right] + I_{z1} \cos\alpha + P_{xz1} \sin\alpha \right\}$$

$$K_{22} = \frac{r_2^2 \cos\alpha}{R} \left\{ \frac{a_1 - l_1}{K_1} \left[- (K_6)(m_1 h_1 + \frac{I_1'}{r_1} \cos\alpha) - \frac{I_2' a_2}{r_2 h_2} \right] + m_1 a_1 - \frac{I_1' \sin\alpha}{r_1} \right\}$$

$$K_{23} = \left\{\left[-\frac{K_{12}K_6}{K_1} + \frac{\cos\alpha}{h_2}\right]\left[a_1h_1m_1 + P_{xz1}\right]\right\}$$

$$K_{24} = \left\{\frac{K_{12}}{K_1}\left[-P_{xz2} - \frac{a_2I_{x2}}{h_2}\right] + \frac{K_{12}K_6}{K_1}\left[a_1h_1m_1 \sin\alpha\right.\right.$$

$$\left. - h_1^2m_1 \cos\alpha - I_{x1} \cos\alpha + P_{xz1} \sin\alpha\right]$$

$$- \frac{1}{h_2}\left[\cos\alpha\,(a_1h_1m_1 \sin\alpha - m_1h_1^2 \cos\alpha\right.$$

$$\left.\left. - I_{x1} \cos\alpha + P_{xz1} \sin\alpha) - m_2h_2^2 - I_{x2}\right]\right\}$$

$$K_{25} = \frac{K_{12}}{K_1}\left[\cos\alpha - \frac{a_2 \sin\alpha}{h_2}\right] + \frac{\sin\alpha}{h_2}$$

$$K_{26} = \frac{1}{K_4}\left\{-\left(\frac{K_3}{\sin\alpha} + h_2\right)\left[\frac{K_{12}K_9}{K_1} + 1 + \frac{d_2}{h_2}\right.\right.$$

$$\left.\left. + \frac{K_{10} \cos^2\alpha}{h_2 \sin\alpha} + \frac{K_{11} \cos\alpha}{h_2}\right]\right\} + \frac{\cos\alpha}{h_2}$$

$$- \frac{K_{12}K_6}{K_1}\left(\frac{K_3}{\sin\alpha} + h_2 - \frac{(a_1 - l_1)}{\sin\alpha}\right)$$

$$K_{27} = \frac{c_1r_2^2}{K_4}\left[\frac{K_3}{\sin\alpha} + a_1 \sin\alpha - h_1 \cos\alpha - (b_1\right.$$

$$\left. - h_1 \cos\alpha)\right]\left\{\frac{K_{12}K_9}{K_1} + \frac{K_{11} \cos\alpha}{h_2} + 1 + \frac{d_2}{h_2}\right.$$

$$\left. + \frac{K_{10} \cos^2\alpha}{h_2 \sin\alpha}\right\} + c_2r_2^2\left\{-(b_2 - h_2)\left[\frac{1}{K_4}\left(\frac{K_{12}K_9}{K_1}\right.\right.\right.$$

$$+\frac{K_{11}\cos\alpha}{h_2}+1+\frac{d_2}{h_2}+\frac{K_{10}\cos^2\alpha}{h_2\sin\alpha})+\frac{K_{12}K_6}{K_1}$$

$$-\frac{\cos\alpha}{h_2}\Bigg]+\frac{K_{10}}{\sin\alpha}\left(\frac{K_{12}K_6}{K_1}-\frac{\cos\alpha}{h_2}\right)\Bigg\}$$

$$K_{28} = \Bigg\{-\frac{g_c}{K_4}\left[\frac{K_{12}K_9}{K_1}+\frac{K_{11}\cos\alpha}{h_2}+1+\frac{d_2}{h_2}\right.$$

$$\left.+\frac{K_{10}\cos^2\alpha}{h_2\sin\alpha}\right]\left[m_1a_1\cos\alpha+m_1h_1\sin\alpha-m_2a_2\right]$$

$$-g_c\left[\frac{K_{12}K_6}{K_1}-\frac{\cos\alpha}{h_2}\right]\left[m_1h_1\sin\alpha-m_2a_2\right]\Bigg\}$$

$$K_{29} = \Bigg\{\frac{\sin\alpha(a_1-l_1)}{K_1h_2}\left[I_2'+I_1'\frac{r_2}{r_1}\sin\alpha(K_6)\right]$$

$$-\frac{I_1'r_2\sin\alpha\cos\alpha}{h_2r_1}\Bigg\}$$

$$K_{30} = g_c\left\{h_1m_1\left[-\frac{K_{12}K_6}{K_1}+\frac{\cos\alpha}{h_2}\right]+m_2\right\}$$

$$K_{31} = \Bigg\{\left[-\frac{K_{12}K_6}{K_1}+\frac{\cos\alpha}{h_2}\right]\Bigg\{\left[\frac{h_1m_1r_2\cos\alpha}{R}\right]\Bigg[a_1\cos\alpha$$

$$+h_1\sin\alpha\Bigg]+\left[\frac{r_2\cos\alpha}{R}\right]\Bigg[I_{x1}\sin\alpha$$

$$+P_{xz1}\cos\alpha\Bigg]-r_2\left[m_1h_1\cos\alpha+\frac{I_1'}{r_1}\right]\Bigg\}$$

$$-\frac{K_{12}r_2\cos\alpha}{K_1R}\left[\frac{P_{xz2}a_2}{h_2}+I_{z2}\right]+\frac{r_2\cos\alpha}{R}\Bigg[\frac{P_{xz2}}{h_2}$$

$$-m_2a_2\Bigg]\Bigg\}$$

$$K_{32} = \left\{ \frac{r_2^2 \cos\alpha}{R} \left[\frac{K_{12}K_6}{K_1} - \frac{\cos\alpha}{h_2} \right] \left[h_1 m_1 + \frac{I_1' \cos\alpha}{r_1} \right] + \frac{r_2 \cos\alpha}{R} \left[\frac{I_2'}{h_2} \left(\frac{K_{12}a_2}{K_1} - 1 \right) - m_2 r_2 \right] \right\}$$

Eliminating F_{g4} and M_3 by substituting Equations (3.30) and (3.31) into Equations (5.1) and (5.2) yields

$$(5.3) \quad K_{13}\ddot{\theta} = K_{14}\ddot{\phi} + \left\{ K_{15}\left[c_3 r_2 \Omega + (c_1 + c_2)(r_2\Omega)^2 \right] + (K_{16} + K_{22})\Omega^2 + K_{17} \right\}\theta + (K_{19}c_5 + K_{21}\Omega)\dot{\theta} + K_{18}\Omega\dot{\phi} + K_{20}\phi$$

$$(5.4) \quad K_{24}\ddot{\phi} = K_{23}\ddot{\theta} + (K_{25}c_5 + K_{31}\Omega)\dot{\theta} + \left\{ K_{26}\left[c_3 r_2 \Omega + (c_1 + c_2)(r_2\Omega)^2 \right] + (K_{27} + K_{32})\Omega^2 + K_{28} \right\}\theta + K_{29}\Omega\dot{\phi} + K_{30}\phi$$

Again defining new coefficients for convenience in writing let

$$(5.5) \quad A_1 = K_{15}\left[c_3 r_2 \Omega + (c_1 + c_2)(r_2\Omega)^2 \right] + (K_{16} + K_{22})\Omega^2 + K_{17}$$

$$(5.6) \quad A_2 = K_{19}c_5 + K_{21}\Omega$$

$$(5.7) \quad A_3 = K_{25}c_5 + K_{31}\Omega$$

$$(5.8) \quad A_4 = K_{26}\left[c_3 r_2 \Omega + (c_1 + c_2)(r_2\Omega)^2 \right]$$

$$+(K_{27} + K_{32})\Omega^2 + K_{28}$$

Equations (5.3) and (5.4) can then be rewritten as

(5.9) $K_{13}\ddot{\theta} = K_{14}\ddot{\phi} + A_1\theta + A_2\dot{\theta} + K_{18}\Omega\dot{\phi} + K_{20}\phi$

(5.10) $K_{24}\ddot{\phi} = K_{23}\ddot{\theta} + A_3\dot{\theta} + A_4\theta + K_{29}\Omega\dot{\phi} + K_{30}\phi$

Here only ϕ and θ remain as time variables.

Transformation to Two Fourth Order Equations

Equations (5.9) and (5.10) can be reduced to two fourth order differential equations by writing the characteristic equations for (5.9) and (5.10) as follows:

(5.11) $K_{13}D^2\theta = K_{14}D^2\phi + A_1\theta + A_2D\theta + K_{18}\Omega D\phi + K_{20}\phi$

(5.12) $K_{23}D^2\phi = K_{23}D^2\theta + A_3D\theta + A_4\theta + K_{29}\Omega D\phi + K_{30}\phi$

Here the operator D represents the derivative and D^2 the second derivative.

Factoring ϕ and θ the equations become

(5.13) $(K_{13}D^2 - A_2D - A_1)\theta = (K_{14}D^2 + K_{18}\Omega D + K_{20})\phi$

(5.14) $(K_{23}D^2 + A_3D + A_4)\theta = (K_{24}D^2 - K_{29}\Omega D - K_{30})\phi$

The left side of the two equations are made identical by selecting the appropriate multiplication term for each equation.

$$(5.15) \quad (K_{13}D^2 - A_2D - A_1)(K_{23}D^2 + A_3D + A_4)\,\theta = (K_{14}D^2 + K_{18}\Omega D + K_{20})(K_{23}D^2 + A_3D + A_4)\,\phi$$

$$(5.16) \quad (K_{23}D^2 + A_3D + A_4)(K_{13}D^2 - A_2D - A_1)\,\theta = (K_{24}D^2 - K_{29}\Omega D - K_{30})(K_{13}D^2 - A_2D - A_1)\,\phi$$

By subtraction

$$(5.17) \quad (K_{24}D^2 - K_{29}\Omega D - K_{30})(K_{13}D^2 - A_2D - A_1) - (K_{14}D^2 + K_{18}\Omega D + K_{20})(K_{23}D^2 + A_3D + A_4)\,\phi = 0$$

Equation (5.17) is the characteristic equation for a fourth order differential equation in terms of ϕ. Note that this algebraic process is the equivalent of subtracting "K_{23} times the second derivative of Equation (5.16) minus A_2 times the derivative of Equation (5.16) minus A_1 times Equation (5.16)" from "K_{23} times the second derivative of Equation (5.15) plus A_3 times the derivative of Equation (5.15) plus A_4 times Equation (5.15)".

If the same procedure is used, with the alternate multiplication factors for Equations (5.15) and (5.16), ϕ can be eliminated and the characteristic equation for θ will be formed. The result of this procedure is an equation identical to Equation (5.17) with θ replacing ϕ, thus

$$(5.18)\quad \Big[(K_{24}D^2 - K_{29}\Omega D - K_{30})(K_{13}D^2 - A_2 D - A_1) - (K_{14}D^2 + K_{18}\Omega D + K_{20})(K_{23}D^2 + A_3 D + A_4)\Big]\theta = 0$$

Rewriting Equations (5.17) and (5.18)

$$(5.19)\quad (D^4 + E_1 D^3 + E_2 D^2 + E_3 D + E_4)\,\phi = 0$$

$$(5.20)\quad (D^4 + E_1 D^3 + E_2 D^2 + E_3 D + E_4)\,\theta = 0$$

Expansion of Equations (5.17) and (5.18) yield

$$(5.21)\quad E_o = K_{14}K_{23} - K_{13}K_{24}$$

$$(5.22)\quad E_1 = \Big[K_{14}A_3 + K_{18}K_{23}\Omega + A_2K_{24} + K_{13}K_{29}\Omega \Big] / E_o$$

$$(5.23)\quad E_2 = \Big[K_{14}A_4 + K_{18}A_3\Omega + K_{20}K_{23} + K_{24}A_1 - K_{29}A_2\Omega + K_{13}K_{30} \Big] / E_o$$

$$(5.24)\quad E_3 = \Big[K_{18}A_4\Omega + K_{20}A_3 - K_{29}A_1\Omega - K_{30}A_2 \Big] / E_o$$

$$(5.25)\quad E_4 = \Big[K_{20}A_4 - K_{30}A_1 \Big] / E_o$$

Substituting the expressions for A_1, A_2, A_3, and A_4 and factoring, the coefficients for Equations (5.19) and (5.20) become

$$(5.26)\quad E_o = K_{14}K_{23} - K_{13}K_{24}$$

$$(5.27)\quad E_1 = \Big\{ K_{14}\Big[K_{25}c_5 + K_{31}\Omega \Big] + K_{18}K_{23}\Omega + K_{24}\Big[K_{19}c_5 + K_{21}\Omega \Big] + K_{13}K_{29}\Omega \Big\} / E_o$$

$$(5.28)\quad E_2 = \Big\{K_{14}\Big[\!\Big[K_{26}\big[c_3r_2\Omega + (c_1 + c_2)(r_2\Omega)^2\big] + (K_{27}+K_{32})\Omega^2 + K_{28}\Big]\!\Big] + K_{18}\Omega\big[K_{25}c_5 + K_{31}\Omega\big] + K_{20}K_{23} + K_{24}\big[K_{15}(c_3r_2\Omega + (c_1 + c_2)(r_2\Omega)^2) + (K_{16}+K_{22})\Omega^2 + K_{17}\big] - K_{29}\Omega\big[K_{19}c_5 + K_{21}\Omega\big] + K_{13}K_{30}\Big\} / E_o$$

$$(5.29)\quad E_3 = \Big\{K_{18}\Omega\big[K_{26}(c_3r_2\Omega + (c_1+c_2)(r_2\Omega)^2) + (K_{27}+K_{32})\Omega^2 + K_{28}\big] + K_{20}\big[K_{25}c_5 + K_{31}\Omega\big] - K_{29}\Omega\big[K_{15}(c_3r_2\Omega + (c_1 + c_2)(r_2\Omega)^2) + (K_{16}+K_{22})\Omega^2 + K_{17}\big] - K_{30}\big[K_{19}c_5 + K_{21}\Omega\big]\Big\} / E_o$$

$$(5.30)\quad E_4 = \Big\{K_{20}\big[K_{26}(c_3r_2\Omega + (c_1 + c_2)(r_2\Omega)^2) + (K_{27}+K_{32})\Omega^2 + K_{28}\big] - K_{30}\big[K_{15}(c_3r_2\Omega + (c_1 + c_2)(r_2\Omega)^2) + (K_{16}+K_{22})\Omega^2 + K_{17}\big]\Big\} / E_o$$

These are the coefficient expressions to be used in the computer solution of Equations (5.19) and (5.20).

Solution Procedure for Motion Equations

The solutions of the fourth order differential equations can be found by solving the corresponding fourth degree polynomials formed by their characteristic equations. Specifically if the four roots of the polynomial Equation (5.19) are [1] $D_1 = \alpha_1$, $D_2 = \alpha_2$, and $D_3 = \alpha_3 \pm i\beta_1$, then if $\alpha_1 \neq \alpha_2$ the solution for the corresponding differential equation becomes

$$(5.31)\quad \phi = B_1 e^{\alpha_1 t} + B_2 e^{\alpha_2 t} + B_3 e^{(\alpha_3 + i\beta_1)t} + B_4 e^{(\alpha_3 - i\beta_1)t}$$

The Euler identity, $e^{i\beta_1 t} = \cos\beta_1 t + 1 \sin\beta_1 t$, can be used to write Equation (5.31) as

$$(5.32)\quad \phi = B_1 e^{\alpha_1 t} + B_2 e^{\alpha_2 t} + e^{\alpha_3 t} (B_5 \cos\beta_1 t + B_6 \sin\beta_1 t)$$

where B_1, B_2, B_5, and B_6 are real coefficients.

A form which is still more convenient for this particular problem can be obtained by using the trigonometric identity

$$(5.33)\quad \sin(\beta_1 t + \phi_1) = \cos\beta_1 t \sin\phi_1 + \sin\beta_1 t \cos\phi_1$$

The use of Equation (5.33) gives the final form of

(1) In general two real and two complex roots were found in the actual computer work.

the solution as

$$(5.34) \quad \phi = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + e^{\alpha_3 t}\left[C_3 \sin(\beta_1 t + \phi_1)\right]$$

The four integration constants which must be determined from the initial condition are C_1, C_2, C_3, and ϕ_1.

