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Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review

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We present canonical linearized equations of motion for a bicycle modeled as four rigid laterally-symmetric ideally-hinged parts: front and rear wheels and frames. The wheels are also axisymmetric and make knife-edge ideal rolling contact with the level ground. The mass distribution, lengths and angles are otherwise arbitrary. The equations are suitable for study of, for example, the self-stability of a bicycle. We derived these equations by hand in two different ways and then checked them against two general-purpose non-linear dynamic simulators in various ways. In the century-old literature we have found a few equations that fully agree with those here. This non-holonomic system has a seven-dimensional accessible configuration space and three velocity degrees of freedom parameterized by rates of frame lean, steer angle and rear-wheel rotation. The terms in the equations are constructed methodically for easiest implementation. For two sets of benchmark parameters we accurately calculate the stability eigenvalues and the speeds at which the bicycle is self-stable. These provide a comparison case for checking alternative formulations of the equations of motion or alternative numerical solutions. The results here can also serve as a check for general-purpose dynamics programs.

Keywords: bicycle, dynamics, linear, stability, nonholonomic, Benchmark.

1. Introduction

In 1818 Karl von Drais (Herlihy 2004) showed that a person riding forward on a contraption with two in-line wheels, a sitting scooter of sorts, could balance by steering the front wheel. Later, the velocipede of the 1860's which had pedals directly driving the front wheel like a child's tricycle, could also be balanced using

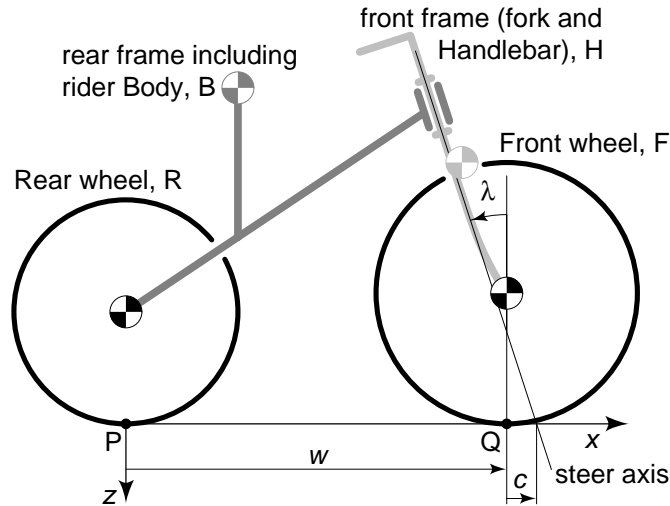


Figure 1. **Bicycle model parameters.** For all four parts (R, B, F and H), centre of mass locations are expressed relative to the $x-z$ coordinates shown (with origin at P and y pointing towards the reader) and in the reference configuration shown. Other parameters include the body masses, the wheel radii, the tilt λ of the steer axis, the wheel base w and the trail c and are listed in Table 1. The figure is drawn to scale using the distances in Table 1. Configuration variables (lean, steer, etc) are defined in figure 2.

active steering control. This so-called boneshaker had equal size wooden wheels and a vertical steering axis passing through the front wheel axle. By the 1890's it was well known that essentially anyone could learn to balance a “safety bicycle”. The safety bicycle had pneumatic tires and a chain drive. More subtly, but more importantly for balance and control, the safety bicycle also had a tilted steer axis and fork offset (bent front fork) like a modern bicycle. French mathematician Emmanuel Carvallo (1898) and Cambridge undergraduate Francis Whipple (1899) then used rigid-body dynamics equations to show in theory what was surely known in practice, that some safety bicycles could, if moving in the right speed range, balance themselves.

Today these same two basic features of bicycle balance are clear:

- A controlling rider can balance a forward-moving bicycle by turning the front wheel in the direction of an undesired lean. This moves the ground-contact points back under the rider, like an inverted broom or stick can be balanced on an open hand by accelerating the support point in the direction of lean.
- Some uncontrolled bicycles can balance themselves. If a good bicycle is given a push (to about 6 m/s), it steadies itself and then progresses stably until its speed gets too low. The torques for the self-correcting steer motions can come from various geometric, inertial and gyroscopic features of the bike.

Beyond these two generalities, there is little that has been solidly established in the literature, perhaps because of the lack of need. Through trial and error bicycles had evolved by 1890 to be stable enough to survive to the present day with essentially no modification. Because bicycle design has been based on tinkering rather than

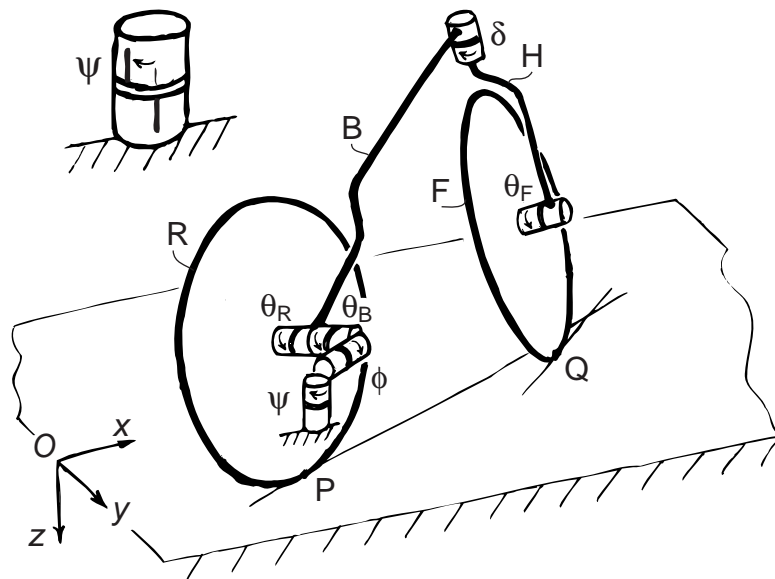


Figure 2. **Configuration and dynamic variables.** The 7-dimensional accessible configuration space is parameterized here by the x and y coordinates of the rear contact P, measured relative to a global fixed coordinate system, and 5 Euler-like angles represented by a sequence of hinges (gimbals). The hinges are drawn as a pair of cans which rotate with respect to each other. For a positive rotation, the can with arrow rotates in the direction of the arrow relative to its mate as shown on the enlarged isolated can at the top left. For example, a clockwise (looking down) change of heading (yaw or steer) ψ of the rear frame B relative to the fixed x axis, a right turn, is positive. The lean ('roll' in aircraft terminology) to the right is ϕ . The rear wheel rotates with θ_R relative to the rear frame, with forward motion being negative. The steer angle is δ with right steer positive. The front wheel rotates with θ_F relative to the front frame. As pictured, ψ , ϕ and δ are all negative. The velocity degrees of freedom are parameterized by $\dot{\phi}$, $\dot{\delta}$ and $\dot{\theta}_R$. The sign convention used is the engineering vehicle dynamics standard (SAE 2001).

equations, there has been little scrutiny of bicycle analyses. So it is perhaps not surprising that the literature contains some incorrect equations and flawed explanations of bicycle stability.

To better satisfy general curiosity about bicycle balance and perhaps contribute to the further evolution of bicycle design, we aim here to firmly establish some basic bicycle stability science. The core of the paper is a set of easy-to-use and thoroughly checked linearized dynamics equations (5.3 and Appendix A) for the motion of a somewhat canonical bicycle model. Future studies of bicycle stability, aimed for example at clarifying the two points above, can be based on these equations.

Many methods can be used to derive the equations using various choices of coordinates, each leading to vastly different looking governing equations. Even matching initial conditions between solution methods can be a challenge. However, stability eigenvalues and the speed-range of stability are independent of all of these differences. So, for example, a computer-based study of a bicycle based on any formulation can be checked for correctness and accuracy by comparing with the benchmark stability results here.

The work here may also have more general use. The bicycle balance problem is close to that for skateboards, skating and perhaps walking and running. Secondly, there is a dearth of non-trivial examples with precisely known solutions that can be used to check general purpose multi-body dynamics simulators (such as are used for machine, vehicle and robot design). This paper provides such a non-trivial benchmark system.

2. Brief literature review

Since their inception bicycles have attracted attention from more-or-less well known scientists of the day including thermodynamicist William Rankine, the mathematicians Carlo Bourlet, Paul Appell and Emmanuel Carvallo, the meteorologist Francis Whipple, the mathematical physicist Joseph Boussinesq, and the physicist Arnold Sommerfeld working with mathematician Felix Klein and engineer Fritz Noether (brother of Emmy). A later peak in the “single track vehicle” dynamics literature began in about 1970, perhaps because digital computers eased integration of the governing equations, because of the increased popularity of large motorcycles (and attendant accidents), and because of an ecology-related bicycle boom. This latter literature includes work by dynamicists such as Neĭmark, Fufaev, Breakwell and Kane. Starting in the mid-1970s the literature increasingly deviates from the rigid-body treatment that is our present focus.