The solution for θ is identical except for the integration constants.

$$(5.35) \quad \theta = C_4 e^{\alpha_1 t} + C_5 e^{\alpha_2 t} + e^{\alpha_3 t}\left[C_6 \sin(\beta_1 t + \theta_1)\right]$$

The method of obtaining solutions for Equations (5.9) and (5.10) can be summarized as follows:

(1) Calculate the coefficients of Equations (5.21) thru (5.25).

(2) Solve for the roots of the polynomial in Equation (5.19).

(3) Impose the initial conditions to find the integration constants.

It should be noted that C_4, C_5, and θ_1, can be expressed in terms of C_1, C_2, and ϕ_1. This calculation is not carried out here however since it is not necessary to actually impose initial conditions for the stability study.

Stability Criterion

The form of Equations (5.34) and (5.35) is particularly adaptable to a study of stability. The necessary and sufficient condition for stability is that all of the real roots α_1, α_2, and α_3 be negative. This condition assures that once the disturbances are removed all terms will decay toward zero as time increases. On the other hand the presence of any positive real root will cause ϕ and θ to increase with time. The margin of stability or instability is of course indicated by the magnitude of the real roots.

The frequency of the oscillation can be determined by the imaginary part of the complex root. The relationship is

(5.36) $f = \beta_1/2\pi$

Since this information can be gained without specifying the initial conditions, the computer program is written to calculate the coefficients designated by Equations (5.27) thru (5.30) and solve for the roots of the polynomial Equation (5.19).

Computer Program

The computer program was written for an IBM 1620 computer using Forgo.(2)

The output for each set of machine constants consists

(2) Forgo is a Fortran language compiler program developed by The Wisconsin College of Engineering Computer Laboratory.

of the coefficients and roots of Equation (5.19) for wheel speeds of 20 to 160 radians per second at increments of 20 radians per second. The root extraction is accomplished with Ferrari's[3] analytical solution of the quartic equation.

This program requires approximately four minutes to feed in and one additional minute of computing time for each set of input data. The details of the program and its use are included in the Appendix.

Input Constants

Application of the stability equations developed here make it necessary to establish the 27 machine constants which physically describe the machine. These constants and representative values for each are listed below:

$a_1 = -0.8$ ft	$h_1 = 1.5$ ft	$l_2 = 2$ ft
$a_2 = 2$ ft	$h_2 = 1.5$ ft	$m_1 = 4$ Sl
$b_1 = 3$ ft	$I_{x1} = 1$ Sl ft^2	$m_2 = 15$ Sl
$b_2 = 2$ ft	$I_{x2} = 3$ Sl ft^2	P_{xz1} 0 Sl ft^2
$c_1 = 0.008$ lb$_f$ sec^2/ft^2	$I_{z1} = 0.5$ Sl ft^2	$P_{xz2} = 0$ Sl ft^2
$c_2 = 0.002$ lb$_f$ sec^2/ft^2	$I_{z2} = 8$ Sl ft^2	$r_1 = 1$ ft
$c_3 = 0.005$ lb$_f$ sec/ft	$I_1' = 1$ Sl ft^2	$r_2 = 1$ ft
$c_5 = 0$ lb$_f$ ft sec	$I_2' = 2$ Sl ft^2	$\sin\alpha = 0.5$
$d_2 = 2$ ft	$l_1 = -0.3$ ft	$\cos\alpha = 0.866$

(3) Nelson B. Conkwright, Introduction to the Theory of Equations, Ginn and Co., Boston (1941).

An explanation of the selection of some of these values is in order:

(1) The road load coefficients c_1, c_2, and c_3 are selected to give a realistic horsepower requirement. The coefficient c_3 contributes negligibly to the road load and in fact represents only a very small force tangent to the front wheel. The bulk of the drag force is assigned to System 1 because of its exposure and the common use of windshields.

(2) The mass of System 2 is relatively large because of the passenger and engine.

(3) The products of inertia are taken as zero because in practice they can assume either positive or negative values depending on the mass distribution.

The values of the machine constants given here are the basis for the numerical results presented in Chapter VI. While these values do not describe any specific machine, they are sufficiently accurate to yield computer results representative of actual machines.

CHAPTER VI

NUMERICAL RESULTS

To interpret the numerical results it is necessary to examine the stabilizing action of motorcycles.

Stabilizing Action

The manual operation of the handle bars illustrates the basic stabilizing process. Falls are avoided by turning the front wheel in the direction of the fall.

For a simplified analysis the overturning moment may be approximated as

(6.1) $M_o = mg_c h \sin\phi$

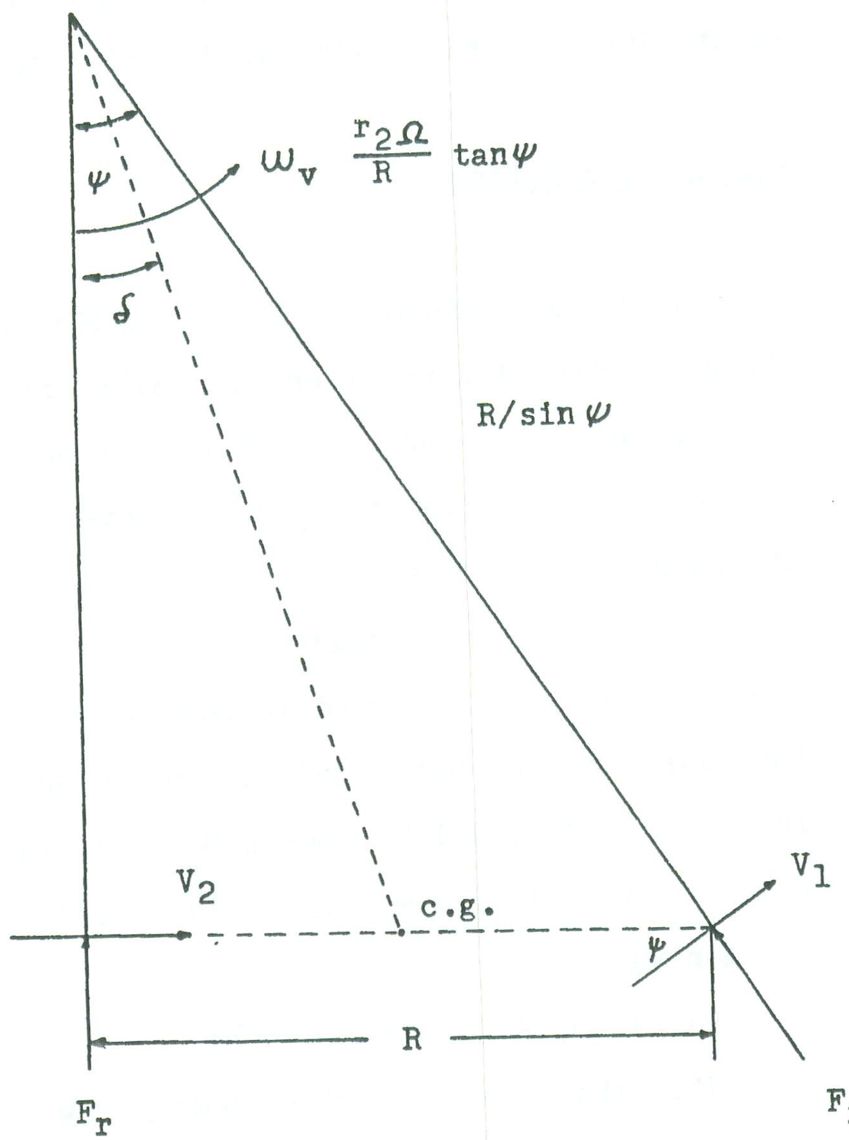
where m and h (as measured in the plane of symmetry) are the mass and c.g. height of the entire unit. The righting moment, M_r , which opposes M_o , is set up by the forces shown in Figure (6.1). In equation form M_r can be written as

(6.2) $M_r = (F_r + F_f \cos\psi)h$

With the exception of extremely low speeds the angle ψ is small hence Equation (6.2) can be written

(6.3) $M_r = (F_r + F_f)h$

In terms of acceleration the reaction forces can be approximated as



Stabilizing Forces
Figure 6.1

$$(6.4)\quad F_f + F_r \approx \frac{mR}{\sin\psi}(\omega_v)^2 + f(\phi) =$$

$$\frac{mR}{\sin\psi}\left(\frac{r_2\Omega\tan\psi}{R}\right)^2 + f(\phi) =$$

$$\frac{m(r_2\Omega)^2}{R}\sin\psi + f(\phi)$$

where $f(\phi)$ is an acceleration term reflecting $\ddot{\phi}$. Again using the small angle approximations

$$(6.5)\quad F_f + F_r = \frac{m(r_2\Omega)^2\psi}{R} + f(\phi)$$

Finally, equating M_r to M_o

$$(6.6)\quad mg_c h \sin\phi = \frac{m(r_2\Omega)^2\psi}{R} h + hf(\phi)$$

or

$$(6.7)\quad f(\phi) = mg_c\left[\sin\phi - \frac{(r_2\Omega)^2}{g_c R}\psi\right]$$

Equation (6.7) shows that for given values of lean angle (ϕ) and speed (Ω), the acceleration term $[f(\phi)]$ can be altered by adjusting the front wheel angle (ψ). Increasing ψ in the direction of ϕ causes the lean angle to accelerate toward zero thus stability is possible only through the proper control of the front wheel.

Further, Equation (6.7) also indicates that the necessary variation of ψ is small compared to the range of ϕ. This can be illustrated by using values of the reference machine described in Chapter V. For an equilibrium turn at 30 mph with a ϕ of 30°, ψ becomes

$$\psi = \frac{(.5)(32.2)(5.5)}{(44)^2} = \frac{1}{22}\text{ radians} < 3^\circ$$

Control of the Front Wheel

At very low speeds a rider can and in general must control the front wheel to keep the vehicle upright. At high speeds, the reaction time is so short that the front wheel control must be designed into the machine itself.

With the geometry shown in Figure (2.1), three factors tend to turn the front wheel directly.

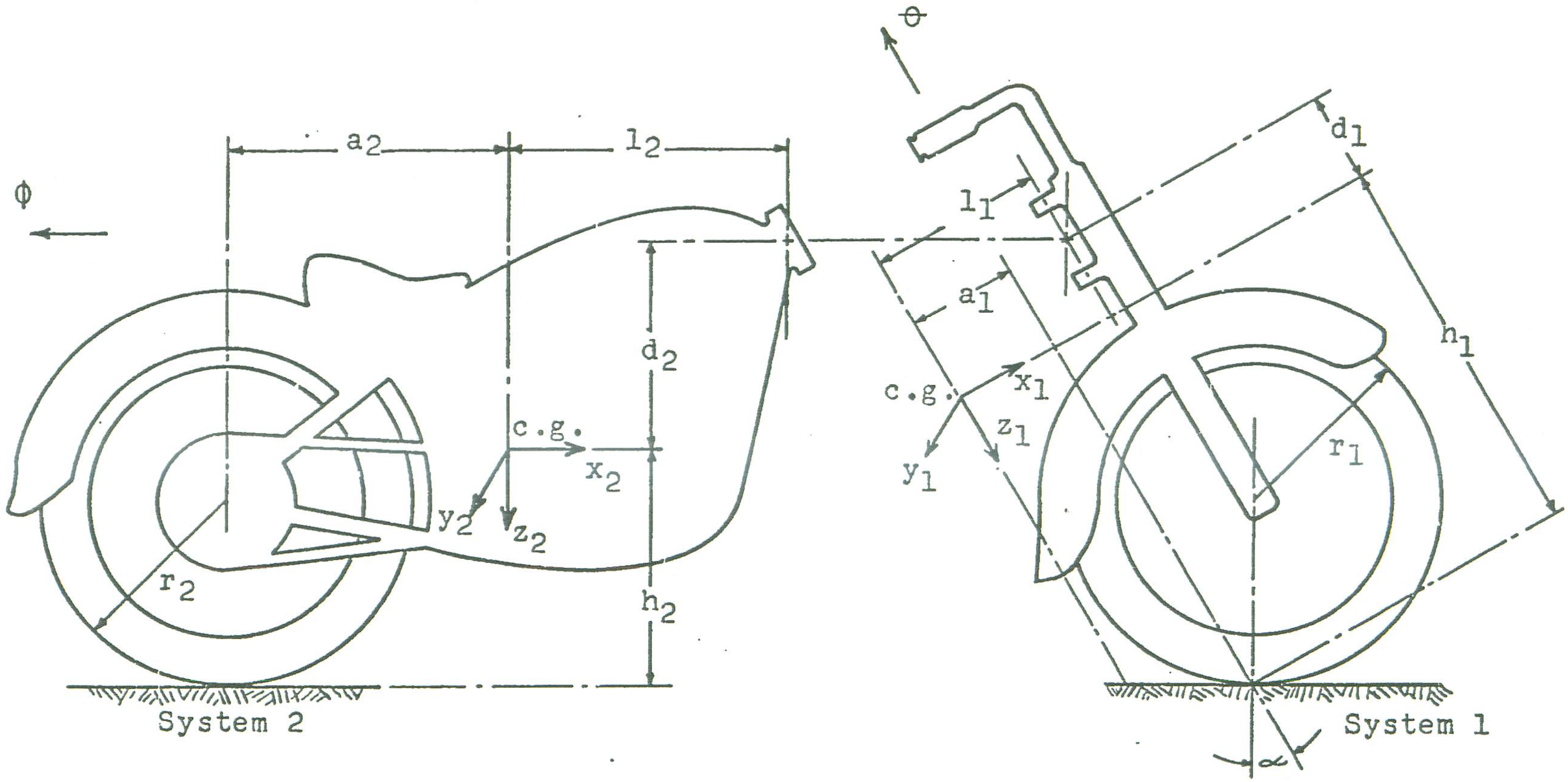
(1) The weight of System 2 acts at the steering axis to turn the front wheel in the direction of the lean angle.

(2) In practice, the c.g. of System 1 is generally in front of the steering axis thus the weight of System 2 also contributes in the same direction.

(3) The magnitude of the front wheel angle is limited by the force F_f as shown in Figure (6.1).

Two factors contribute rate controls to the front wheel motion:

(1) The gyroscopic action of the front wheel causes it to precess according to the external moment on System 1. For an external moment causing a given $\ddot{\phi}$, the front wheel precesses in the direction of $\ddot{\phi}$.



System Breakdown for Analysis

Figure 2.1 (Repeated)

(2) All motion of the front wheel is opposed by the damping in the steering axes.

From these considerations it can be seen that the characteristics of System 1 will have the greatest influence on stability.

Restriction to Small Angles

The linearizing process of Chapter III eliminates the product of ϕ, θ and their first derivatives as secondary terms. The approximations given in Equation (6.7) indicate that $\dot{\phi}$ is the largest of these terms for a frequency $f > \frac{1}{2\pi}$ radians/sec. Assuming a sinusoidal oscillation,

$$\phi = \phi_{max} \sin 2\pi ft$$

$$\therefore \dot{\phi}_{max} = 2\pi f \phi_{max}.$$

This of course indicates that the allowable value of ϕ_{max} must be reduced as the frequency is increased. At the highest frequency encountered in the solutions ($\approx$ 10 cps)

$$\dot{\phi}_{max} = 62.8 \ \phi_{max}$$

or ϕ_{max} must be less than one degree to insure a maximum value of one radian/second for $\dot{\phi}_{max}$.

However, tracing $\dot{\phi}$ through the basic equations in Chapter I indicates numerically that the contributions of $\dot{\phi}$ are still small when compared to the other terms involved.

There is no sharply defined limit but a ϕ_{max} of 2 degrees appears to be allowable and disturbances of this magnitude are representative for upright equilibrium operation.

Presentation of Results

The final computer output consists of the four roots of the polynomial Equation (5.19). With very few exceptions there are two real and two complex roots. One of the real roots is consistently a large negative value (-10 to -80 sec^{-1}) hence it can be safely disregarded. The plotted results then consist of (a) one real root which is referred to as the "Marginal Root", (b) the real part of the complex root which is referred to as the "Damping Root", and (c) the frequency which is directly proportional to the imaginary part of the complex root.

As indicated in Chapter V, all real roots must be negative for stability, and the margin of stability is indicated by the magnitude of these roots. Note that the numerical results are plotted on pairs of curve sheets. The ranges of stability can be read from either sheet by the legend, however both sheets must be examined to establish the margin of stability.

Since the study is dimensional, the results of the reference machine (the physical characteristics given in Chapter V) are shown on each curve sheet.

Discussion of Results

Since the Marginal Root is very small its importance is open to question. With the small growth rate of ϕ resulting from this root, there is a possibility that the rider may overcome this instability by unconsciously inclining his body. Dohring concluded this from his calculations in which he found this root negative for only a very short speed range. The absence of some of the machine constants makes it impossible to reliably duplicate his calculations.

Here, two factors tend to lend credence to the Marginal Root; (1) this root is generally negative for reasonable values of the machine constants and (2) the instability generally[1] introduced by a small value of trail[2] is reflected only in this root.

(1) The Wilson-Jones paper cites an experiment in which a motorcycle with negative trail was ridden with "no-hands" thus instability does not automatically accompany negative trail. However, it is generally agreed among designers that the optimum value of trail is positive and its value depends heavily on the many other characteristics of the machine.

(2) Referring to Figure (2.1); trail is defined as $(l_1 - a_1)/\cos \alpha$.

The following remarks concern the numerical results given in the curve sheets:

Figure 6.2: Some damping improves the margin of stability but excessive damping interferes with the motion of the front wheel to cause instability. This interference is also reflected in the frequency which decreases with increased damping.

The fact that large values of damping can be withstood is apparently the result of the low values of $\dot{\theta}$.

The discontinuities in the Damping Root indicate the presence of four real roots. With this exception, all of the results show two real roots and one complex root.

Figure 6.3: Trail is one of the variables which motorcycle designers generally agree to be of prime importance to stability. Reduction of the trail to zero reduces the restoring force and the action of the weight of System 2 on the front wheel. Since the stability margin of the two roots are changing in opposite directions, an optimum value of trail is indicated. Note that the instability arising from low values of trail are indicated entirely by the Marginal Root.

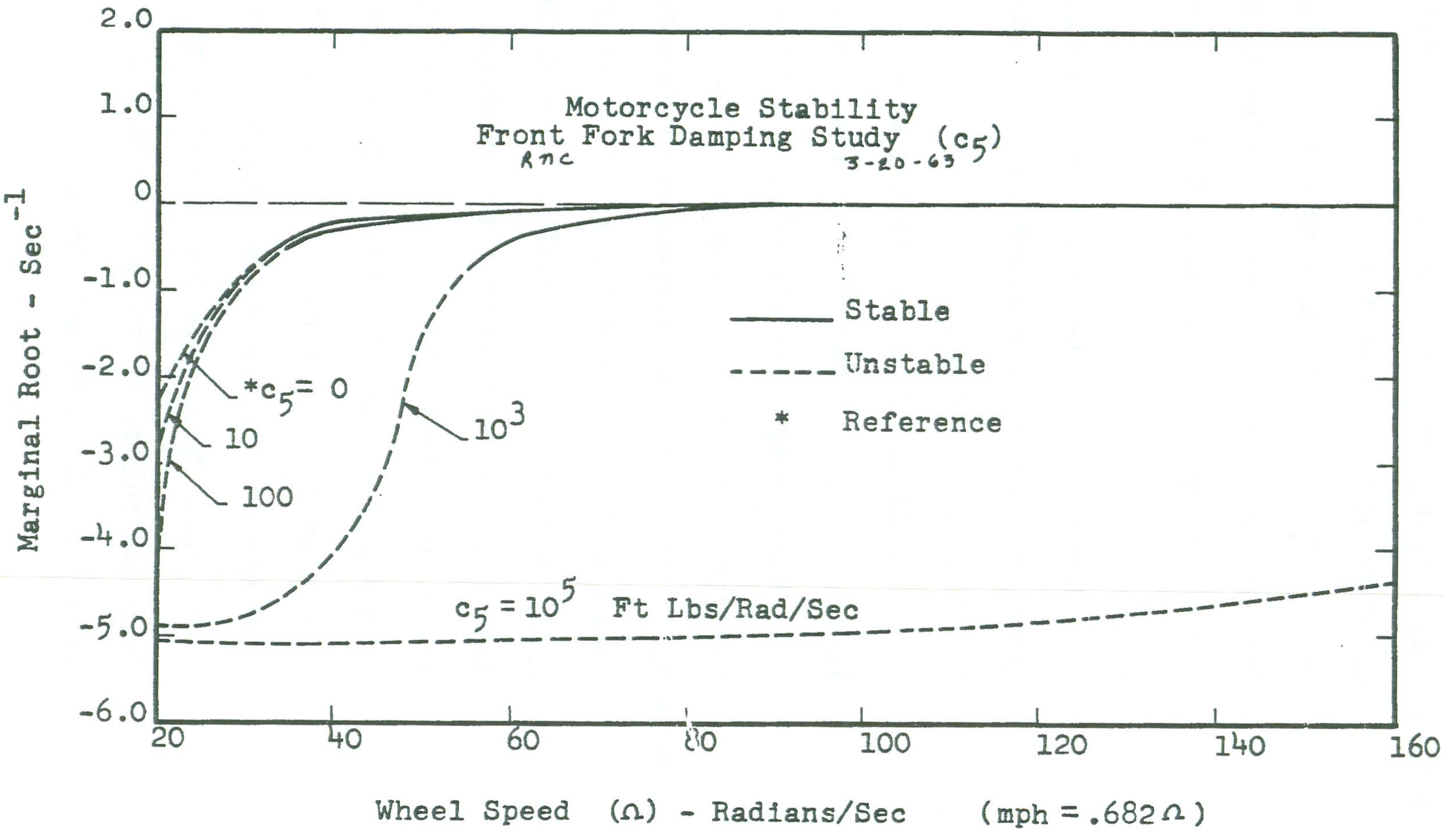

Motorcycle Stability
Front Fork Damping Study (c5)
RNC
3-20-63
Marginal Root - Sec-1
2.0
1.0
0
-1.0
-2.0
-3.0
-4.0
-5.0
-6.0
20
40
60
80
100
120
140
160
Stable
Unstable
* Reference
*c5 = 0
10
100
10^3
c5 = 10^5 Ft Lbs/Rad/Sec
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.2 a

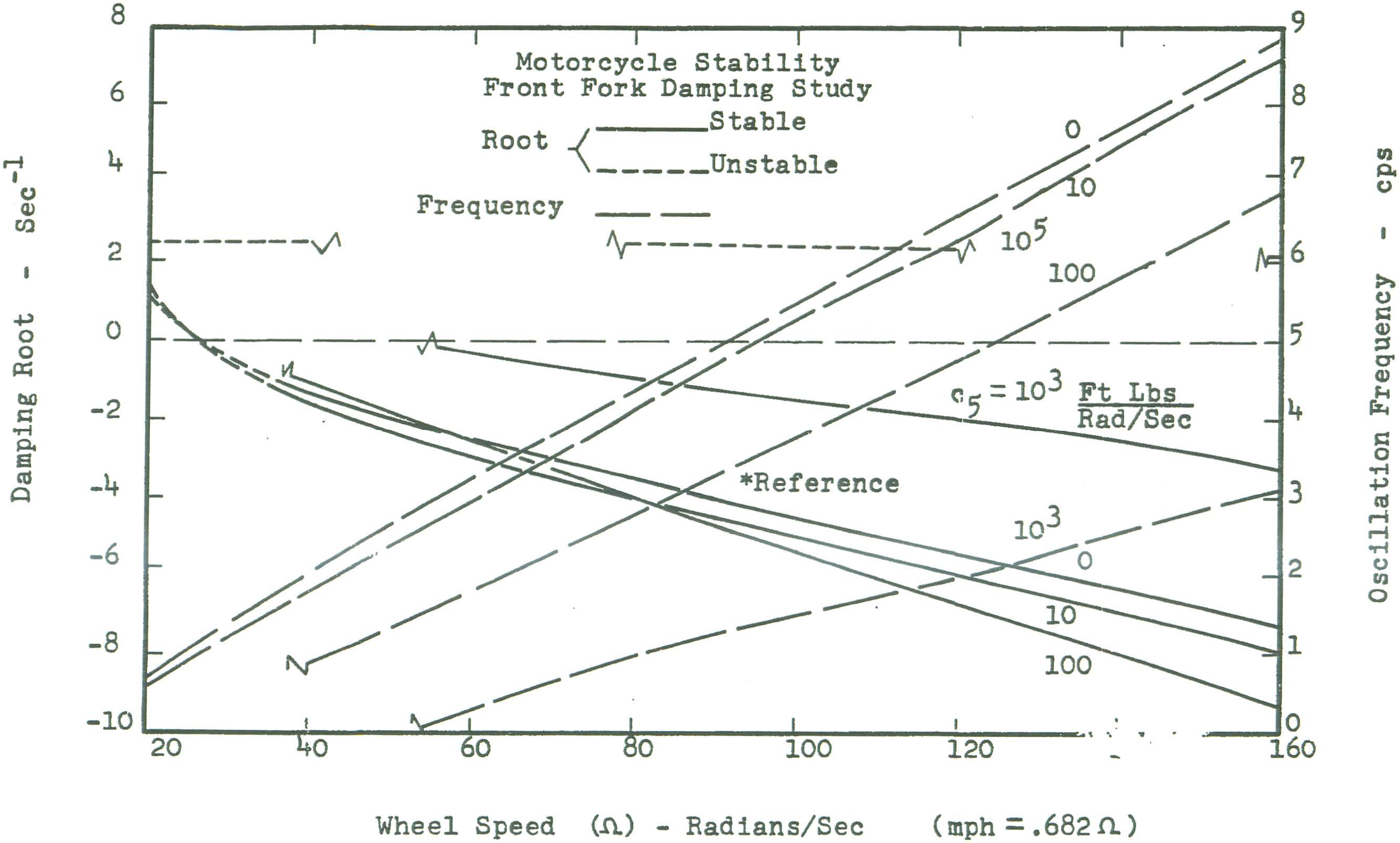
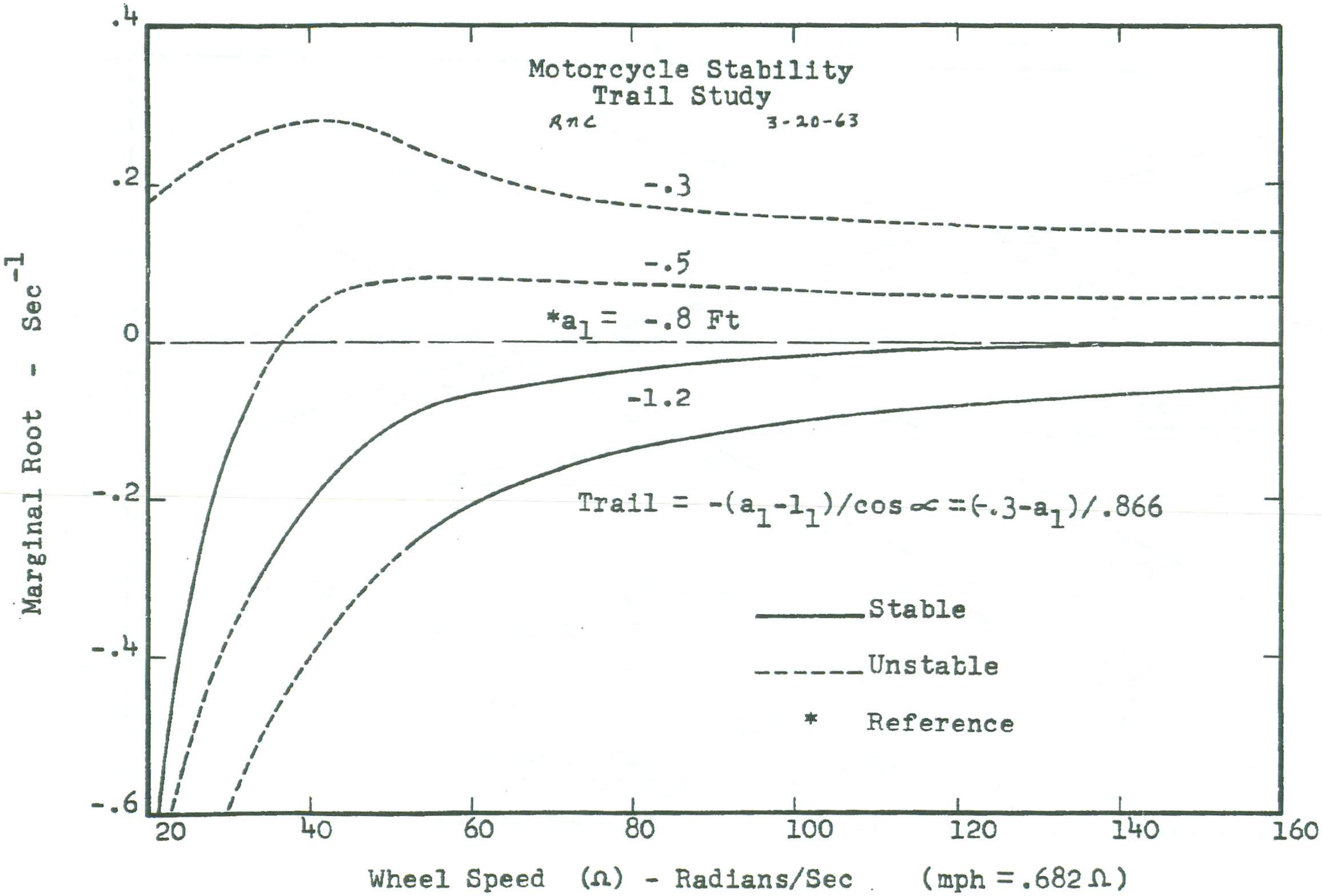

Motorcycle Stability
Front Fork Damping Study
Root
Stable
Unstable
Frequency
Damping Root - Sec^{-1}
Oscillation Frequency - cps
8
6
4
2
0
-2
-4
-6
-8
-10
9
8
7
6
5
4
3
2
1
0
20
40
60
80
100
120
160
0
10
10^5
100
$c_5 = 10^3$ $\frac{\text{Ft Lbs}}{\text{Rad/Sec}}$
*Reference
10^3
0
10
100
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.2 b



Motorcycle Stability
Trail Study
RNC 3-20-63
-.3
-.5
*a1 = -.8 Ft
-1.2
Trail = -(a1-l1)/cos ∝ =(-.3-a1)/.866
Stable
Unstable
* Reference
.4
.2
0
-.2
-.4
-.6
Marginal Root - Sec-1
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.3 a

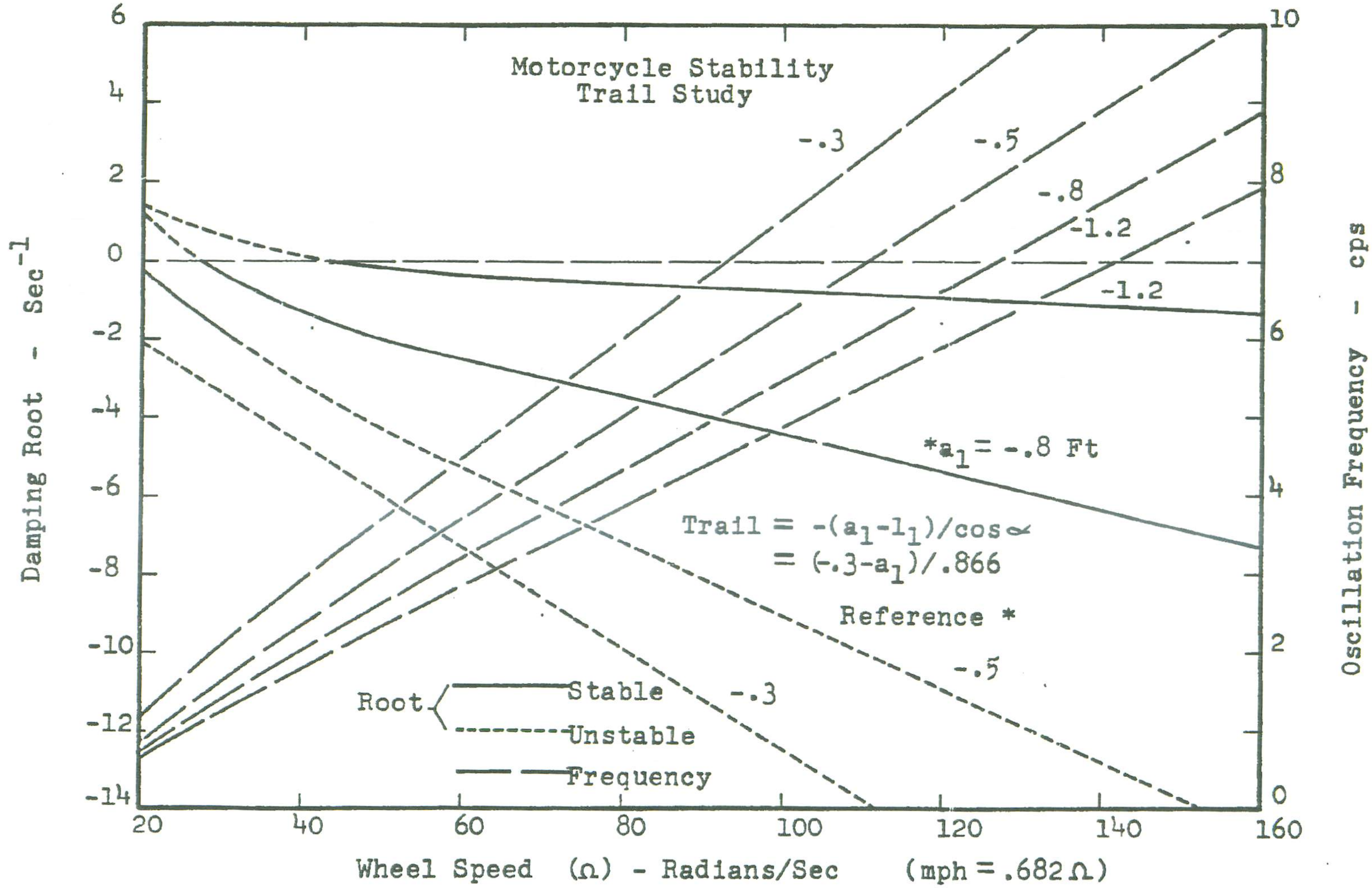

Motorcycle Stability
Trail Study
-.3
-.5
-.8
-1.2
-1.2
*a1 = -.8 Ft
Trail = -(a1-l1)/cos α
= (-.3-a1)/.866
Reference *
-.5
-.3
Root
Stable
Unstable
Frequency
Damping Root - Sec^-1
Oscillation Frequency - cps
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.3 b

Figure 6.4: The indicated stability is very sensitive to the position of the front c.g. along the x_1 axis. Note that this machine is unstable if the c.g. is moved back as far as the steering axis. This result is surprising in view of the absence of any particular stress on this point in the literature.