Over the past 140 years scores of other people have studied bicycle dynamics, either for a dissertation, a hobby, or sometimes as part of a life’s work on vehicles. This sparse and varied research on the dynamics of bicycles modeled as linked rigid bodies was reviewed in Hand (1988). Supplementary Appendix I, summarized below, expands on Hand’s review. A more general but less critical review, that also includes models with compliance, is in Sharp (1985).

Many bicycle analyses are based on qualitative dynamics discussions that are too simple to catch the most interesting bicycle phenomenology, namely the ability of a moving bicycle to balance itself. The Physics Today paper by David Jones (1970) is the best-known of this lot. The paper by Maunsell (1946) is amongst the most careful and accurate. Such qualitative dynamics discussions can also be found in: Lallement (1866), Rankine (1869), Sharp (1896), Appell (1896), Wallace (1929), A.T. Jones (1942), Den Hartog (1948), Higbie (1974), Kirshner (1980), Le Hénaff (1987), Olsen & Papadopoulos (1988), Patterson (1993), Cox (1998), Wilson (2004), and Åström *et al.* (2005).

A second class of papers does use analysis to study the dynamics, but are still too restricted to capture self-stability. Some investigate only models with overly simple geometry and/or mass distribution that can never be self-stable. Others, even if using a bicycle model that is sufficiently general, use rules for the control of the steer and thus skip the equation for self-steer dynamics. Such restricted approaches are found in Bourlet (1894, 1899), Boussinesq (1899), Routh (1899), Bouasse (1910), Bower (1915), Pearsall (1922), Loĭcjanskiĭ & Lur’e (1934), Timoshenko & Young (1948), Haag (1955), Neĭmark & Fufaev (1967), Lowell & McKell (1982), Getz & Marsden (1995), Fajans (2000) and Åström *et al.* (2005).

Rigid-body dynamics models that have general-enough geometry and mass distribution for self-stability to be possible, and whose analyses also include uncontrolled steer dynamics are in Carvallo (1898), Whipple (1899), Klein & Sommer-

feld (1910), Manning (1951), Döhning (1953, 1955), Collins (1963), Singh (1964), Neimark & Fufaev (1967), Rice & Roland (1970), Roland & Massing (1971), Sharp (1971), Singh & Goel (1971), Roland & Lynch (1972), Weir (1972), Eaton (1973), Roland (1973), Rice (1974), Van Zytveld (1975), Singh & Goel (1975), Rice (1976), Weir & Zellner (1978), Lobas (1978), Psiaki (1979), Dikarev et al. (1981), Koenen (1983), Papadopoulos (1987), Hand (1988), Mears (1988), Franke *et al.* (1990), Meijaard (2004), Schwab *et al.* (2005), and Åström *et al.* (2005). Because the governing equations are complex and generally not reduced, and the models frequently differ with respect to modelling assumptions, parameterizations for the bicycle design, or choices for the dynamic variables, most authors have not checked or built upon the work of others. Actually, most authors seem unaware of most of the previous research. Until Hand (1988), who found and checked many papers, only Weir (1972) and Eaton (1973) explicitly compared their equations with any others. Klein and Sommerfeld (1910) claim to have compared with Carvallo (1898) and Whipple (1899), and van Zytveld compared with Breakwell (unpublished).

Of these dozens of dynamics equations seven independent derivations are at least as general as the model here, and also present linearized equations that seem perfectly correct: Döhning (1953), Weir (1972), Eaton (1973), Dikarev *et al.* (1981), Papadopoulos (1987), Hand (1988) and Mears (1988). Six others had minor and easily corrected errors or were just slightly less general: Carvallo (1898), Whipple (1899), Klein & Sommerfeld (1910), Sharp (1971) and Van Zytveld (1975). Neimark & Fufaev (1967) had more substantial but still correctable errors. Psiaki (1979) is likely to be correct but his equations were too complex for us to check in detail.

Other papers have missing terms, disagreed with the equations here other ways, or are too complex to verify in detail but are suspect for one reason or another. More papers continue to turn up which we have not yet checked in detail, e.g. Manning (1951), Kondo (1963), Ge (1966) and Seffen (1999).

3. The Bicycle Model

Our model represents a compromise between realism and generality on the one hand and simple definition on the other. Within the constraint of overall lateral (left-right) symmetry and circular symmetry of the wheels, the shape and mass distributions we use are fully general (Caveat: the symmetry allows toroidal wheels while we assume knife-edge wheels). We neglect the motion of the rider relative to the frame, structural compliances and dampers, joint friction, tire compliance and tire “slip”. The bicycle consists of four rigid bodies: a Rear wheel R, a rear frame B with the rider Body rigidly attached to it, a front frame H consisting of the Handlebar and fork assembly, and a Front wheel F (figure 1).

For rigid-body dynamics the bicycle design is fully characterized by 25 parameters described below. Table 1 lists the numerical values used for the numerical benchmark. The numerical values are mostly fairly realistic, but some values (e.g., wheel thickness) are exaggerated to guarantee a significant role in the benchmark.

The bicycle design parameters are defined using an upright reference configuration with both wheels on the level flat ground and with zero steer angle. The reference coordinate origin is at the rear wheel contact point P. We use the slightly odd conventions of vehicle dynamics (SAE 2001) with positive x pointing towards

the front contact point, positive z pointing down and the y axis pointing to the rider's right.

The radii of the circular wheels are r_R and r_F . The wheel masses are m_R and m_F with their centres of mass at the wheel centres. The moments of inertia of the rear and front wheels about their axles are I_{Ryy} and I_{Fyy} . The moments of inertia of the wheels about any diameter in the xz plane are I_{Rxx} and I_{Fxx} . The wheel mass distribution need not be planar so any positive inertias are allowed with $I_{Ryy} \leq 2I_{Rxx}$ and $I_{Fyy} \leq 2I_{Fxx}$. All front wheel parameters can be different from those of the rear so, for example, it is possible to investigate separately the importance of angular momentum of the front and rear wheels.

Narrow high-pressure high-friction tire contact is modeled as non-slipping rolling contact between the ground and the knife-edge wheel perimeters. The frictionless wheel axles are orthogonal to the wheel symmetry planes and are located at the wheel centres.

In the reference configuration the front wheel ground contact Q is located at a distance w (the ‘‘wheel base’’) in front of the rear wheel contact P. The front wheel ground contact point trails a distance c behind the point where the steer axis intersects with the ground. Although $c > 0$ for most bicycles, the equations allow a ‘negative trail’ ($c < 0$) with the wheel contact point in front of the steer axis. The steer axis can pass behind (as with the conventional forward-bent front fork of figure 1) or in front of the wheel centre.

The rear wheel R is connected to the rear frame assembly B (which includes the rider body) at the rear hub. The centre of mass of B is located at $(x_B, y_B = 0, z_B < 0)$. The moment of inertia of the rear frame about its centre of mass is represented by a 3×3 moment of inertia matrix with components

$$\mathbf{I}_B = \begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix} \quad (3.1)$$

where all mass is symmetrically distributed relative to the xz plane, but not necessarily on the plane, so $I_{Byy} \leq I_{Bxx} + I_{Bzz}$.

The rear frame assembly B connects with the handlebar assembly H at the steer (‘‘head’’) axis which allows frictionless relative rotation. The steer axis tilt angle λ is measured back from the upwards vertical, positive when tipped back as on a conventional bicycle with $-\pi/2 < \lambda < \pi/2$. The steer tilt is $\pi/2$ minus the conventional ‘‘head angle’’; a bicycle with head angle of 72° has $\lambda = 18^\circ = \pi/10$. The combination of wheel base w , trail c and steer axis tilt angle λ implicitly locates the hinge connecting the frame B with the handlebar assembly H.

The centre of mass of the front frame assembly (fork and handlebar) H is at $(x_H, y_H = 0, z_H < 0)$ relative to the rear contact P. H has mass m_H and moment of inertia matrix about its centre of mass

$$\mathbf{I}_H = \begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix} \quad (3.2)$$

where, as for the B frame, $I_{Hyy} \leq I_{Hxx} + I_{Hzz}$.

Two non-design parameters are the downwards gravitational acceleration g and the nominal forward speed v .

This model, or slight simplifications of it, is a common idealization of a bicycle (see supplementary Appendix 4). Motorcycle modeling is often based on an extension of this model using toroidal wheels, tire-compliance and tire slip.

(a) *How many parameters describe a bicycle?*

The bicycle model here is defined completely by the 25 design parameters above (see table 1). This is not a minimal description for dynamic analysis, however. For example, the inertia properties of the rear frame B and the lateral inertial properties of the rear wheel R can be combined, reducing the number of parameters by 2. Similarly for the front frame, reducing the number of parameters to 21. The polar inertia of the front wheel can be replaced with a gyrostat constant which gives spin angular momentum in terms of forward velocity. This does not reduce the number of parameters in non-linear modeling. But in linear modeling the radius of the wheels is irrelevant for lean and steer geometry and their effect on angular momentum is embodied in the gyrostat constants. This reduces the number of parameters by 2 to 19. Finally, in the linearized equations of motion the polar (yy components) of the moments of inertia of the two frames are irrelevant, reducing the necessary number of design parameters to 17. In their most reduced form (below) the linearized equations of motion have 11 arbitrary independent matrix entries, each of which is a complex combination of the 17 parameters just described. Non-dimensionalization might reduce this to 8 free constants. That is, for example, the space of nondimensional root-locus plots is possibly only 8 dimensional. For the purposes of simpler comparison, we use the 25 design parameters above.

(b) *How many degrees of freedom does a bicycle have?*

Because this system has non-holonomic kinematic constraints, the concept of “degree of freedom” needs clarification. The holonomic (hinges and ground contact) and non-holonomic (non-slip rolling) constraints restrict this collection of 4 linked bodies in three dimensional space as follows. Start with the 24 degrees of freedom of the 4 rigid bodies, each with 3 translational and 3 rotational degrees of freedom in physical space ($4 \times (3 + 3) = 24$). Then subtract out 5 degrees of freedom for each of the three hinges and one more for each wheel touching the ground plane: $24 - 3 \times 5 - 2 = 7$. Thus, before we consider the non-slipping wheel-contact constraints, the accessible configuration space is 7 dimensional. The 4 non-holonomic rolling constraints (two for each wheel ground-contact) do not further restrict this accessible configuration space: kinematically allowable parallel-parking-like moves can translate and steer the bicycle on the plane in arbitrary ways and also can rotate the wheels relative to the frame with no net change of overall bicycle position or orientation. Thus the accessible configuration space for this model is 7 dimensional.

(i) *Description of the 7-dimensional configuration space*

This 7-dimensional configuration space can be parameterized as follows (see figure 2). The location of the rear-wheel contact with the ground is (x_P, y_P) relative to a global fixed coordinate system with origin O. The orientation of the rear frame with respect to the global reference frame $O-xyz$ is given by a sequence of angular rotations (Euler angles) depicted in figure 2 with fictitious hinges (represented as

cans) in series, mounted at the rear hub: a yaw rotation, ψ , about the z -axis, a lean rotation, ϕ , about the rotated x -axis, and a pitch rotation, θ , about the rotated y -axis. Note that the pitch θ is not one of the 7 configuration variables because it is determined by a 3-D trigonometric relation that keeps the front wheel on the ground. The steering angle δ is the rotation of the front handlebar frame with respect to the rear frame about the steering axis. A right turn of a forwards-moving bicycle has $\delta > 0$. Finally, the rotation of the rear R and front F wheels with respect to their respective frames B and H are θ_R and θ_F . In summary, the configuration space is parameterized here with $(x_P, y_P, \psi, \phi, \delta, \theta_R, \theta_F)$. Quantities such as wheel-center coordinates, and rear-frame pitch, are all determined by these.

(ii) *Velocity degrees of freedom*

With motions akin to parallel parking, consistent with all of the hinge and rolling constraints (but not necessarily consistent with the equations of motion), it is possible to move from any point in this 7-dimensional space to any other point. However, the 4 non-holonomic rolling constraints reduce the 7 dimensional accessible configuration space to $7 - 4 = 3$ velocity degrees of freedom.

This 3-dimensional kinematically-accessible velocity space can conveniently be parameterized by the lean rate $\dot{\phi}$ of the rear frame, the steering rate $\dot{\delta}$ and the rotation rate $\dot{\theta}_R$ of the rear wheel R relative to the rear frame B.

4. Basic features of the model, equations and solutions

(a) *The system behavior is unambiguous*

The dynamics equations for this model follow from use of linear and angular momentum balance and the assumption that the kinematic constraint forces follow the rules of action and reaction and do no net work. These equations may be assembled into a set of ordinary differential equations, or differential-algebraic equations by various methods. One can assemble governing differential equations using the Newton–Euler rigid-body equations, using Lagrange equations with Lagrange multipliers for the in-ground-plane rolling-contact forces, or one can use methods based on the principle of virtual velocities (e.g., Kane’s method), etc. But the subject of mechanics is sufficiently well defined that we know that all standard methods will yield equivalent sets of governing differential equations. Therefore, a given consistent-with-the-constraints initial state (positions and velocities of all points on the frames and wheels) will always yield the same subsequent motions of the bicycle parts. So, while the choice of variables and the recombination of governing equations leads to vast differences in the appearance of the governing equations, any difference between dynamics predictions can only be due to errors. Despite this unique specification, prior to this paper there is no established-as-sound set of agreed equations, or calculated motion, in the literature (see supplementary Appendix 4).

(b) *The system is conservative but not Hamiltonian*

The only friction forces in this model are the lateral and longitudinal forces at the ground-contact points. Because of the no-slip condition these friction forces

<u>Parameter</u>	<u>Symbol</u>	<u>Value for benchmark</u>
Wheel base	w	1.02 m
Trail	c	0.08 m
Steer axis tilt ($\pi/2 - \text{head angle}$)	λ	$\pi/10$ ($90^\circ - 72^\circ$)
Gravity	g	9.81 N/kg
Forward speed	v	various m/s, see tables 2–4
<u>Rear wheel R</u>		
Radius	r_R	0.3 m
Mass	m_R	2 kg
Mass moments of inertia	(I_{Rxx}, I_{Ryy})	(0.0603, 0.12) kgm ²
<u>Rear Body and frame assembly B</u>		
Position centre of mass	(x_B, z_B)	(0.3, -0.9) m
Mass	m_B	85 kg
Mass moments of inertia	$\begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix}$	$\begin{bmatrix} 9.2 & 0 & 2.4 \\ 0 & 11 & 0 \\ 2.4 & 0 & 2.8 \end{bmatrix}$ kgm ²
<u>Front Handlebar and fork assembly H</u>		
Position centre of mass	(x_H, z_H)	(0.9, -0.7) m
Mass	m_H	4 kg
Mass moments of inertia	$\begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix}$	$\begin{bmatrix} 0.05892 & 0 & -0.00756 \\ 0 & 0.06 & 0 \\ -0.00756 & 0 & 0.00708 \end{bmatrix}$ kgm ²
<u>Front wheel F</u>		
Radius	r_F	0.35 m
Mass	m_F	3 kg
Mass moments of inertia	(I_{Fxx}, I_{Fyy})	(0.1405, 0.28) kgm ²

Table 1. Parameters for the benchmark bicycle depicted in figure 1 and described in the text. The values given are accurate to 15 decimal places (0.12 means 0.120 000 000 000 00) for validation and comparison purposes. The inertia components and angles are such that the principal inertias (eigenvalues of the inertia matrix) are also exactly described with only a few digits. The tangents of the angles that the inertia eigenvectors make with the global reference are rational fractions. Note that moment-of-inertia matrix entries must satisfy the triangle inequalities that no one principal value is bigger than the sum of the other two and all principal values are positive.

are non-dissipative. The hinges and ground contact are all workless kinematic constraints. In uncontrolled bicycle motion the only external applied forces are the conservative gravity forces on each part. That is, there are no dissipative forces and the system is energetically conservative; the sum of the gravitational and kinetic energies is a constant for any free motion. But the non-holonomic kinematic constraints preclude writing the governing equations in standard Hamiltonian form, so theorems of Hamiltonian mechanics do not apply. One result, surprising to some cultured in Hamiltonian systems, is that the bicycle equations can have asymptotic (exponential) stability (see figure 4) even with no dissipation. This apparent contradiction of the stability theorems for Hamiltonian systems is because the bicycle, while conservative, is, by virtue of the non-holonomic wheel contacts, not Hamiltonian.

(c) Symmetries in the solutions

Some of the configuration variables do not appear in any expression for the forces, moments, potential energies, or kinetic energies of any of the parts (these are so-called cyclic or ignorable coordinates). In particular the location of the bicycle on the plane (x_P, y_P) , the heading of the bicycle ψ , and the rotations (θ_R, θ_F) of the two wheels relative to their respective frames do not show up in any of the dynamics equations for the velocity degrees of freedom. So one can write a reduced set of dynamics equations that do not include these ignorable coordinates. The full configuration as a function of time can be found afterwards by integration of the kinematic constraint equations, as will be discussed. Also, these ignorable coordinates cannot have asymptotic stability; a small perturbation of, say, the heading ψ will lead to a different ultimate heading.

The lateral (left-right) symmetry of the bicycle-design along with the lateral symmetry of the equations implies that the straight-ahead unsteered and untipped ($\delta = 0, \phi = 0$) rolling motions are necessarily solutions for any forward or backward speed v . Moreover, the longitudinal and the lateral motion, relative to these symmetric solutions, must be decoupled from each other to first order (linearly decoupled): by lateral symmetry a perturbation to the right must cause the same change in speed as a perturbation to the left, by linearity the effects must be the negative of each other, therefore there can be no first-order change in speed due to lean; similarly speed change cannot cause lean. So the linearized fore-aft equations of motion are entirely decoupled from the lateral equations of motion (see also supplementary Appendix 4).