The frequency is not greatly effected within the c.g. positions allowed for stability.

Figure 6.5: An improvement in the stability is indicated by both roots as the height of the c.g. is lowered along the z_1 axis.

Figure 6.6: An optimum value of front wheel inertia is indicated. Removal of the gyroscopic effect gives instability according to the Damping Root. The Marginal Root shows instability for excessive values of I_1'.

The frequency increases with I_1'.

Figure 6.7: A reduction in rake angle (α) indicates an improvement in the margin of stability. This is somewhat surprising in view of the popular rake angle design of 30 degrees. It is possible however, that this angle is selected more for cornering response that for stability.

Figure 6.8: The frequency is generally insensitive to

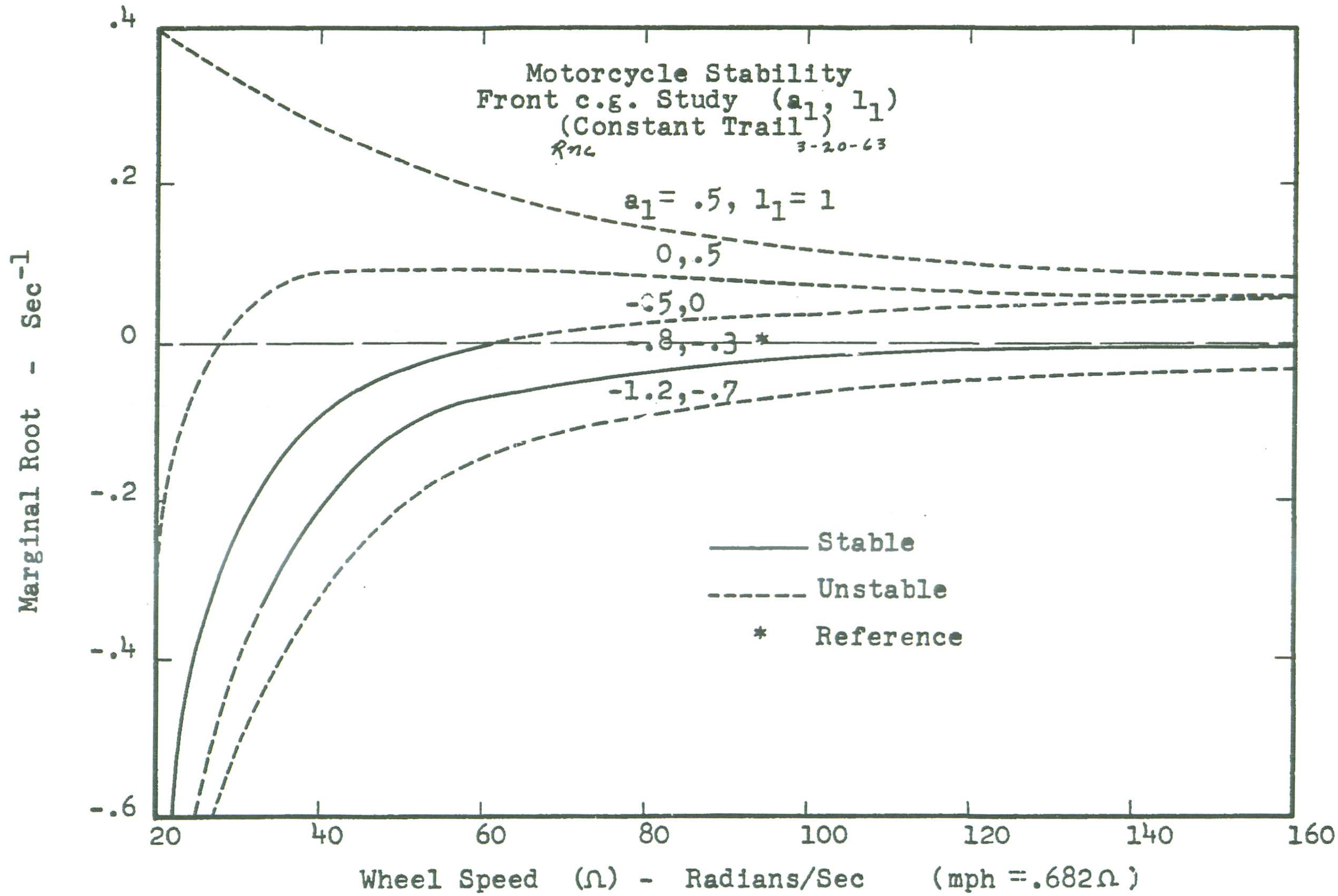

Motorcycle Stability
Front c.g. Study (a_1, l_1)
(Constant Trail)
3-20-63
a_1 = .5, l_1 = 1
0, .5
-.5, 0
-.8, -.3 *
-1.2, -.7
Stable
Unstable
* Reference
.4
.2
0
-.2
-.4
-.6
Marginal Root - Sec^{-1}
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.4 a

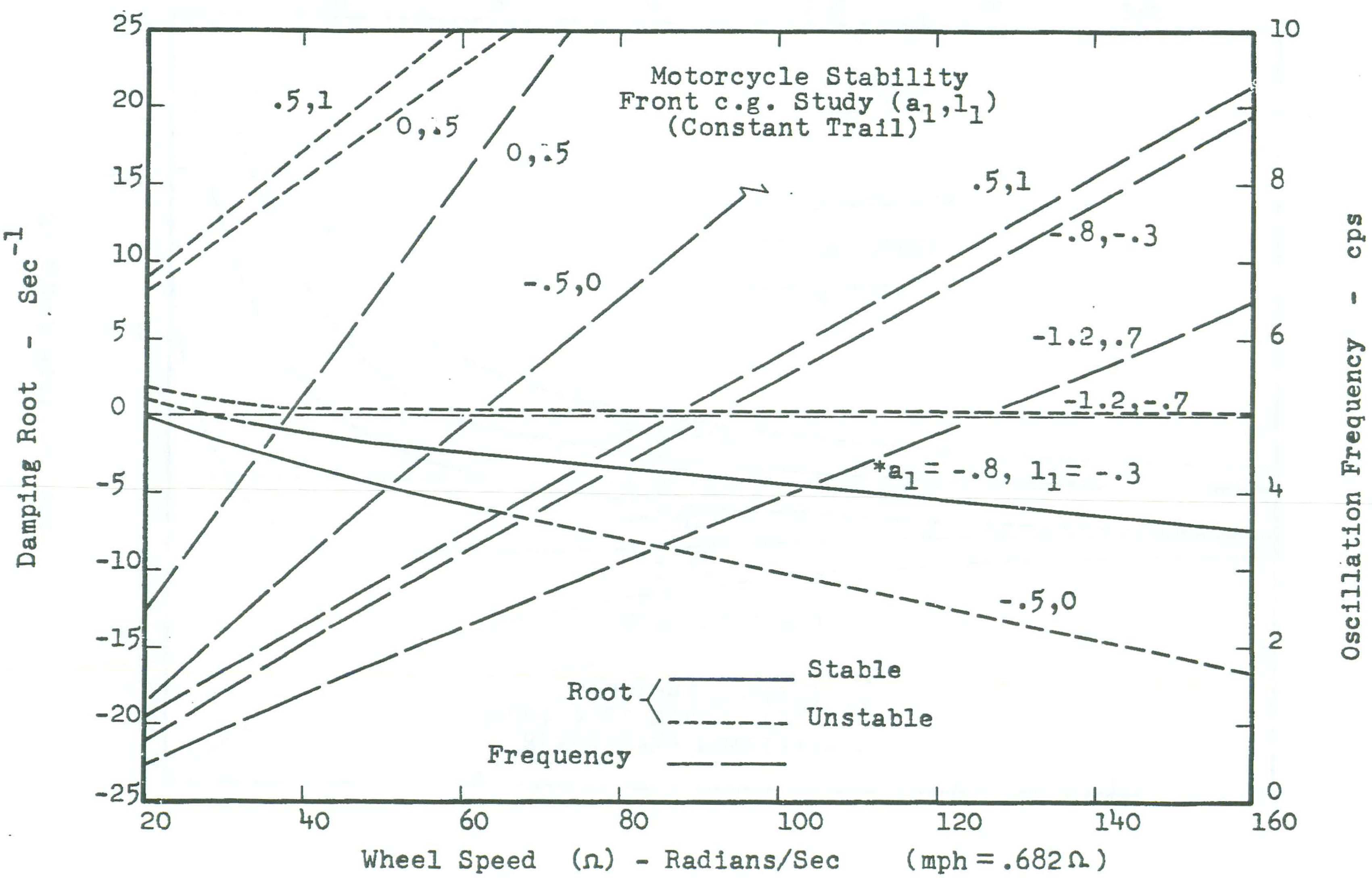

Motorcycle Stability
Front c.g. Study (a_1,l_1)
(Constant Trail)
.5,1
0,.5
0,.5
-.5,0
.5,1
-.8,-.3
-1.2,.7
-1.2,-.7
*a_1 = -.8, l_1 = -.3
-.5,0
Root
Stable
Unstable
Frequency
Damping Root - Sec^{-1}
Oscillation Frequency - cps
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)
25
20
15
10
5
0
-5
-10
-15
-20
-25
10
8
6
4
2
0
20
40
60
80
100
120
140
160