Because all of the equations of such frictionless kinematically constrained systems are time reversible, any bicycle motion is also a solution of the equations when moving backwards, with all particle trajectories being traced at identical speeds in the reverse direction. Thus a bicycle that is exponentially stable in balance when moving forwards at speed $v > 0$ must be exponentially unstable when moving at $-v$ (backwards at the same speed). And a fore-aft symmetric bicycle (having a vertical central steering axis and a handle-bar assembly, front mass distribution and front wheel that mirrors that of the rear assembly) cannot be exponentially stable for any forward or backward motions. If a bike has exponentially decaying solutions in one direction it must have exponentially growing solutions in the opposite direction because of time reversal. By symmetry it must therefore also have exponentially growing solutions in the (supposedly stable) original direction.

(d) The non-linear equations have no simple expression

In contrast with the linear equations we present below, there seems to be no reasonably compact expression of the full non-linear equations of motion for this model. The kinematic loop, from rear-wheel contact to front-wheel contact, determines the rear frame pitch through a quartic equation (Psiaki 1979), so there is no simple expression for rear frame pitch for large lean and the steer angle. Thus the writing of non-linear governing differential equations in a form that various researchers can check against alternative derivations, is a challenge that is not addressed here, and might never be addressed. An algorithmic derivation of non-linear equations, suitable for numerical calculation and benchmark comparison is

presented in (Basu-Mandal & Chatterjee, 2006) where no-hands circular motions and their stability are studied in detail.

5. Linearized equations of motion

Here we present a set of linearized differential equations for the bicycle model, slightly perturbed from upright straight-ahead motion, in a canonical form. To aid in organizing the equations we include applied roll and steer torques which are later set to zero for study of uncontrolled motion.

(a) Derivation of governing equations

Mostly-correct derivations and presentations of the equations of motion for a relatively general bicycle model, although not finally expressed in the canonical form of equation 5.3, are found in Carvallo (1898) and Whipple (1899), Sommerfeld & Klein (1910), Döhning (1953), Sharp (1971), Weir (1972), Eaton (1973) and Van Zytveld (1975). Dikarev *et al.* (1981) have a derivation of equation 5.3 based on correcting the errors in Neimark and Fufaev (1967) as does Hand (1988) which just precedes Mears (1988). Meijaard (2004) also has a derivation of the linearized equations (generated in preparation for this paper).

The derivations above are generally long and frequently lead to equations that are unwieldy or even inscrutable. A minimal derivation of the equations using angular momentum balance about various axes, based on Papadopoulos (1987), is given in Appendix B. Note that this derivation, as well as all of the verified equations from the literature, are not based on a systematic linearization of full non-linear differential equations but on ad-hoc small-angle approximations. Thus far, systematic linearizations have not achieved analytical expressions for the linearized-equation coefficients in terms of the 25 bicycle parameters. However, part of the validation process here includes comparison with full non-linear simulations and also comparison with numerical values of the linearized-equation coefficients as determined by these same programs.

(b) Forcing terms

For numerical benchmark purposes, where eigenvalues are paramount, we neglect control forces or other forcing (except gravity which is always included). However, the forcing terms help to organize the equations. And forcing terms are needed for study of disturbances and control, so they are included in the equations of motion.

Consider an arbitrary distribution of forces \mathbf{F}_i acting on bike points which are added to the gravity forces. Their net effect is to contribute to the forces of constraint (the ground reaction forces and the action-reaction pairs between the parts at the hinges) and to contribute to the accelerations $(\ddot{\phi}, \ddot{\delta}, \ddot{\theta}_R)$ associated with the three velocity degrees of freedom (lean, steer and forward motion).

Three generalized forces can be defined by writing the power of the applied forces associated with arbitrary perturbation of the velocities that are consistent with the hinge-assembly and ground-wheel contact constraints. The power necessarily factors

into a sum of three terms

$$P = \sum \mathbf{F}_i \cdot \mathbf{v}_i = T_\phi \dot{\phi} + T_\delta \dot{\delta} + T_{\theta_R} \dot{\theta}_R \quad (5.1)$$

because the velocities \mathbf{v}_i of all material points are necessarily linear combinations of the generalized velocities $(\dot{\phi}, \dot{\delta}, \dot{\theta}_R)$. The generalized forces $(T_\phi, T_\delta, T_{\theta_R})$ are thus each linear combinations of the components of the various applied forces \mathbf{F}_i .

The generalized forces $(T_\phi, T_\delta, T_{\theta_R})$ are energetically conjugate to the generalized velocities. The generalized forces can be visualized by considering special loadings each of which contributes to only one generalized force when the bicycle is in the reference configuration. In this way of thinking

1. T_{θ_R} is the propulsive “force”, expressed as an equivalent moment on the rear wheel. In practice pedal torques or a forward push on the bicycle contribute to T_{θ_R} and not to T_ϕ and T_δ .
2. T_ϕ is the right lean torque, summed over all the forces on the bicycle, about the line between the wheel ground contacts. A sideways force on the rear frame located directly above the rear contact point contributes only to T_ϕ . A sideways wind gust, or a parent holding a beginning rider upright contributes mainly to T_ϕ .
3. T_δ is an action-reaction steering torque. A torque causing a clockwise (looking down) action to the handle bar assembly H along the steer axis and an equal and opposite reaction torque on the rear frame contributes only to T_δ . In simple modeling, T_δ would be the torque that a rider applies to the handlebars. Precise description of how general lateral forces contribute to T_δ depends on the projection implicit in equation 5.1. Some lateral forces make no contribution to T_δ , namely those acting at points on either frame which do not move when an at-rest bicycle is steered but not leaned. Lateral forces applied to the rear frame directly above the rear contact point make no contribution to T_δ . Nor do forces applied to the front frame if applied on the line connecting the front contact point with the point where the steer axis intersects the vertical line through the rear contact point. Lateral forces at ground level, but off the two lines just described, contribute only to T_δ . Lateral forces acting at the wheel contact points make no contribution to any of the generalized forces.

Just as for a pendulum, finite vertical forces (additional to gravity) change the coefficients in the equations of motion but do not contribute to the forcing terms. Similarly, propulsive forces also change the coefficients but have no first order effect on the lateral forcing. Thus the equations presented here only apply for small ($\ll mg$) propulsive and vertical forces.

(c) *The first linear equation: with no forcing forward speed is constant*

The governing equations describe a linear perturbation of a constant-speed straight-ahead upright solution: $\phi = 0$, $\delta = 0$, and the constant forward speed is $v = -\dot{\theta}_R r_R$. As explained above and in more detail in supplementary Appendix 4, lateral symmetry of the system, combined with the linearity in the equations precludes any first-order coupling between the forward motion and the

lean and steer. Therefore the first linearized equation of motion is simply obtained from two-dimensional (xz plane) mechanics as:

$$\left[r_R^2 m_T + I_{Ryy} + (r_R/r_F)^2 I_{Fyy} \right] \ddot{\theta}_R = T_{\theta_R}, \quad (5.2)$$

where m_T is total bike mass (see Appendix A). That is, in cases with no propulsive force the nominal forward speed $v = -r_R \dot{\theta}_R$ is constant (to first order).

(d) *Lean and steer equations*

The linearized equations of motion for the two remaining degrees of freedom, the lean angle ϕ and the steer angle δ , are two coupled second-order constant-coefficient ordinary differential equations. Any such set of equations can be linearly combined to get an equivalent set. We define the canonical form below by insisting that the right-hand sides of the two equations consist only of T_ϕ and T_δ , respectively. The first equation is called *the lean equation* and the second is called *the steer equation*. That we have a rigid-body system requires that the linear equations have the form (Papadopoulos 1987):

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}, \quad (5.3)$$

where and the time-varying variables are

$$\mathbf{q} = \begin{bmatrix} \phi \\ \delta \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} T_\phi \\ T_\delta \end{bmatrix}.$$

The form of equation 5.3 is identical to that derived in Appendix B thus we can generalize the specific applied loading used in the derivation to the work-conjugate loads defined above. The terms in the constant coefficient matrices \mathbf{M} , \mathbf{C} , and \mathbf{K} are defined in terms of the design parameters in Appendix A. Briefly, \mathbf{M} is a symmetric mass matrix which gives the kinetic energy of the bicycle system at zero forward speed by $\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$. The damping-like (there is no real damping) matrix $\mathbf{C} = v \mathbf{C}_1$ is linear in the forward speed v and captures captures gyroscopic torques due to steer and lean rate, as well as effects of lateral acceleration of the front contact due to steer rate. The stiffness matrix $\mathbf{K} = g \mathbf{K}_0 + v^2 \mathbf{K}_2$ is the sum of two parts: a velocity-independent symmetric part $g \mathbf{K}_0$ proportional to the gravitational acceleration, which can be used to calculate changes in potential energy with $\mathbf{q}^T [g \mathbf{K}_0] \mathbf{q}$, and a part $v^2 \mathbf{K}_2$ which is quadratic in the forward speed and is due to gyroscopic and centrifugal effects in steady turning.

In Appendix A the coefficients in the governing equations (5.3) are expressed analytically in terms of the 25 design parameters. Equation (5.3) above is the core of this paper.

6. Benchmark model and solutions

The same form of governing equations can be found by various means even with, perhaps, the constant coefficients being derived numerically. Further, motion of a bicycle model can be found without ever explicitly writing the governing equations, using more direct numerical methods. To facilitate comparisons to results from these less explicit approaches we have defined a specific benchmark bicycle with values given to all parameters in table 1. The parameter values were chosen to minimize the

possibility of fortuitous cancellation that could occur if used in an incorrect model. On the other hand we wanted numbers that could be easily described precisely. In the benchmark bicycle the two wheels are different in all properties and no two angles, masses or distances match.

(a) *Coefficients of the linearized equations of motion*

Substitution of the values of the design parameters for the benchmark bicycle from table 1 in the expressions from Appendix A results in the following values for the entries in the mass matrix from (A 20),

$$\mathbf{M} = \begin{bmatrix} 80.817\ 22 & 2.319\ 413\ 322\ 087\ 09 \\ 2.319\ 413\ 322\ 087\ 09 & 0.297\ 841\ 881\ 996\ 86 \end{bmatrix}, \quad (6.1)$$

the constant stiffness matrix from (A 22) which are to be multiplied by gravity g ,

$$\mathbf{K}_0 = \begin{bmatrix} -80.95 & -2.599\ 516\ 852\ 498\ 72 \\ -2.599\ 516\ 852\ 498\ 72 & -0.803\ 294\ 884\ 586\ 18 \end{bmatrix}, \quad (6.2)$$

the coefficients of the stiffness matrix from (A 24) which is are to be multiplied by the square of the forward speed v^2 ,

$$\mathbf{K}_2 = \begin{bmatrix} 0 & 76.597\ 345\ 895\ 732\ 22 \\ 0 & 2.654\ 315\ 237\ 946\ 04 \end{bmatrix}, \quad (6.3)$$

and finally the coefficients of the “damping” matrix from (A 26) which are to be multiplied by the forward speed v ,

$$\mathbf{C}_1 = \begin{bmatrix} 0 & 33.866\ 413\ 914\ 924\ 94 \\ -0.850\ 356\ 414\ 569\ 78 & 1.685\ 403\ 973\ 975\ 60 \end{bmatrix}. \quad (6.4)$$

(b) *Linearized stability, eigenvalues for comparison*

A useful property of linearized equations is that the stability eigenvalues are independent of coordinate choice and even independent of the form of the equations. Any non-singular change of variables yields equations with the same linearized stability eigenvalues. Thus stability eigenvalues serve well as convenient benchmark results permitting comparison between different approaches.

The stability eigenvalues are calculated by assuming an exponential solution of the form $\mathbf{q} = \mathbf{q}_0 \exp(\lambda t)$ for the homogeneous equations ($\mathbf{f} = \mathbf{0}$) in equations (5.3). This leads to a characteristic polynomial, e.g. as

$$\det(\mathbf{M}\lambda^2 + v\mathbf{C}_1\lambda + g\mathbf{K}_0 + v^2\mathbf{K}_2) = 0, \quad (6.5)$$

which is quartic in λ . After substitution of the expressions from Appendix A, the coefficients in this quartic polynomial become complicated expressions of the 25 design parameters, gravity, and speed v . The solutions λ of the characteristic polynomial for a range of forward speeds are shown in figure 3. Eigenvalues with a positive real part correspond to unstable motions whereas eigenvalues with a negative real part correspond to asymptotically stable motions for the corresponding

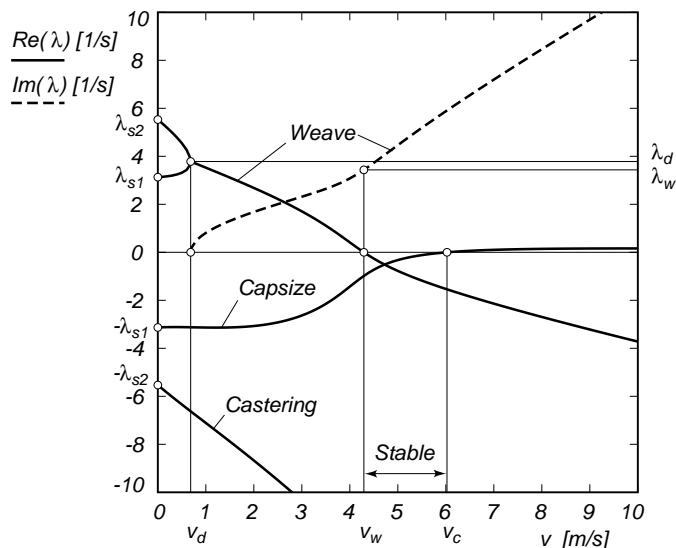


Figure 3. Eigenvalues λ from the linearized stability analysis for the benchmark bicycle from figure 1 and table 1 where the solid lines correspond to the real part of the eigenvalues and the dashed line corresponds to the imaginary part of the eigenvalues, in the forward speed range of $0 \leq v \leq 10$ m/s. The speed range for the asymptotic stability of the benchmark bicycle is $v_w < v < v_c$. The zero crossings of the real part of the eigenvalues are for the weave motion at the weave speed $v_w \approx 4.2$ m/s and for the capsize motion at capsize speed $v_c = 6.0$ m/s, and there is a double real root at $v_d = 0.68$ m/s. For accurate eigenvalues and transition speeds see tables 2, 3, and 4.

mode. Imaginary eigenvalues correspond to oscillatory motions. As mentioned earlier, the time-reversal nature of these conservative dynamical equations leads to symmetry in the characteristic equation (6.5) and in the parameterized solutions: if (v, λ) is a solution then $(-v, -\lambda)$ is also a solution. This means that figure 3 is point symmetric about the origin as revealed by Astrom *et al.* (2005).

This fourth order system has four distinct eigenmodes except at special parameter values leading to multiple roots. A complex (oscillatory) eigenvalue pair is associated with a pair of complex eigenmodes. At high-enough speeds, the two modes most significant for stability are traditionally called the *capsize mode* and *weave mode*. The capsize mode corresponds to a real eigenvalue with eigenvector dominated by lean and with steer proportional to lean: when unstable, a capsizing bicycle leans progressively into a tightening spiral as it falls over. The weave mode is an oscillatory motion in which the bicycle steers sinusously about the headed direction with a slight phase lag relative to leaning. The third eigenvalue is large, real and negative. It corresponds to the *castering mode* which is dominated by steer in which the front ground contact follows a tractrix-like trajectory, something like the straightening of a grocery-cart wheel.

At very low speeds, typically $0 < v < 0.5$ m/s, there are two pairs of real eigenvalues. Each pair consists of a positive and a negative eigenvalue and corresponds to a double-inverted-pendulum-like motion of the bicycle. The positive root in each pair corresponds to falling. The negative root corresponds to the time reversal of this falling. For one pair lean and steer have the same sign during the fall, for the

v [m/s]	λ [1/s]
$v = 0$	$\lambda_{s1} = \pm 3.131\ 643\ 247\ 906\ 56$
$v = 0$	$\lambda_{s2} = \pm 5.530\ 943\ 717\ 653\ 93$
$v_d = 0.684\ 283\ 078\ 892\ 46$	$\lambda_d = 3.782\ 904\ 051\ 293\ 20$
$v_w = 4.292\ 382\ 536\ 341\ 11$	$\lambda_w = 0 \pm 3.435\ 033\ 848\ 661\ 44\ i$
$v_c = 6.024\ 262\ 015\ 388\ 37$	0

Table 2. Some characteristic values for the forward speed v and the eigenvalues λ from the linearized stability analysis for the benchmark bicycle from figure 1 and table 1 in the forward speed range of $0 \leq v \leq 10$ m/s. There are double roots at the speed v_d , the weave speed v_w , and the capsizes speed v_c .

v [m/s]	$\text{Re}(\lambda_{\text{weave}})$ [1/s]	$\text{Im}(\lambda_{\text{weave}})$ [1/s]
0	–	–
1	3.526 961 709 900 70	0.807 740 275 199 30
2	2.682 345 175 127 45	1.680 662 965 906 75
3	1.706 756 056 639 75	2.315 824 473 843 25
4	0.413 253 315 211 25	3.079 108 186 032 06
5	–0.775 341 882 195 85	4.464 867 713 788 23
6	–1.526 444 865 841 42	5.876 730 605 987 09
7	–2.138 756 442 583 62	7.195 259 133 298 05
8	–2.693 486 835 810 97	8.460 379 713 969 31
9	–3.216 754 022 524 85	9.693 773 515 317 91
10	–3.720 168 404 372 87	10.906 811 394 762 87

Table 3. Complex eigenvalues λ_{weave} from the linearized stability analysis for the oscillatory weave motion for the benchmark bicycle from figure 1 and table 1 in the forward speed range of $0 \leq v \leq 10$ m/s.

other pair lean falls one way while steer falls the other. When speed is increased to about $v_d \approx 0.684$ m/s two real eigenvalues become identical and form a complex conjugate pair; this is where the oscillatory weave motion emerges. At first this motion is unstable but at $v_w \approx 4.292$ m/s, the weave speed, these eigenvalues cross the real axis in a Hopf bifurcation and the weave motion becomes stable up to infinite speed. At high speeds the frequency of the weave motion is approximately proportional to the forward speed, meaning that the wavelength of the oscillation approaches a constant. Meanwhile the capsizes motion, which was stable for low speed, crosses the real axis in a pitchfork bifurcation at $v_c \approx 6.024$ m/s, the capsizes speed, and the motion becomes mildly unstable. With further increase in speed, the unstable capsizes eigenvalue attains its maximum value, followed by a decrease towards zero from above. The speed range for which the uncontrolled bicycle shows asymptotically stable behaviour, with all eigenvalues having negative real part, is $v_w < v < v_c$. For comparison by future researchers, all four eigenvalues are presented with 15 significant digits at equidistant forward speeds in table 3 and table 4.