Figure 6.4 b

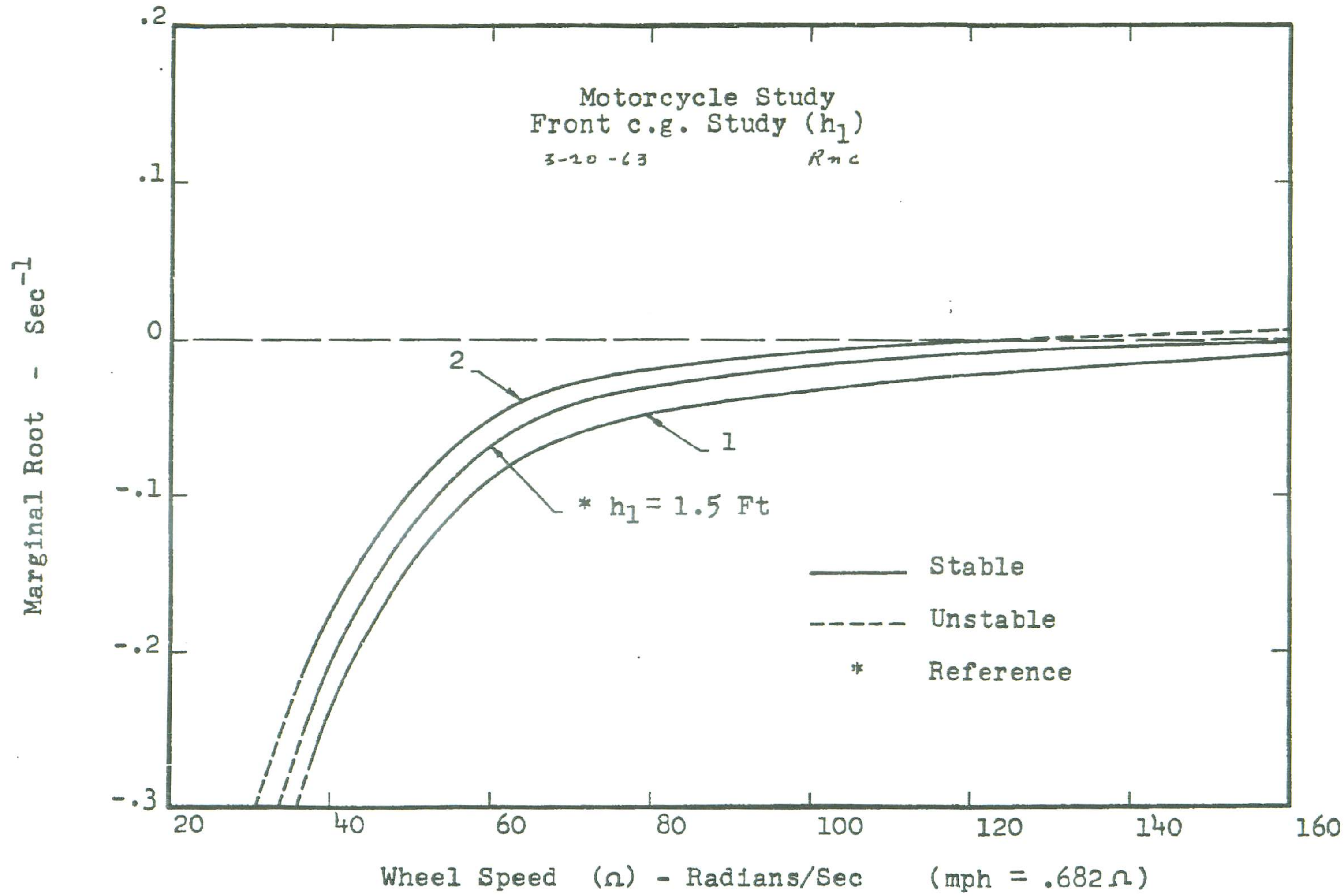
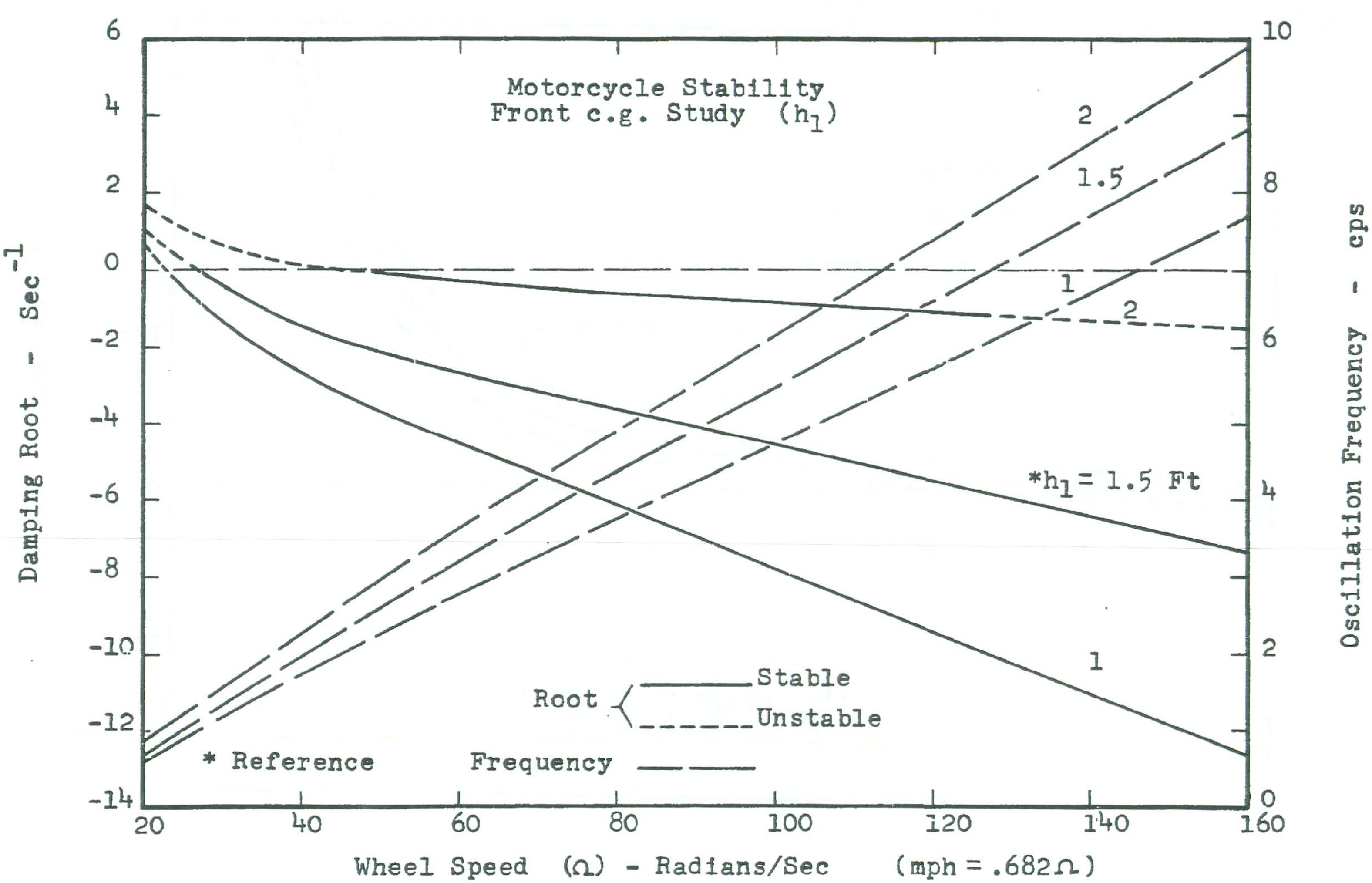
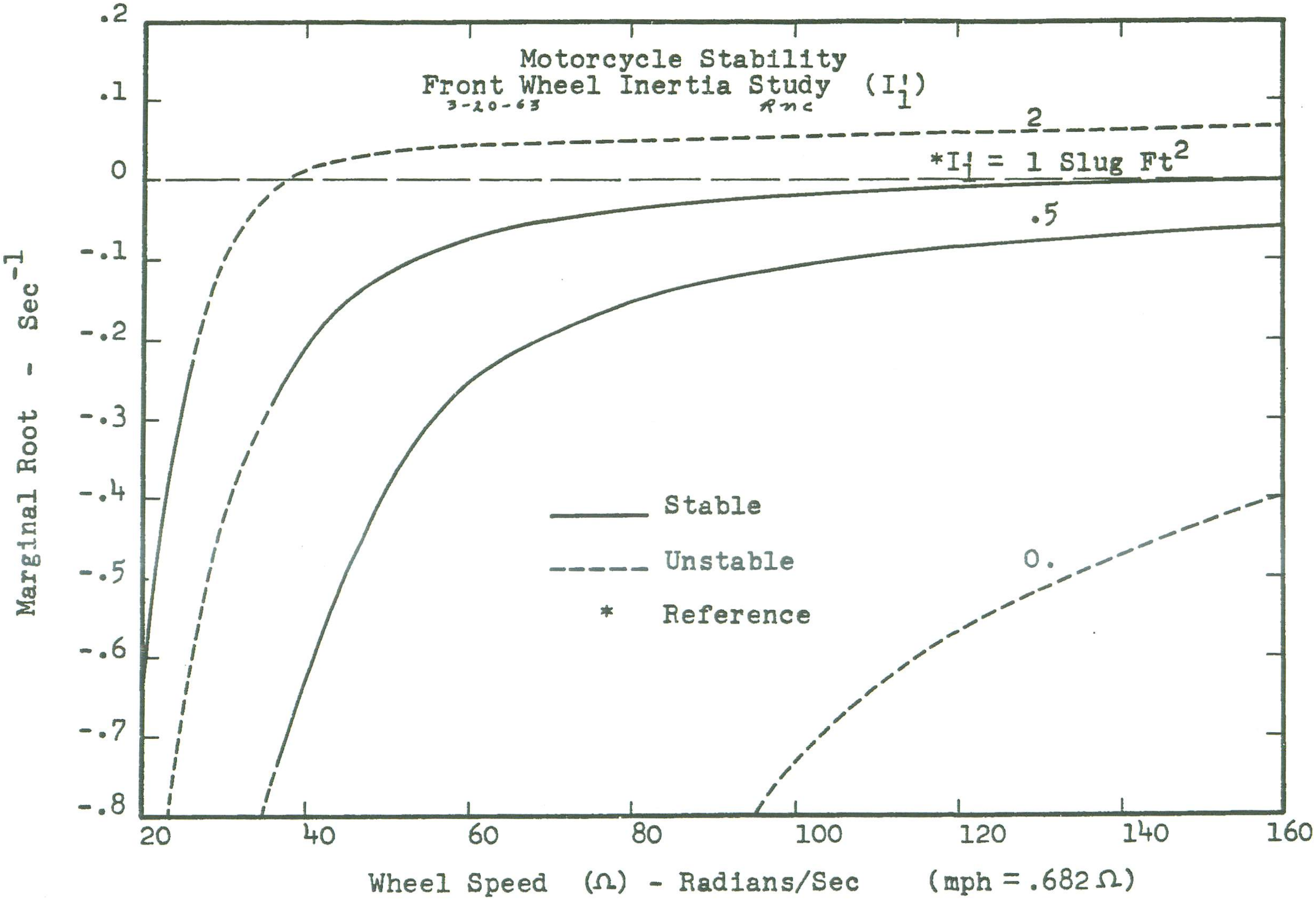

Motorcycle Study
Front c.g. Study (h_1)
3-20-63
Rmc
.2
.1
0
-.1
-.2
-.3
Marginal Root - Sec^{-1}
2
1
* h_1 = 1.5 Ft
Stable
Unstable
* Reference
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.5 a



Motorcycle Stability
Front c.g. Study (h_1)
Damping Root - Sec^{-1}
Oscillation Frequency - cps
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
2
1.5
1
2
$*h_1 = 1.5$ Ft
1
Root
Stable
Unstable
* Reference
Frequency
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.5 b



Motorcycle Stability
Front Wheel Inertia Study (I'1)
3-20-63
RMC
2
*I'1 = 1 Slug Ft²
.5
0.
Stable
Unstable
* Reference
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
Marginal Root - Sec⁻¹
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.6 a

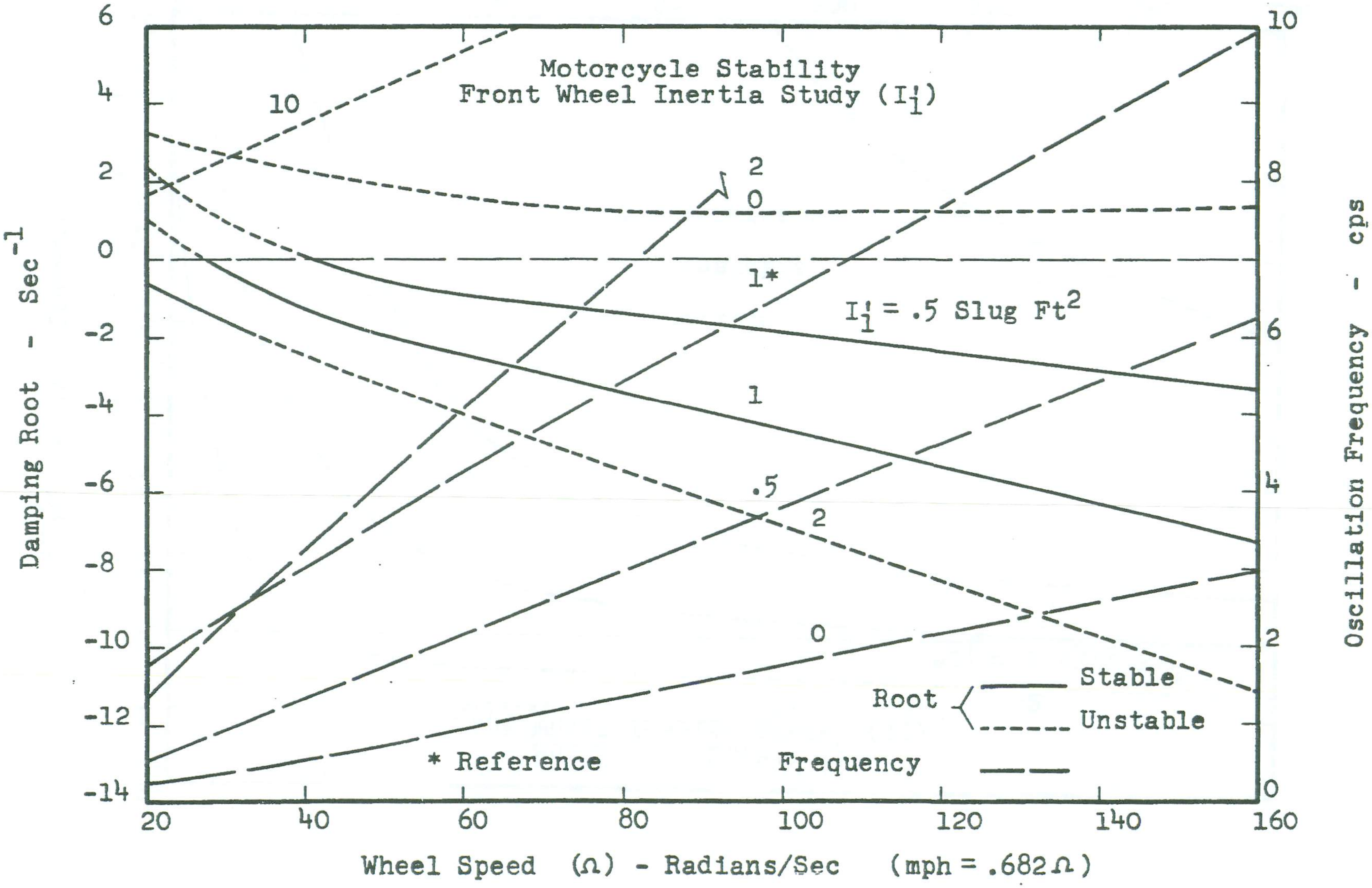
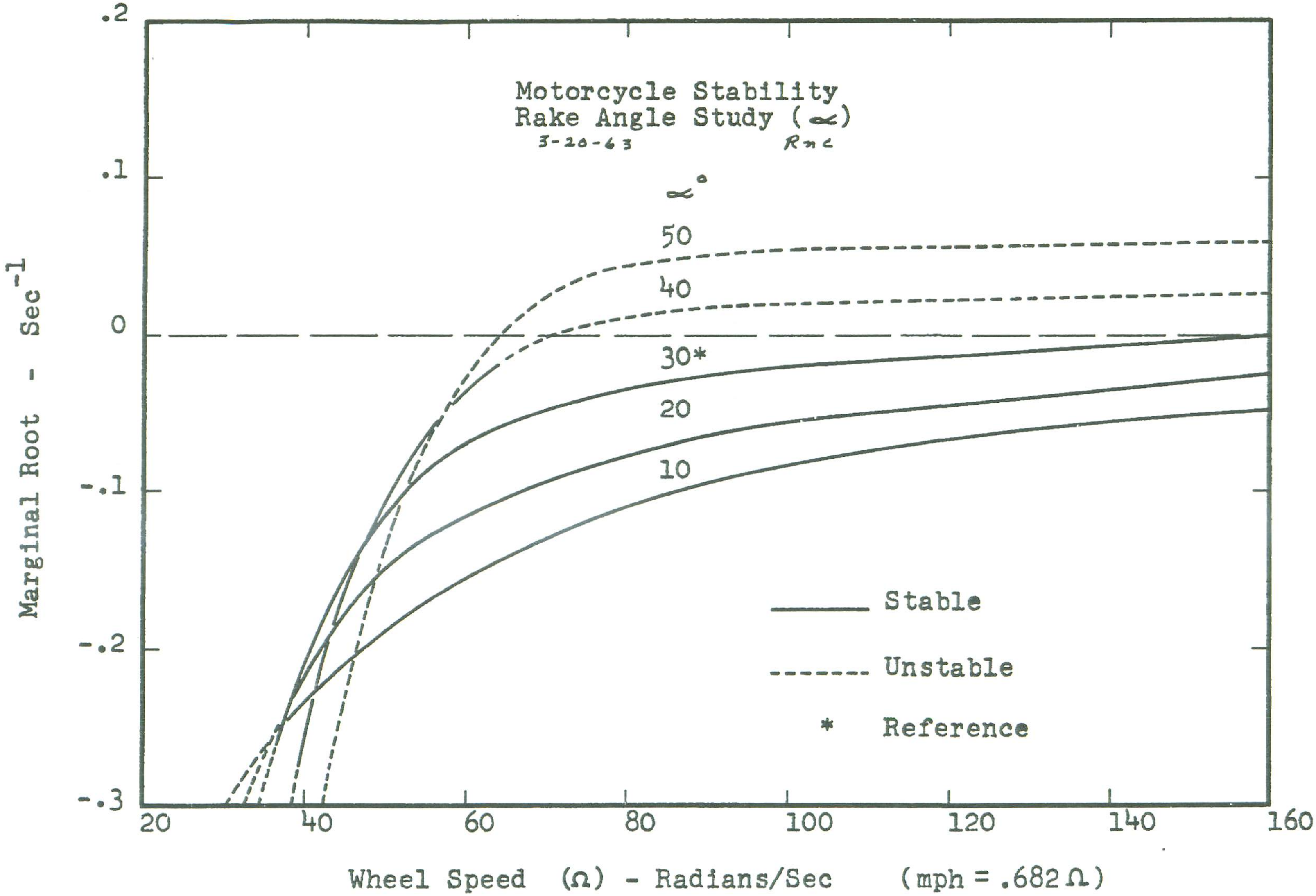

Motorcycle Stability
Front Wheel Inertia Study (I_1')
I_1' = .5 Slug Ft^2
10
2
0
1*
1
.5
2
0
Root
Stable
Unstable
Frequency
* Reference
Damping Root - Sec^{-1}
Oscillation Frequency - cps
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
20
40
60
80
100
120
140
160
10
8
6
4
2
0

Figure 6.6 b



Motorcycle Stability
Rake Angle Study (∝)
3-20-63
RMC
∝°
50
40
30*
20
10
Stable
Unstable
* Reference
.2
.1
0
-.1
-.2
-.3
20
40
60
80
100
120
140
160
Marginal Root - Sec^{-1}
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.7 a