7. Validation of the linearized equations of motion

The linearized equations of motion here, equation (5.3) with the coefficients as presented in Appendix A, have been derived by pencil and paper two ways (Pa-

v [m/s]	λ_{capsize} [1/s]	$\lambda_{\text{castering}}$ [1/s]
0	-3.131 643 247 906 56	-5.530 943 717 653 93
1	-3.134 231 250 665 78	-7.110 080 146 374 42
2	-3.071 586 456 415 14	-8.673 879 848 317 35
3	-2.633 661 372 536 67	-10.351 014 672 459 20
4	-1.429 444 273 613 26	-12.158 614 265 764 47
5	-0.322 866 429 004 09	-14.078 389 692 798 22
6	-0.004 066 900 769 70	-16.085 371 230 980 26
7	0.102 681 705 747 66	-18.157 884 661 252 62
8	0.143 278 797 657 13	-20.279 408 943 945 69
9	0.157 901 840 309 17	-22.437 885 590 408 58
10	0.161 053 386 531 72	-24.624 596 350 174 04

Table 4. Real eigenvalues λ from the linearized stability analysis for the capsize motion and the castering motion for the benchmark bicycle from figure 1 and table 1 in the forward speed range of $0 \leq v \leq 10$ m/s.

padopoulos 1987, Meijaard 2004), and agree exactly with some of the past literature, see §2, most especially Döhring (1953), Weir (1972) and Hand (1988). We have also checked equation coefficients via the linearization capability of two general non-linear dynamics simulation programs. The first, SPACAR, is a numeric multibody dynamics program and the second, AutoSim, is symbolic software for multibody dynamic analysis. Since a term-by-term algebraic comparison was not possible in these programs, the validation was performed by comparing the numerical entries in the coefficient matrices of the linearized equations of motion for the specific benchmark bicycle. Finally, the transient motions after a small perturbation were predicted by the linearized equations at one forward speed and compared with the motions predicted by the non-linear simulations.

(a) *Equations of Motion Derived with the Numeric Program SPACAR*

SPACAR is a program system for dynamic simulation of multibody systems, based on a finite element approach. The first version Van der Werff (1977) was based on principles laid out by Besseling (1964). SPACAR has been further developed since then (Jonker (1988, 1990), Meijaard (1991), and Schwab (2002, 2003)). The SPACAR program handles systems of rigid and flexible bodies connected by various joints in both open and closed kinematic loops, and where parts may have rolling contact. SPACAR generates numerically, and solves, full non-linear dynamics equations using minimal coordinates (constraints are eliminated). SPACAR can also find the numeric coefficients for the linearized equations of motion based on a systematic numeric linearization of the non-linear equations. Used for the simulations here are the rigid body, point mass, hinge, and rolling-wheel contact features of the program (Schwab & Meijaard 1999, 2003).

As determined by SPACAR the entries in the matrices of the linearized equations of motion (5.3) agree to 14 digits with the values presented in §6 a. More details about the SPACAR implementation are in supplementary Appendix 2.

(b) *Equations of motion derived with the Symbolic Program AutoSim*

We also derived the non-linear governing equations using the multibody dynamics program AutoSim (Sayers 1991a, 1991b). AutoSim is a Lisp (Steele 1990) program mostly based on Kane's (1968) approach. It consists of function definitions and data structures allowing the generation of symbolic equations of motion of rigid-body systems. AutoSim is mainly for systems of objects connected with prismatic and revolute joints arranged with the topology of a tree (no loops).

AutoSim generates equations in the form

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{S}(\mathbf{q}, t)\mathbf{u}, \\ \dot{\mathbf{u}} &= [\mathbf{M}(\mathbf{q}, t)]^{-1}\mathbf{Q}(\mathbf{q}, \mathbf{u}, t).\end{aligned}\tag{7.1}$$

Here, \mathbf{q} are the generalized coordinates, \mathbf{u} are the generalized velocities, \mathbf{S} is the kinematic matrix that relates the rates of the generalized coordinates to the generalized speeds, \mathbf{M} is the system mass matrix, and \mathbf{Q} contains all force terms and velocity dependent inertia terms.

Additional constraints are added for closed kinematic loops, special joints or non-holonomic constraints. For example, the closed loop holonomic constraint for both bicycle wheels touching the ground cannot be solved simply in symbolic form for the dependent coordinates (requires the solution of a quartic polynomial). An iterative numerical solution for this constraint was used, destroying the purely symbolic nature of the equations.

Strictly speaking, standard AutoSim linearization is not applicable for our system due to the kinematic closed loop of the wheel ground contact. Fortunately, with the laterally symmetric bicycle here the dependent coordinate (the pitch angle) remains zero to first order, for which special case the linearization works. The final AutoSim-based linearization output consists of a MatLab script file that numerically calculates the matrices of the linearized equations.

The entries in the matrices of the linearized equations of motion (5.3) as determined by the program AutoSim agree to 14 digits with the values presented in §6 a. More details about the AutoSim verification are in supplementary Appendix 3.

8. Energy conservation and asymptotic stability

When an uncontrolled bicycle is within its stable speed range, roll and steer perturbations die away in a seemingly damped fashion. However, the system conserves energy. As the forward speed is affected only to second order, linearized equations do not capture this. Therefore a non-linear dynamic analysis with SPACAR was performed on the benchmark bicycle model to demonstrate the leakage of the energy from lateral perturbations into forward speed. The initial conditions at $t = 0$ are the upright reference position $(\phi, \delta, \theta_R) = (0, 0, 0)$ at a forward speed of $v = 4.6$ m/s, which is within the stable speed range of the linearized analysis, and an initial angular roll velocity of $\dot{\phi} = 0.5$ rad/s. In the full non-linear equations the final upright forward speed is augmented from the initial speed by an amount determined by the energy in the lateral perturbation. In this case the speedup was about 0.022 m/s.

Figure 4 shows the expected small increase of the forward speed v while the lateral motions die out. The same figure also shows that the period for the roll and the steer motion is approximately $T_0 = 1.60$ s, which compares well with the

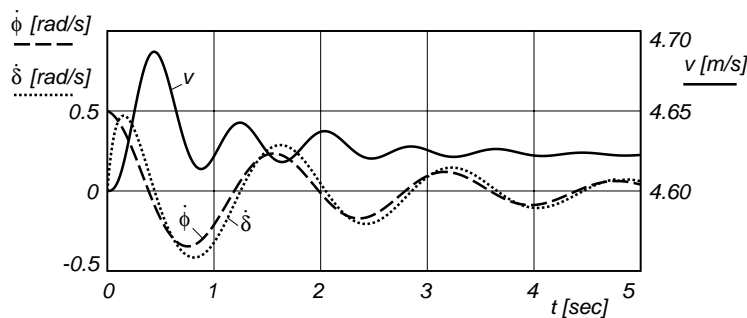


Figure 4. Non-linear dynamic response of the benchmark bicycle from figure 1 and table 1, with the angular roll velocity $\dot{\phi}$, the angular steering velocity $\dot{\delta}$, and the forward speed $v = -\dot{\theta}_R r_R$ for the initial conditions: $(\phi, \delta, \theta_R)_0 = (0, 0, 0)$ and $(\dot{\phi}, \dot{\delta}, v)_0 = (0.5 \text{ rad/s}, 0, 4.6 \text{ m/s})$ for a time period of 5 seconds.

1.622 s from the linearized stability analysis. The lack of agreement in the second decimal place is from finite-amplitude effects, not numerical accuracy issues. When the initial lateral velocity is decreased by a factor of 10 the measured period of motion already settles down to 1.622 s. The steering motion $\dot{\delta}$ has a small phase lag relative to the roll motion $\dot{\phi}$ in the solution presented.

9. Conclusions, discussion and future work

We have presented reliable equations of motion for a fairly general bicycle model and compared them to the century-old literature. These equations can be used as a basis for future research. Alternatively, they can be used as a check for others who derive their own equations by comparison with: a) the analytic form of the coefficients in equation (5.3), or b) the numerical value of the coefficients in equation (5.3) using either the general benchmark bicycle parameters of table 1, or the simpler set in the Supplementary Appendices, or c) the tabulated stability eigenvalues, or d) the speed range of self-stability for the benchmark parameters.

This paper only narrowly addresses the question “How does an uncontrolled bicycle stay up?” by firming up the 108 year old discovery that self-stability is explicable with a sufficiently complex rigid body dynamics model.

Rather, this paper presents reliable equations for a well-delineated model for studying controlled and uncontrolled stability of a bicycle more deeply. The model delineation is not by selecting the most important terms for describing real bicycle balance. For example, some terms included here have very small effects, like the non-planarity of the inertia of the real wheel. And other neglected terms may be paramount, like the rider’s flexibility and control reflexes. Even for the study of uncontrolled stability, tire deformation and frame flex are necessary for understanding wobble (shimmy). Rather, the model here includes all the sharply-defined rigid-body terms, while leaving out terms that require more-subtle and less-well-defined modeling.

In a future paper we will use the equations presented here to address various features of bicycle self-stability and how these depend, or do not depend, on the bicycle design parameters. For example, we will dispel some bicycle mythology about the need for mechanical trail or gyroscopic wheels for bicycle stability.

Acknowledgements

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Appendices

These main appendices include A) definitions of the coefficients used in the equations of motion, and B) a brief derivation of the governing equations. Additional supplementary appendices not included with the main paper include 1) a chronological review of the history of bicycle dynamics studies, 2) a detailed description of the SPACAR validation, 3) a detailed description of the AutoSim validation, 4) a detailed explanation of how lateral symmetry decouples lateral and forward motion in the linearized equations, and 5) a reduced benchmark for use by those who have a less-general bicycle simulation and want to use the results here for validation.

Appendix A. Coefficients of the linearized equations

Here we define the coefficients in equation (5.3). These coefficients and various intermediate variables are expressed in terms of the 25 design parameters (as well as v and g) in table 1 and figure 1. Some intermediate terms defined here are also used in the derivation of the equations of motion in Appendix B.

We use the subscript R for the rear wheel, B for the rear frame incorporating the rider Body, H for the front frame including the Handlebar, F for the front wheel, T for the Total system, and A for the front Assembly which is the front frame plus the front wheel.

Of the quantities defined in A1 - A19, only 12 are used explicitly in the matrix entries of the equations of motion A20 - A27. A2 is used only in A19 and B2. A6 is used only in A7. A8 - A13 are used in A14 - A16, A19, and B3.

The total mass and the corresponding centre of mass location (with respect to the rear contact point P) are

$$m_T = m_R + m_B + m_H + m_F, \quad (\text{A } 1)$$

$$x_T = (x_B m_B + x_H m_H + w m_F) / m_T, \quad (\text{A } 2)$$

$$z_T = (-r_R m_R + z_B m_B + z_H m_H - r_F m_F) / m_T. \quad (\text{A } 3)$$

For the system as a whole, the relevant mass moments and products of inertia with respect to the rear contact point P along the global axes are

$$I_{Txx} = I_{Rxx} + I_{Bxx} + I_{Hxx} + I_{Fxx} + m_R r_R^2 + m_B z_B^2 + m_H z_H^2 + m_F r_F^2, \quad (\text{A } 4)$$

$$I_{Txx} = I_{Bxx} + I_{Hxx} - m_B x_B z_B - m_H x_H z_H + m_F w r_F. \quad (\text{A } 5)$$

The dependent moments of inertia for the axisymmetric rear wheel and front wheel are

$$I_{Rzz} = I_{Rxx}, \quad I_{Fzz} = I_{Fxx}. \quad (\text{A } 6)$$

Then the moment of inertia for the whole bike along the z axis is

$$I_{Tzz} = I_{Rzz} + I_{Bzz} + I_{Hzz} + I_{Fzz} + m_B x_B^2 + m_H x_H^2 + m_F w^2. \quad (\text{A } 7)$$

The same properties are similarly defined for the front assembly A (the front frame combined with the front wheel):

$$m_A = m_H + m_F, \quad (\text{A } 8)$$

$$x_A = (x_H m_H + w m_F) / m_A, \quad z_A = (z_H m_H - r_F m_F) / m_A. \quad (\text{A } 9)$$

The relevant mass moments and products of inertia for the front assembly with respect to the centre of mass of the front assembly along the global axes are

$$I_{Axx} = I_{Hxx} + I_{Fxx} + m_H (z_H - z_A)^2 + m_F (r_F + z_A)^2, \quad (\text{A } 10)$$

$$I_{Axx} = I_{Hxx} - m_H (x_H - x_A)(z_H - z_A) + m_F (w - x_A)(r_F + z_A), \quad (\text{A } 11)$$

$$I_{Azz} = I_{Hzz} + I_{Fzz} + m_H (x_H - x_A)^2 + m_F (w - x_A)^2. \quad (\text{A } 12)$$

Let $\boldsymbol{\lambda} = (\sin \lambda, 0, \cos \lambda)^T$ be a unit vector pointing down along the steering axis where λ is the counter-clockwise angle relative about the $+y$ axis of the steering axis relative to the $+z$ direction. The centre of mass of the front assembly is ahead of the steering axis by perpendicular distance

$$u_A = (x_A - w - c) \cos \lambda - z_A \sin \lambda. \quad (\text{A } 13)$$

For the front assembly three special inertia components are needed: the moment of inertia about the steer axis and the products of inertia relative to crossed, skew axes, taken about the points where they intersect. The latter give the torque about one axis due to angular acceleration about the other. For example, the λx component is taken about the point where the steer axis crosses the ground. It includes a part from I_A operating on unit vectors along the steer axis and along x and also a parallel axis term based on the distance of m_A from each of those axes.

$$I_{A\lambda\lambda} = m_A u_A^2 + I_{Axx} \sin^2 \lambda + 2I_{Axx} \sin \lambda \cos \lambda + I_{Azz} \cos^2 \lambda, \quad (\text{A } 14)$$

$$I_{A\lambda x} = -m_A u_A z_A + I_{Axx} \sin \lambda + I_{Axx} \cos \lambda, \quad (\text{A } 15)$$

$$I_{A\lambda z} = m_A u_A x_A + I_{Axx} \sin \lambda + I_{Azz} \cos \lambda. \quad (\text{A } 16)$$

The ratio of the mechanical trail (i.e., the perpendicular distance that the front wheel contact point is horizontally behind the steering axis) to the wheel base is

$$\mu = (c/w) \cos \lambda. \quad (\text{A } 17)$$

The rear and front wheel angular momenta along the y -axis, divided by the forward speed, together with their sum form the gyrostatic coefficients

$$S_R = I_{Ryy}/r_R, \quad S_F = I_{Fyy}/r_F, \quad S_T = S_R + S_F. \quad (\text{A } 18)$$

We define a frequently appearing static moment term as

$$S_A = m_A u_A + \mu m_T x_T. \quad (\text{A } 19)$$

The entries in the linearized equations of motion can now be formed. The mass moments of inertia are

$$\begin{aligned} M_{\phi\phi} &= I_{Txx} \quad , \quad M_{\phi\delta} = I_{A\lambda x} + \mu I_{Txz}, \\ M_{\delta\phi} &= M_{\phi\delta} \quad , \quad M_{\delta\delta} = I_{A\lambda\lambda} + 2\mu I_{A\lambda z} + \mu^2 I_{Tzz}. \end{aligned} \quad (\text{A } 20)$$

which are elements of the symmetric mass matrix

$$\mathbf{M} = \begin{bmatrix} M_{\phi\phi} & M_{\phi\delta} \\ M_{\delta\phi} & M_{\delta\delta} \end{bmatrix}. \quad (\text{A } 21)$$

The gravity-dependent stiffness terms (to be multiplied by g) are

$$\begin{aligned} K_{0\phi\phi} &= m_T z_T \quad , \quad K_{0\phi\delta} = -S_A, \\ K_{0\delta\phi} &= K_{0\phi\delta} \quad , \quad K_{0\delta\delta} = -S_A \sin \lambda, \end{aligned} \quad (\text{A } 22)$$

which form the stiffness matrix \mathbf{K}_0 as

$$\mathbf{K}_0 = \begin{bmatrix} K_{0\phi\phi} & K_{0\phi\delta} \\ K_{0\delta\phi} & K_{0\delta\delta} \end{bmatrix}. \quad (\text{A } 23)$$

The velocity-dependent stiffness terms (to be multiplied by v^2) are

$$\begin{aligned} K_{2\phi\phi} &= 0 \quad , \quad K_{2\phi\delta} = ((S_T - m_T z_T)/w) \cos \lambda, \\ K_{2\delta\phi} &= 0 \quad , \quad K_{2\delta\delta} = ((S_A + S_F \sin \lambda)/w) \cos \lambda. \end{aligned} \quad (\text{A } 24)$$

which form the stiffness matrix \mathbf{K}_2 as

$$\mathbf{K}_2 = \begin{bmatrix} K_{2\phi\phi} & K_{2\phi\delta} \\ K_{2\delta\phi} & K_{2\delta\delta} \end{bmatrix}. \quad (\text{A } 25)$$

In the equations we use $\mathbf{K} = g\mathbf{K}_0 + v^2\mathbf{K}_2$. Finally the ‘‘damping’’ terms are

$$\begin{aligned} C_{1\phi\phi} &= 0, \quad C_{1\phi\delta} = \mu S_T + S_F \cos \lambda + (I_{Txz}/w) \cos \lambda - \mu m_T z_T, \\ C_{1\delta\phi} &= -(\mu S_T + S_F \cos \lambda), \quad C_{1\delta\delta} = (I_{A\lambda z}/w) \cos \lambda + \mu(S_A + (I_{Tzz}/w) \cos \lambda) \end{aligned} \quad (\text{A } 26)$$

which form the matrix \mathbf{C}_1 as

$$\mathbf{C}_1 = \begin{bmatrix} C_{1\phi\phi} & C_{1\phi\delta} \\ C_{1\delta\phi} & C_{1\delta\delta} \end{bmatrix} \quad \text{where we use} \quad \mathbf{C} = v\mathbf{C}_1. \quad (\text{A } 27)$$

Appendix B. Derivation of the linearized equations of motion

The following brief derivation of the linearized equations of motion is based on Papadopoulos (1987). A longer derivation using Lagrange methods and based on a correction of Neřmark and Fufaev (1967) is in Hand (1988). These derivations involve *ad hoc* linearization as opposed to linearization of full nonlinear equations. To date, no-one has linearized the full implicit non-linear equations (implicit because there is no reasonably simple closed form expression for the closed kinematic chain) into an explicit analytical form either by hand or computer algebra. However, we note again that these linear equations are now well supported by various comparisons described in the text.