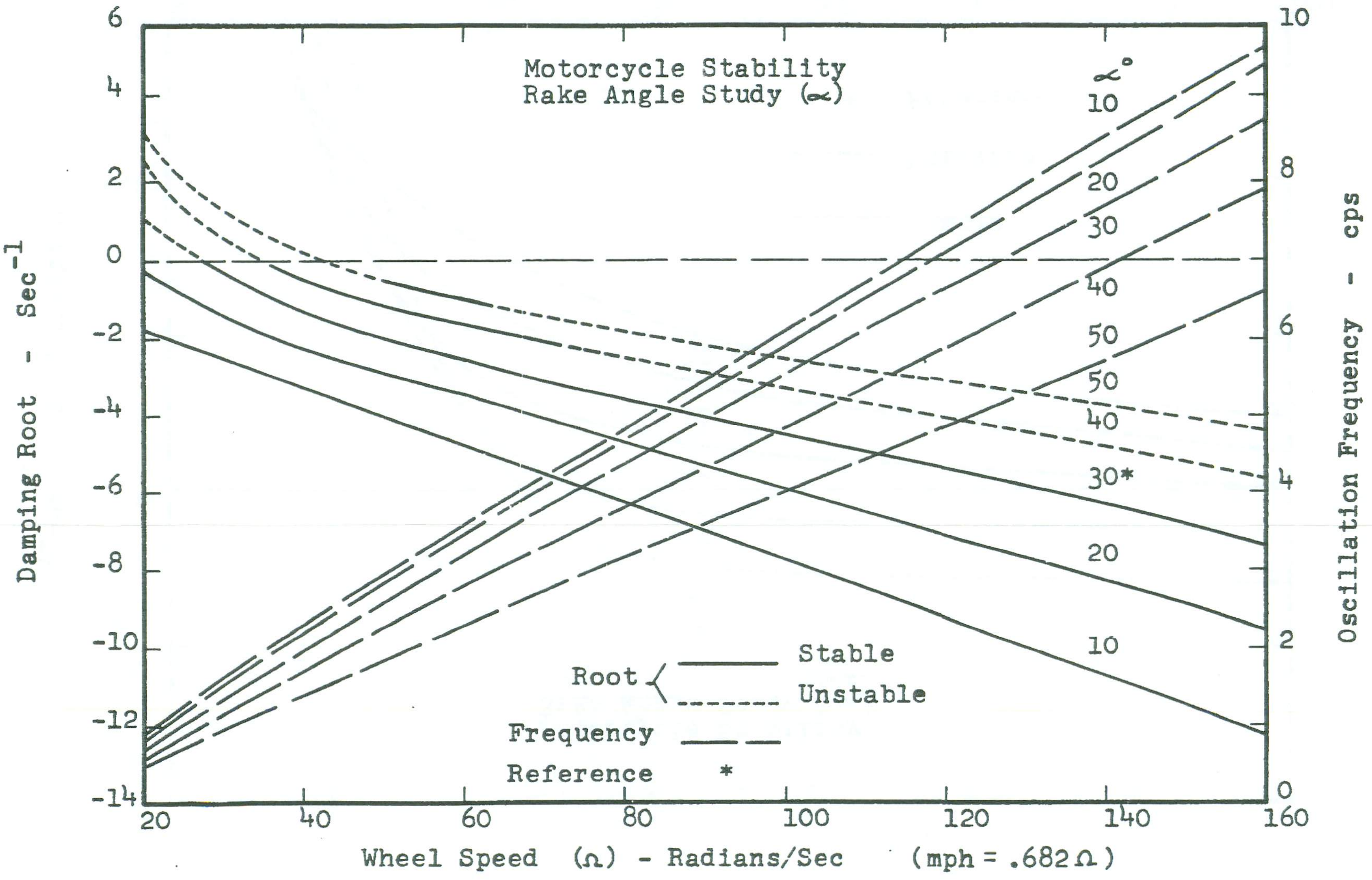

Motorcycle Stability
Rake Angle Study (α)
α°
10
20
30
40
50
50
40
30*
20
10
Root
Stable
Unstable
Frequency
Reference *
Damping Root - Sec^{-1}
Oscillation Frequency - cps
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.7 b

reasonable changes in mass and mass distribution. This is exemplified by the changes in I_{z1}.

It should be noted that I_{z1} has the capacity to alter the Damping Root with no appreciable effect on the Marginal Root.

Figure 6.9: Since the roots change in opposite directions an optimum value of the c.g. height of System 2 is indicated. In practice a low c.g. is apparently desirable for the vehicle itself because of the location of the rider mass.

Figure 6.10: The stability is relatively insensitive to movement of the rear c.g. along the x_2 axis.

The stability is considerably more sensitive when the rear c.g. is shifted along both the x_2 and z_2 axes simultaneously. This combines the changes shown in Figures (6.9) and (6.10).

Figure 6.11: P_{xz1}, like I_{z1}, shows the capacity to alter the Damping Root without appreciably changing the Marginal Root.

In addition to the machine constants shown on the preceding pages, b_1, b_2, c_1, c_2, I_2', m_1, m_2, and P_{xz2} were studied. Only the extreme values of these constants had any appreciable effect on either the stability or the

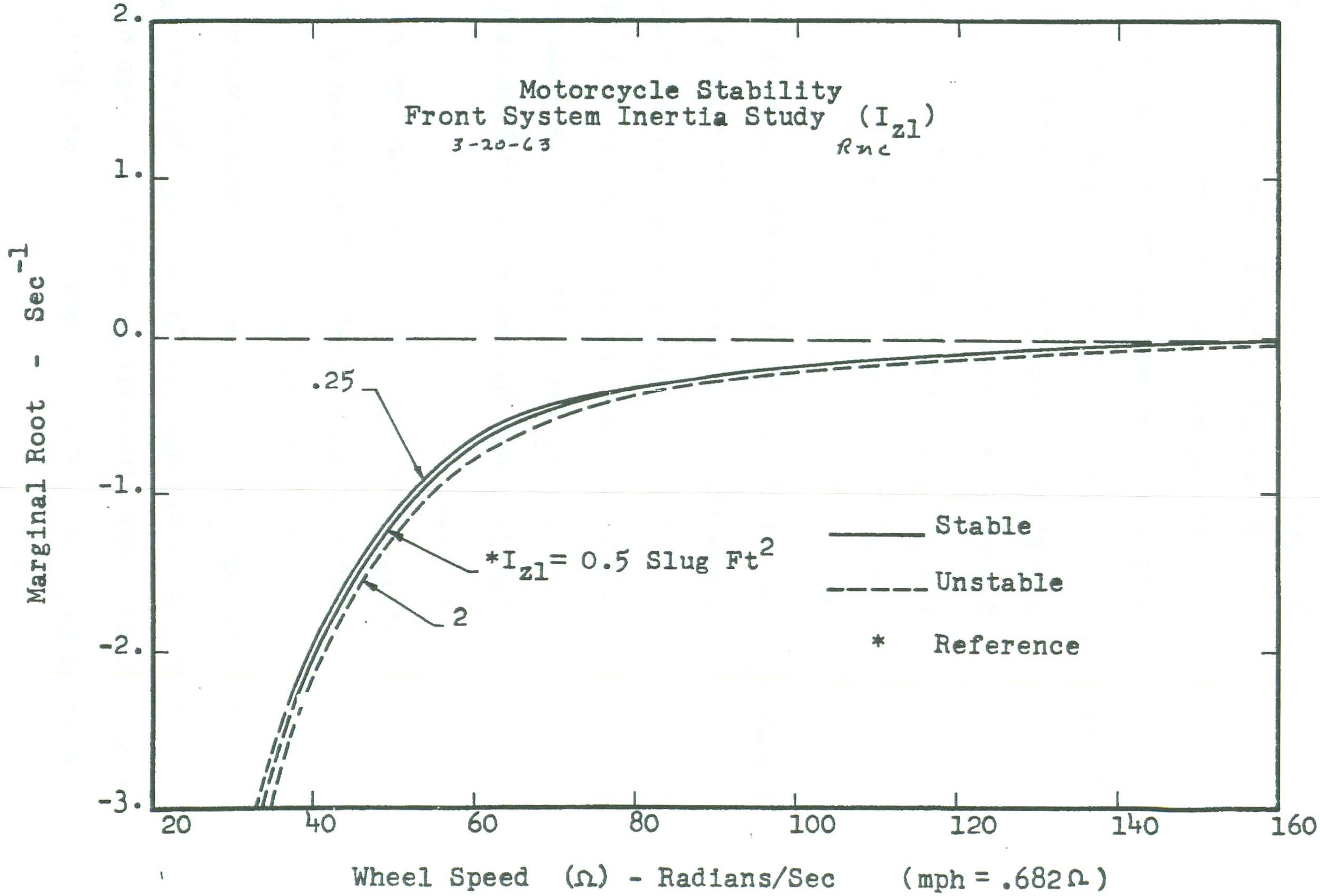

Motorcycle Stability
Front System Inertia Study (I_{z1})
3-20-63
Rnc
2.
1.
0.
-1.
-2.
-3.
Marginal Root - Sec^{-1}
.25
*I_{z1}= 0.5 Slug Ft^2
2
Stable
Unstable
* Reference
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.8 a

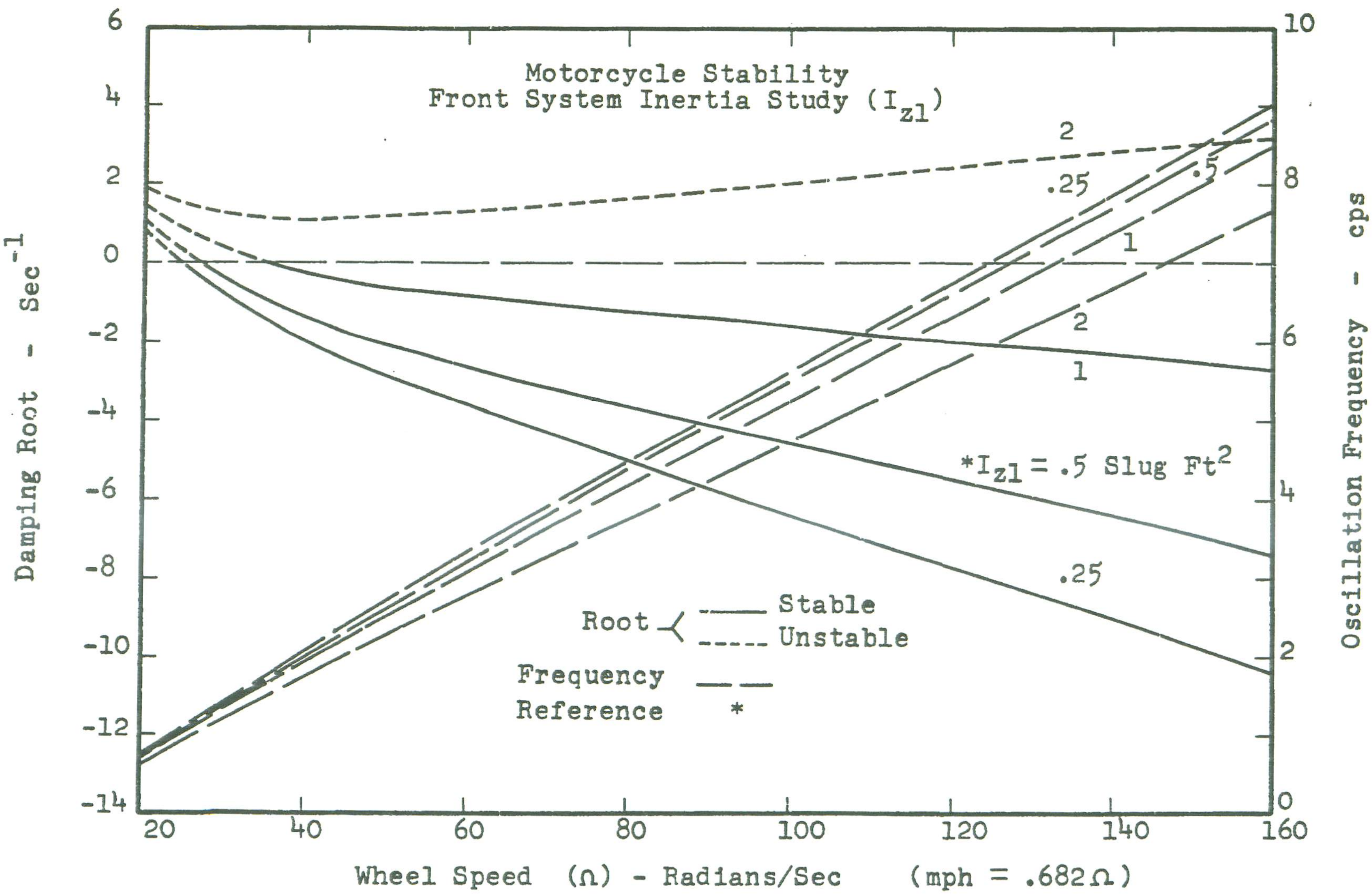

Motorcycle Stability
Front System Inertia Study (I_{z1})
Damping Root - Sec^{-1}
Oscillation Frequency - cps
Root Stable
Unstable
Frequency
Reference *
*I_{z1} = .5 Slug Ft^2
2
.25
.5
1
2
1
.25
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.8 b

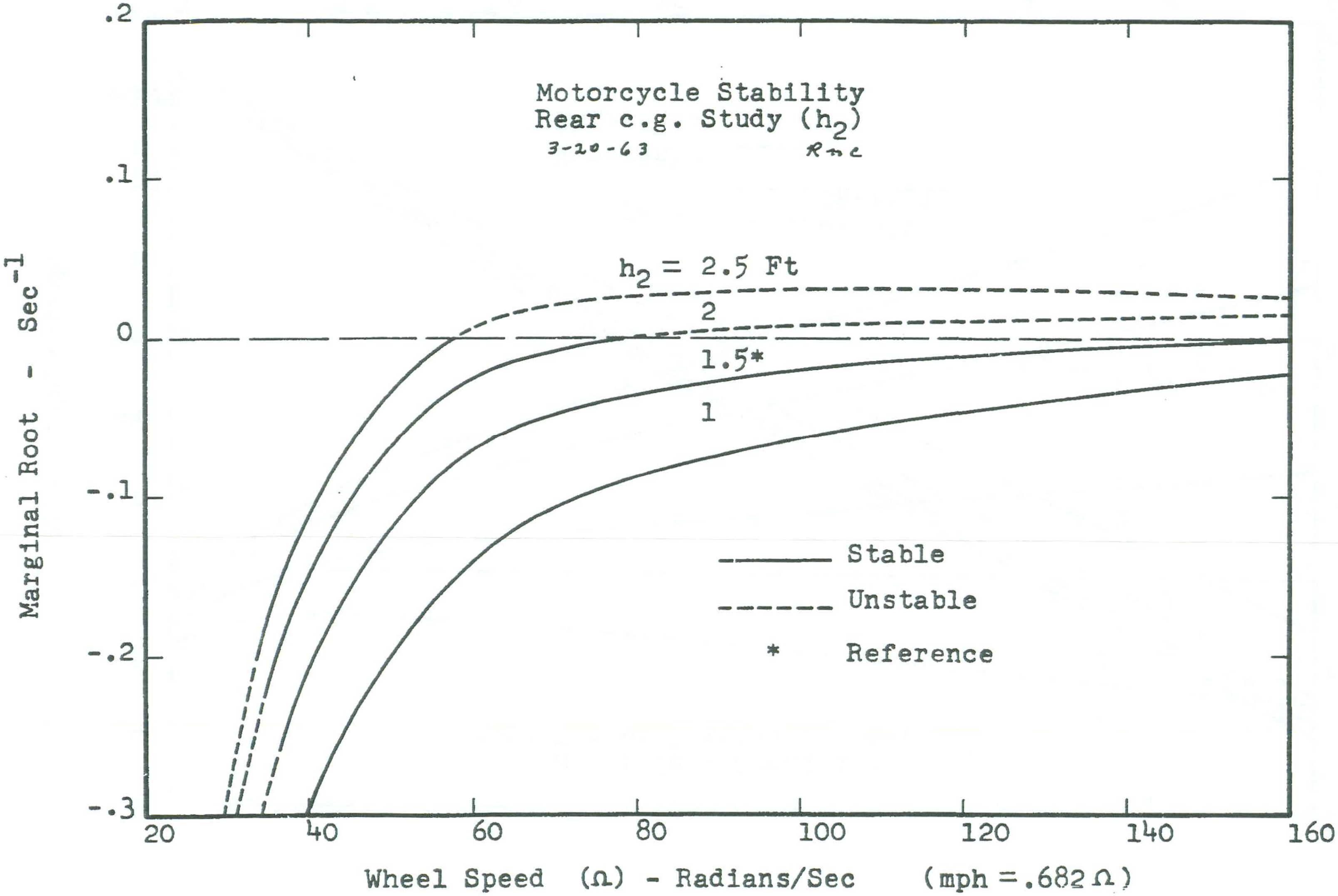

Motorcycle Stability
Rear c.g. Study (h_2)
3-20-63
Rnc
h_2 = 2.5 Ft
2
1.5*
1
Stable
Unstable
* Reference
.2
.1
0
-.1
-.2
-.3
Marginal Root - Sec^{-1}
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)

Figure 6.9 a

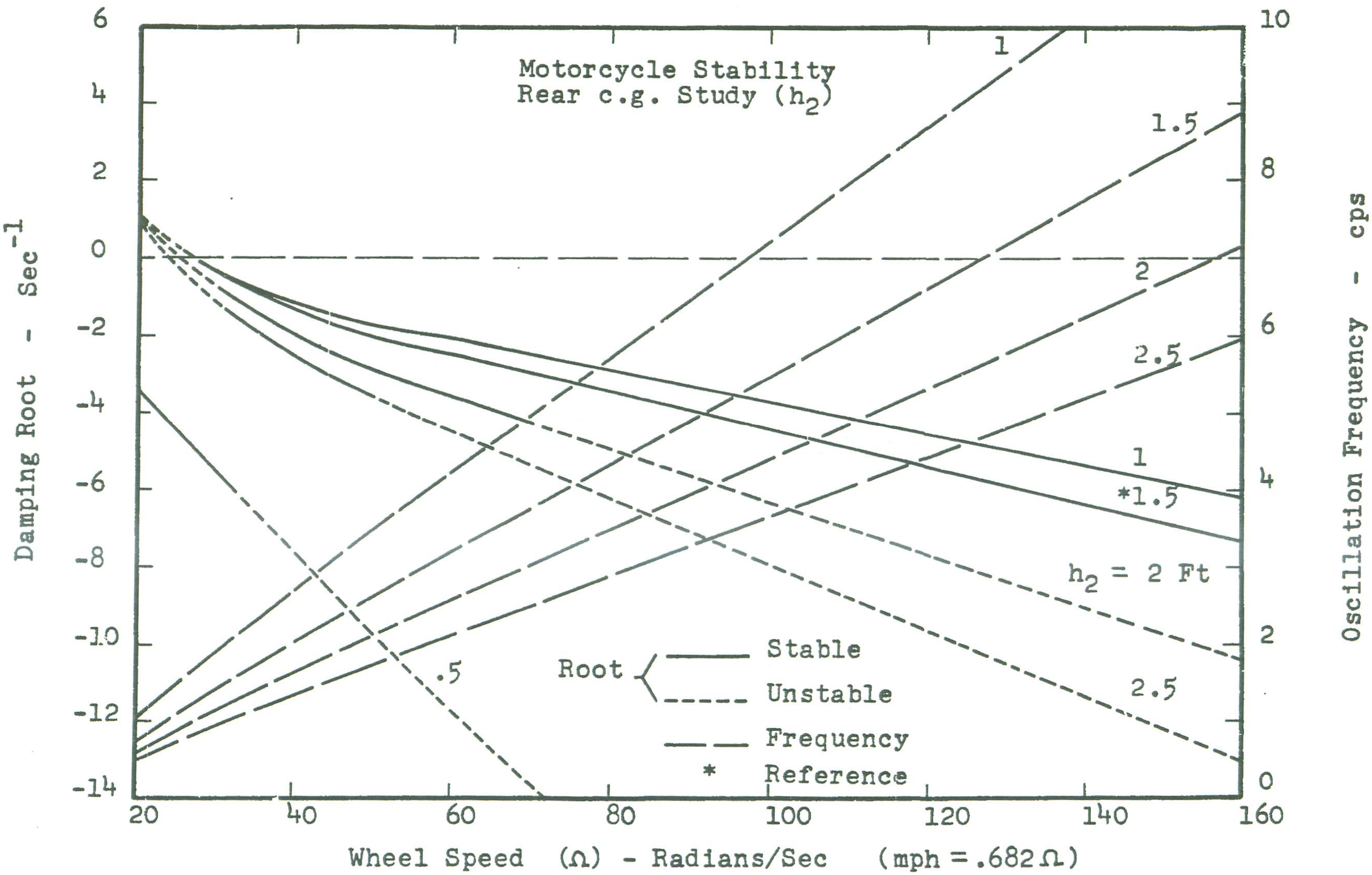

Motorcycle Stability
Rear c.g. Study (h_2)
Damping Root - Sec^{-1}
Oscillation Frequency - cps
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
1
1.5
2
2.5
1
*1.5
h_2 = 2 Ft
2.5
.5
Root
Stable
Unstable
Frequency
* Reference

Figure 6.9 b

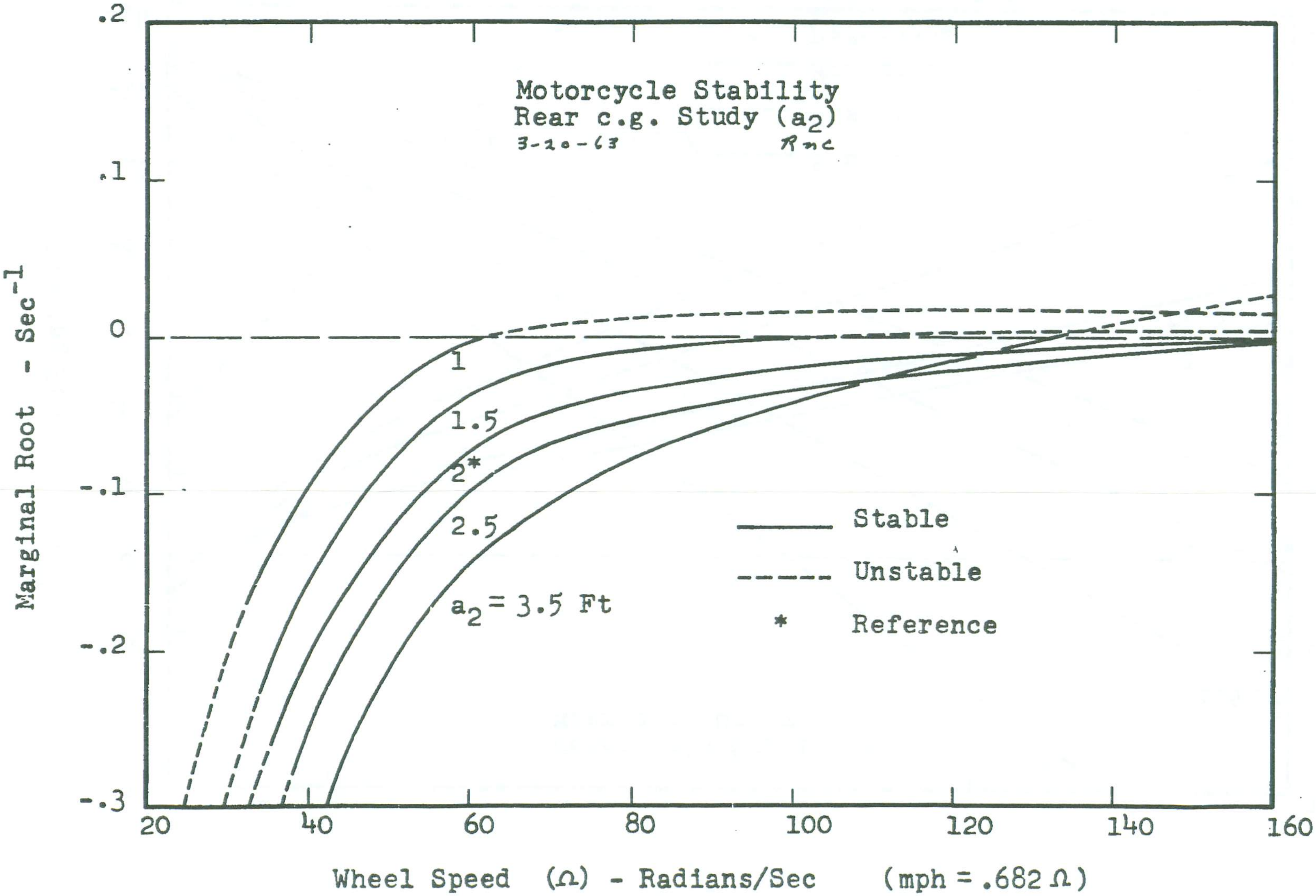

Motorcycle Stability
Rear c.g. Study (a_2)
3-20-63
RMC
.2
.1
0
-.1
-.2
-.3
20
40
60
80
100
120
140
160
Marginal Root - Sec^{-1}
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)
1
1.5
2*
2.5
a_2 = 3.5 Ft
Stable
Unstable
* Reference

Figure 6.10 a

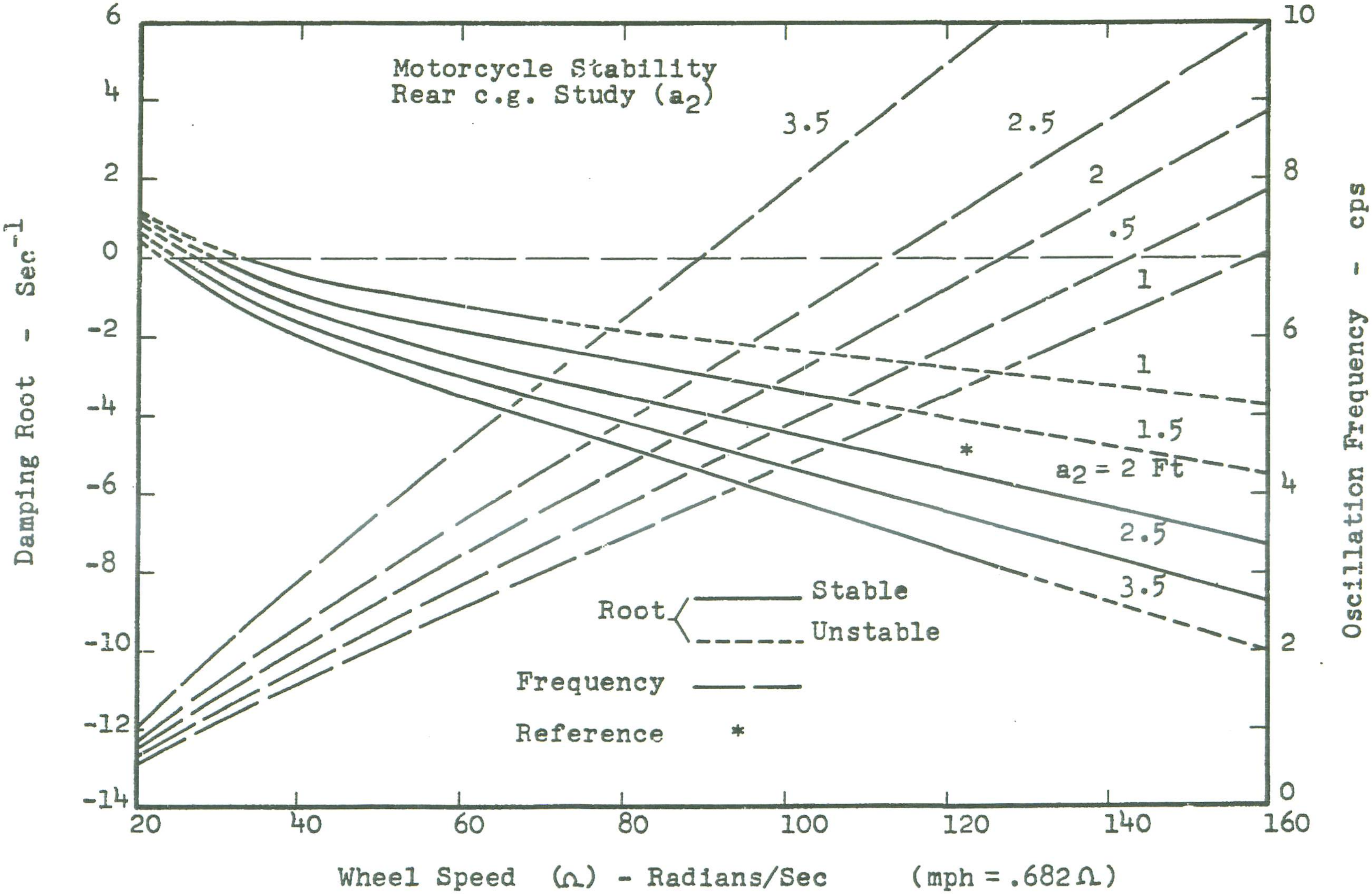

Motorcycle Stability
Rear c.g. Study (a_2)
3.5
2.5
2
.5
1
1
1.5
a_2 = 2 Ft
2.5
3.5
*
Root
Stable
Unstable
Frequency
Reference *
Damping Root - Sec^{-1}
Oscillation Frequency - cps
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.10 b

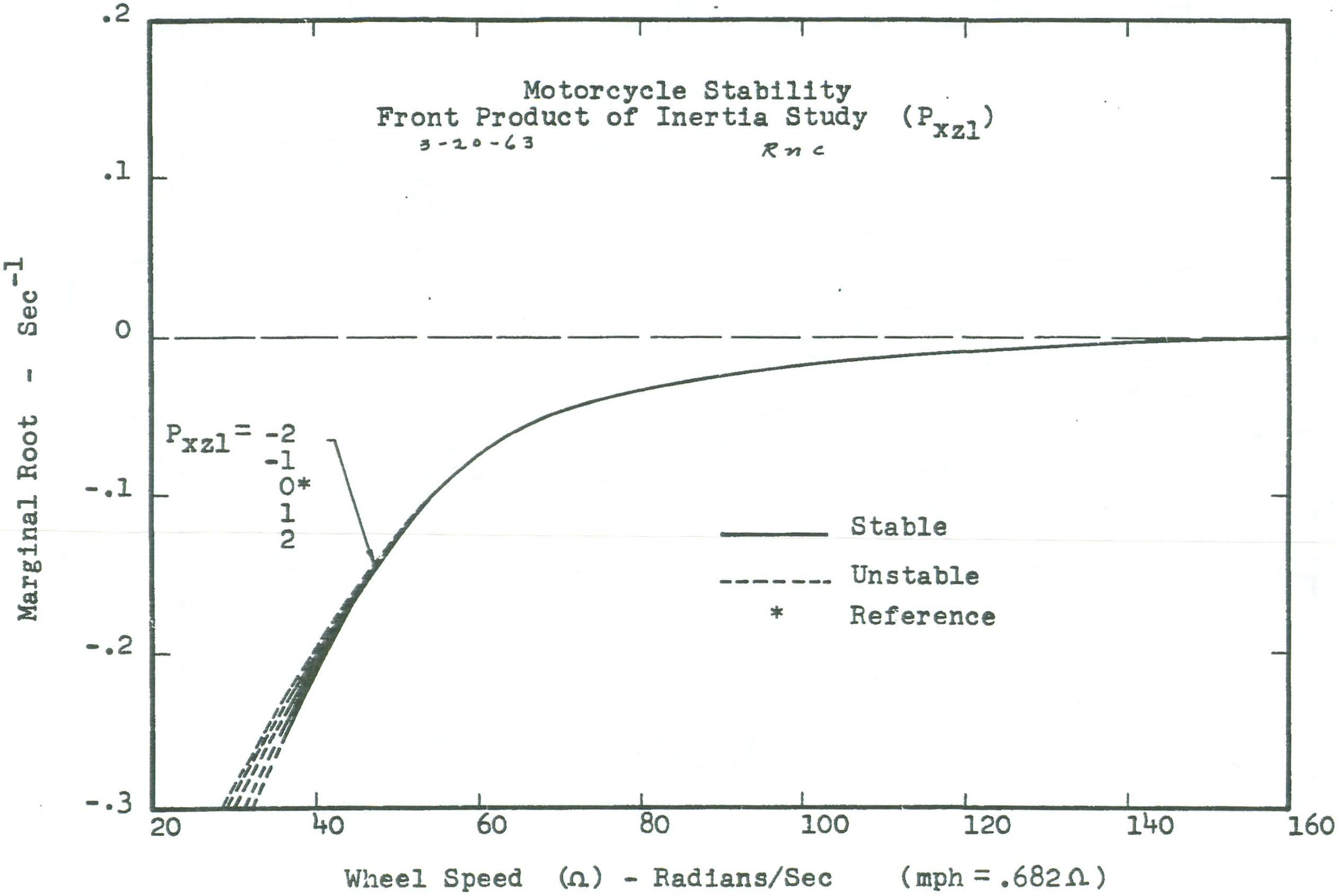
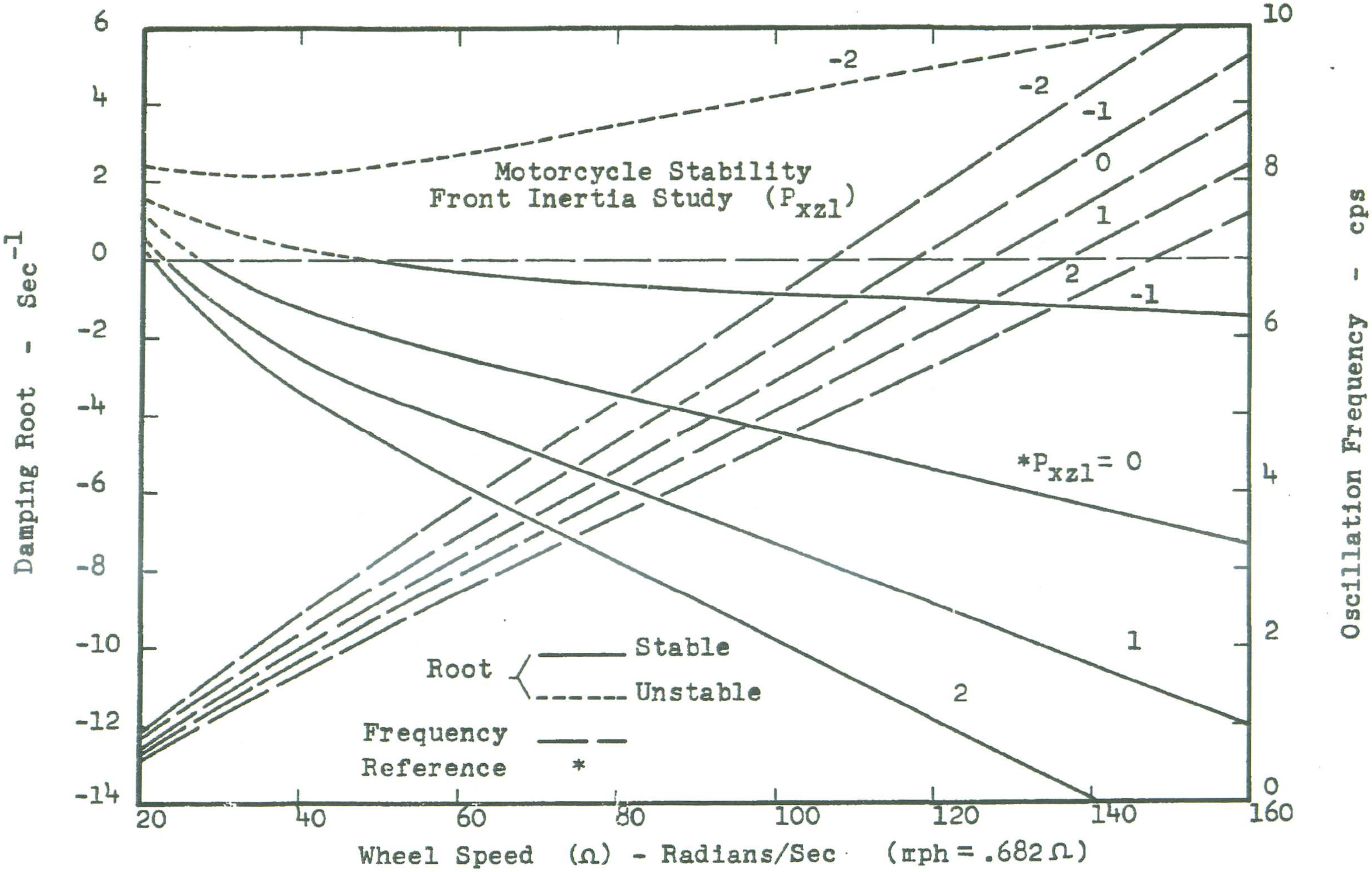

Motorcycle Stability
Front Product of Inertia Study (P_{xz1})
3-20-63
RNC
.2
.1
0
-.1
-.2
-.3
Marginal Root - Sec^{-1}
P_{xz1} = -2
-1
0*
1
2
Stable
Unstable
* Reference
20
40
60
80
100
120
140
160
Wheel Speed (Ω) - Radians/Sec (mph = .682Ω)

Figure 6.11 a



Motorcycle Stability
Front Inertia Study (P_{xz1})
Damping Root - Sec^{-1}
Oscillation Frequency - cps
Wheel Speed (Ω) - Radians/Sec (mph = .682 Ω)
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
10
8
6
4
2
0
20
40
60
80
100
120
140
160
-2
-2
-1
0
1
2
-1
*P_{xz1} = 0
1
2
Root
Stable
Unstable
Frequency
Reference *

Figure 6.11 b

frequency of oscillation.

Conclusions

The numerical results presented in this chapter are from an application of the computer program to an assumed motorcycle model. The analysis leading to the program is shown in the first five chapters; the program and its use are detailed in the Appendix; the characteristics of the hypothetical model are given on page 89 with reference to Figure (2.1).

The results predict the stability effect on this specific model of the steering axis damping, trail (see page 98 for definition), location of the c.g. of the front and rear ends, front wheel inertia, rake angle (see Figure 2.1 for definition), moment of inertia of the front end about the steering axis, and the product of inertia of the front end.

No experimental data is available to accurately check these calculated results; however they are generally consistent with the qualitative results given in the literature. Four specific areas of agreement can be cited in support of the validity of these equations:

(1) Consistent with common experience, instability is predicted for low speeds.

(2) Consistent with the literature and the quali-

tative analysis in the first part of this chapter, the results indicate an optimum value of trail and instability for extreme values of trail.

(3) Consistent with widely accepted ideas on the subject, the results indicate that the gyroscopic effect of the front wheel is critical for stability. Instability is indicated for extreme values of the front wheel moment of inertia.

(4) Consistent with the qualitative analysis in the first part of this chapter, the stability is most effected by the characteristics of the front end.

For the assumed model, the greatest effect on stability is indicated for trail and the location of the front c.g. along the x_1 axis (see Figure 2.1). An optimum value of trail between 0.6 and 1 feet is indicated. The optimum position of the front c.g. is indicated as approximately 0.3 feet in front of the steering axis. Note however that these values are varied from the basic set of machine characteristics and in no instance were combinations of two or more variables studied.

In addition to these variables, the results indicate increased stability for low values of rake angle with instability occurring at rake angles greater than forty degrees. Notice that the change in rake angle automatically changes the trail since the remaining vehicle characteristics were unchanged.

Caution should be exercised in trying to generalize the results of the limited calculations presented here. The variety of opinions(3) on the effects of trail and rake angle bear witness to the complexity of the problem. The effect of any one design variable obviously depends on the remaining characteristics of the particular machine being studied.

Recommendations

The practical value of these equations can only be established by extensive experimental work. A realistic approach appears to be the use of this program in conjunction with test riding. Rider opinion should be adequate to establish the relative importance of the two roots. However, the rider should be cautioned to distinguish between cornering and stability performance. There is little doubt that any final design must compromise between these two characteristics.

In practice, two additional factors which are known to be important to stability are tires and machine flexi-

(3) See the discussion of the Wilson-Jones paper.

bility. With considerable effort it appears practical to develop equations to account for both of these factors. This step should, of course, be preceded by a thorough test of the equations developed here.

In view of the earlier assumptions of a rigid machine and thin disc wheels without slip, the tire and flexibility factors should be minimized on the test machine.

APPENDIX

The program shown in this section is written for the IBM 1620 Computer using the Forgo compiler program.

The following statements are to clarify the program for future use:

(1) The second and third lines are the input constants which physically describe the machine being investigated.

(2) The fourth line (statement 6) designates the specific machine constants to be studied. In this case a_1 and l_1 are to assume the values shown at the end of the program. The term XX is the control which either ends the program or causes the computer to read a new input card according to its value of one or zero respectively.

(3) The calculations for the coefficients of the fourth degree polynomial are concluded at statement 4 plus 5 lines. The evaluation of the roots are concluded at statement 11 plus 3 lines.

(4) Statement 12 prints out the coefficients.

(5) Statement 14 prints out the roots. Eight columns are printed for the roots but only the four columns given in exponential form

(having an E as part of the answer) are significant. The first four columns are real roots. The last four columns give the complex roots with columns five and seven being the real parts and columns six and eight their respective imaginary parts.

(6) Each input card yields eight rows of coefficients and the corresponding roots. These rows correspond to the speeds of 20 to 160 radians per second at increments of 20 radians per second.

```
C   C  127817     R N COLLINS   MOTORCYCLE COEFFICIENTS AND ROOTS

       READ,A1,A2,B1,B2,C1,C2,C5,D2,GC,H1,H2,XIX1,XIX2,XIZ1,XIZ2
       READ,XI1P,XI2P,XL1,XL2,XM1,XM2,PXZ1,PXZ2,R1,R2,SI,CO,C3
     6 READ,A1,XL1,XX
       C9=C1+C2
       C10=A1-XL1
       D1=(H2+D2+C10*SI-H1*CO)/CO
       R=C10*CO+A2+XL2+(H1+D1)*SI
       C11=A2/H2
       C12=C11*SI-CO
       X1=R+C10*C12
       X2=R*SI+H2*CO
       X3=R*CO-H2*SI
       X4=R/SI
       X5=X2-D1
       X6=C11*CO+SI
       C13=CO**2
       C14=R*CO-C10
       X7=X4*(1.+C13)-H2*CO
       X9=((-C11)/SI)*C14
       X10=C14-H2*SI
       X11=X2-H1-D1
       X12=C10*SI/H2
       C15=XM1*A1*H1
       C16=XM1*A1**2
       C17=XM1*H1**2
       C18=XM1*A1
       C19=XM1*H1
       C20=C10/X1
       C21=XM2*A2
       C22=X12/X1
       C23=C22*X6
       C24=C22*X9
       C25=R2**2
```

```
C26=C1*C25
C27=C2*C25
C28=C15+PXZ1
X13=XIZ1+C16-C20*X6*(C28)
X14P=C20*(X6*(C28*SI-C17*CO-XIX1*CO)-PXZ2-C11*XIX2)
X14=X14P-(C16+XIZ1)*SI+(C15+PXZ1)*CO
X15=C20*C14
X16P=(C11*C14/R)*(C26*((R*CO/SI)-H2+A1*SI-B1)-C27*(B2-H2))
X16=C20*(X16P-(C27/SI)*X6*(C14-B2*SI))
X17P=(-C11/R)*C14*(C19*SI+C18*CO-C21)+X6*(C19*SI-C21)
X17=GC*(C20*X17P-C18*SI)
X18=C20*(-XI2P-X6*XI1P*R2*SI/R1)+XI1P*R2*CO/R1
X19=C20*C12-1.
X20=XM1*GC*(C20*H1*X6-A1)
X21P=R2*X6*(C15*CO+C17*SI-C19*R+XIX1*SI+PXZ1*CO-XI1P*R/(R1*CO))
X21Q=(C20*CO/R)*(X21P+R2*(C11*PXZ2+XIZ2))
X21=X21Q-(R2*CO/R)*(C16*CO+C15*SI-C18*R+XIZ1*CO+PXZ1*SI)
X22P=C20*(-X6*(C19+(XI1P*CO/R1))-XI2P*C11/R2)
X22=(R2**2)*(CO/R)*(X22P+C18-XI1P*SI/R1)
C29=(CO/H2)-C23
X23=C29*C28
C30=C28*SI-(C17+XIX1)*CO
X24P=(-C22)*(PXZ2+C11*XIX2)+C23*C30
X24Q=(1./H2)*(CO*C30-(XM2*H2**2)-XIX2)
X24=X24P-X24Q
X25=C22*(-C12)+SI/H2
C31=(X3/SI)+H2
C32=C24+1.+(D2/H2)+(X10*C13/(H2*SI))+X11*CO/H2
X26=(L./X4)*(-C31)*C32+C29*(C31-C10/SI)
X27P=(C26/X4)*((X3/SI)+A1*SI-H1*CO-(B1-H1*CO))*C32
X27Q=C27*((H2-B2)*((1./X4)*C32-C29)+(X10/SI)*(-C29))
X27=X27P+X27Q
X28=(-GC/X4)*C32*(C18*CO+C19*SI-C21)+GC*C29*(C19*SI-C21)
X29=(C20*SI/H2)*(XI2P+(XI1P*R2*SI/R1)*X6)-XI1P*SI*CO*R2/(H2*R1)
X30=GC*(C19*C29+XM2)
```

```
      C33=R2*CO/R
      X31P=C33*(C19*(A1*CO+H1*SI)+(XIX1*SI+PXZ1*CO))-R2*(C19*CO+XI1P/R1)
      X31=C29*X31P-C33*(C22*(PXZ2*C11+XIZ2)-((PXZ2/H2)-C21))
      X32P=R2*C33*C29*(C19+XI1P*CO/R1)
      X32=(-X32P)+C33*((XI2P/H2)*(C22*A2-1.)-XM2*R2)
      EOR=1./(X14*X23-X13*X24)
      E1P=(X14*X25+X19*X24)*C5
      E1Q=X14*X31+X18*X23+X21*X24+X13*X29
      E2P=X14*X28+X20*X23+X17*X24+X13*X30
      E2Q=(X18*X25-X19*X29)*C5
      E2R=(X14*X26+X15*X24)*C9*C25
      E2S=X16*X24+X14*(X27+X32)+X18*X31+X22*X24-X21*X29
      E2T=E2R+E2S
      E2U=(X14*X26+X15*X24)*C3*R2
      E3P=(X20*X25-X19*X30)*C5
      E3Q=X18*X28+X20*X31-X17*X29-X21*X30
      E3R=(X18*X26-X15*X29)*C9*C25+X18*(X27+X32)-X29*(X16+X22)
      E3S=(X18*X26-X15*X29)*C3*R2
      E4P=X20*X28-X17*X30
      E4Q=(X20*X26-X15*X30)*C9*C25+X20*(X27+X32)-(X16+X22)*X30
      E4R=(X20*X26-X15*X30)*C3*R2
      W=20.
    4 C40=W*W*W
      C41=W*W
      E1=(E1P+E1Q*W)*EOR
      E2=(E2P+(E2Q+E2U)*W+E2T*C41)*EOR
      E3=(E3P+(E3Q+E3S)*W+E3R*C40)*EOR
      E4=(E4P+E4R*W+E4Q*C41)*EOR
      Y1=-E2*.5
      Y2=(E1*E3*.25)-E4
      Y3=(4.*E2*E4-E1*E1*E4-E3*E3)*.125
      G=((2.*Y1*Y1-9.*Y2)*Y1/27.)+Y3
      H=(3.*Y2-Y1*Y1)/9.
      F=ABS(G*G+4.*H*H*H)
      UP=(-G+SQRT(F))*.5
```

```
      IF(UP)62,61,61
   61 U=(UP)**.33333
      GO TO 63
   62 U=-(-UP)**.33333
   63 Y=U-(H/U)-Y1/3.
      F1=2.*Y+(E1*E1*.25)-E2
      IF(F1)65,64,64
   64 A=SQRT(F1)
      GO TO 66
   65 F1P=-F1
      A=SQRT(F1P)
   66 B=(Y*E1-E3)/(2.*A)
      DES1=(((E1*.5)-A)**2)-4.*(Y-B)
      DES2=(((E1*.5)+A)**2)-4.*(Y+B)
      IF(DES1)8,7,7
    7 DR1=(-(E1*.5-A)+SQRT(DES1))*.5
      DR2=(-(E1*.5-A)-SQRT(DES1))*.5
      DR1R=0.
      DR1IM=0.
      GO TO 9
    8 DR1R=-(E1*.5-A)*.5
      DR1IM=SQRT(-DES1)*.5
      DR1=0.
      DR2=0.
    9 IF(DES2)11,10,10
   10 DR3=(-((E1*.5)+A)+SQRT(DES2))*.5
      DR4=(-((E1*.5)+A)-SQRT(DES2))*.5
      DR2R=0.
      DR2IM=0.
      GO TO 12
   11 DR2R=(E1*.5+A)*.5
      DR2IM=SQRT(-DES2)*.5
      DR3=0.
      DR4=0.
   12 PUNCH,E1,E2,E3,E4
```

```
14 PUNCH,DR1,DR2,DR3,DR4,DR1R,DR1IM,DR2R,DR2IM
   W1=W+20.
   W=W1
   IF(W-160.)4,4,5
 5 IF(XX)13,6,13
13 STOP
   END
   .8,2.,3.,2.,.008,.002,0.,2.,32.2,1.5,1.5,1.,3.,.5,8.
   1.,2.,-.3,2.,4.,15.,0.,0.,1.,1.,.5,.866,.005
   -.8,-.3,0.
   -.5,0.,0.
   -.3,.2,1.
```

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TITLE OF THESIS A Mathematical Analysis of the Stability of Two Wheeled Vehicles

Full Name Robert Neil Collins

Place and Date of Birth Honey Grove, Texas, 12-7-25

Elementary and Secondary Education Selfs Grade School

Honey Grove High School

Colleges and Universities: Years attended and degrees Midwestern University, 49-50; University of Oklahoma, 50-53 B. S. M.E.; University of Wisconsin, M. S. M.E. 53-54; University of Wisconsin 56-63

Membership in Learned or Honorary Societies Pi Tau Sigma, Sigma Tau, Tau Beta Pi

Publications A Study Diesel Ignition Delay Using the Hot Motored Technique

Major Department M. E.

Minor(s) C. E.

Date May 16, 1963

Signed A. H. Easton
Professor in charge of thesis