For a bicycle freely rolling forward on a plane, slightly perturbed from upright straight ahead motion, we wish to find the linear equations of motion governing the two lateral degrees of freedom: rightward lean ϕ of the rear frame, and rightward steer δ of the handlebars. The linearized equation of motion for forward motion is trivial and has already been given in equation (5.2).

We take the bicycle to be near to and approximately parallel to the global x -axis. The bicycle's position and configuration, with respect to lateral linearized dynamics, are defined by the variables y_P , ψ , ϕ and δ . In this derivation we assume not only ϕ and δ but also $\dot{y}_P/v \approx \psi$ are small, such that only first order consequences of the configuration variables need be kept.

Forces of importance to lateral linearized dynamics include: gravity at each body's mass centre, positive in z ; vertical ground reaction force at the front wheel: $-m_T g x_T/w$; horizontal ground reaction force F_{Fy} at the front wheel, approximately in the y direction; a roll moment $T_{R\phi}$ applied to the rear frame and tending to roll the bicycle to the right; a steer torque T_δ , applied positively to the handlebars so as to urge them rightward, and also applied negatively to the rear frame.

Initially we replace the non-holonomic rolling constraints with to-be-determined horizontal forces at the front and rear contacts that are perpendicular to the wheel headings. We apply angular momentum balance to various subsystems about one or another axis \mathbf{u} . On the left side of each equation is the rate of change of angular momentum about the given axis:

$$\sum_{\text{bodies}} \left[\vec{r}_i \times \vec{a}_i m_i + \mathbf{I}_i \dot{\vec{\omega}}_i + \vec{\omega}_i \times (\mathbf{I}_i \vec{\omega}_i) \right] \cdot \mathbf{u}$$

where the positions of the bodies' centres of mass \mathbf{r}_i are relative to a point on the axis and the bodies' angular velocities and accelerations $\vec{\omega}_i, \dot{\vec{\omega}}_i$ and \vec{a}_i are expressed in terms of the time-derivatives of the lean and steer. The right side of each equation is the torque of the applied loads and gravity forces about the given axis.

Roll angular momentum balance for the whole bicycle about a fixed axis in the ground plane that is instantaneously aligned with the line where the frame plane intersects the ground (this axis does not generally go through the front ground contact point) gives:

$$\begin{aligned} -m_T \ddot{y}_P z_T + I_{Txx} \ddot{\phi} + I_{Tzx} \ddot{\psi} + I_{A\lambda x} \ddot{\delta} + \dot{\psi} v S_T + \dot{\delta} v S_F \cos \lambda \\ = T_{R\phi} - g m_T z_T \phi + g S_A \delta. \end{aligned} \quad (\text{B } 1)$$

In addition to the applied $T_{R\phi}$ the right hand side has a lean moment from gravitational forces due to lateral lean-induced sideways displacement of the bicycle parts, and a term due to lateral displacement of front-contact vertical ground reaction relative to the axis.

Next, yaw angular momentum balance for the whole bicycle about a fixed vertical axis that instantaneously passes through the rear wheel contact gives

$$m_T \ddot{y}_P x_T + I_{Tzx} \ddot{\phi} + I_{Tzz} \ddot{\psi} + I_{A\lambda z} \ddot{\delta} - \dot{\phi} v S_T - \dot{\delta} v S_F \sin \lambda = w F_{Fy}. \quad (\text{B } 2)$$

The only yaw torque arises from the yet-to-be-eliminated lateral ground force at the front contact.

Lastly, steer angular momentum balance for the front assembly about a fixed axis that is instantaneously aligned with the steering axis gives

$$\begin{aligned} m_A \ddot{y}_P u_A + I_{A\lambda x} \ddot{\phi} + I_{A\lambda z} \ddot{\psi} + I_{A\lambda\lambda} \ddot{\delta} + v S_F (-\dot{\phi} \cos \lambda + \dot{\psi} \sin \lambda) \\ = T_\delta - c F_{Fy} \cos \lambda + g(\phi + \delta \sin \lambda) S_A. \end{aligned} \quad (\text{B } 3)$$

In addition to the applied steering torque T_δ there are torques from both vertical (gravity reactions) and lateral (yet to be determined from constraints) forces at the front contact, and from downwards gravity forces on the front assembly.

The final steps are to combine equations (B 2) and (B 3) in order to eliminate the unknown front-wheel lateral reaction force F_{Fy} , leaving two equations; and then to use the rolling constraints to eliminate ψ and y_P and their time derivatives, leaving just two unknown variables.

Each rolling-contact lateral constraint is expressed as a rate of change of lateral position due to velocity and heading. For the rear,

$$\dot{y}_P = v\psi. \quad (\text{B } 4)$$

Equivalently for the front, where $y_Q = y_P + w\psi - c\delta \cos \lambda$, and the heading angle is augmented by steer angle:

$$d(y_P + w\psi - c\delta \cos \lambda)/dt = v(\psi + \delta \cos \lambda). \quad (\text{B } 5)$$

We subtract (B 4) from (B 5) to get an expression for $\dot{\psi}$ in terms of δ and $\dot{\delta}$,

$$\dot{\psi} = ((v\delta + c\dot{\delta})/w) \cos \lambda. \quad (\text{B } 6)$$

We differentiate (B 6) to get an expression for $\ddot{\psi}$,

$$\ddot{\psi} = ((v\dot{\delta} + c\ddot{\delta})/w) \cos \lambda. \quad (\text{B } 7)$$

Finally we differentiate (B 4) and substitute (B 6) to get an expression for \ddot{y}_P ,

$$\ddot{y}_P = ((v^2\delta + vc\dot{\delta})/w) \cos \lambda. \quad (\text{B } 8)$$

Substituting (B 6), (B 7), (B 8) into (B 1), we get an expression in ϕ , $\ddot{\phi}$ and δ , $\dot{\delta}$ and $\ddot{\delta}$, with a right hand side equal to $T_{R\phi}$. This is called the *lean equation*. Eliminating F_{Fy} from (B 2) and (B 3), then again substituting (B 6), (B 7), (B 8), we will have another expression in ϕ and δ and their derivatives, where the right hand side is T_δ (the steer torque). This is called the *steer equation*. These two equations are presented in matrix form in (5.3).

Since ψ and y_P do not appear in the final equation, there is no need for the bicycle to be aligned with the global coordinate system used in figure 2. Thus x , y and ψ can be arbitrarily large and the bicycle can be at any position on the plane at any heading. This situation is somewhat analogous to, say, the classical elastica where the lateral displacements and angles used in the strain calculation are small yet the lateral displacements and angles of the elastica overall can be arbitrarily large.

For simulation and visualization purpose we can calculate the ignorable coordinates x_P , y_P and ψ by integration. The first order differential equations for the yaw angle ψ is (B 6). Then the rear contact point is described by

$$\dot{x}_P = v \cos \psi \quad \dot{y}_P = v \sin \psi \quad (\text{B } 9)$$

Note the difference in \dot{y}_p from the small angle approximation used in equation B 4 for the equations-of-motion derivation.

For those interested in constraint forces and tire modeling intermediate results may be used. For example equation (B 2) can be used to find the horizontal lateral force at the front contact. Lateral linear momentum balance can be added to find the horizontal lateral force at the rear contact.

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Supplementary Appendices

to accompany the

Linearized Bicycle Dynamics Paper

by

Meijaard, Papadopoulos, Ruina and Schwab

Appendices in the main body of the paper (not below) include

- A) Definitions of the coefficients used in the equations of motion, and
- B) A brief derivation of the governing equations.

These supplementary appendices (below) include

1. A history of bicycle dynamics studies,
2. An explanation of the verification of the linearized equations using the numerical dynamics package SPACAR, and
3. An explanation of the verification of the linearized equations using the symbolic algebra package AutoSim.
4. An detailed explanation of how lateral symmetry implies that for unforced forward motion $\dot{v} = 0$ is one of the linearized equations of motion.
5. A reduced benchmark for use by those who have a less-general bicycle simulation and want to use the results here for validation.

1. History of bicycle steer and dynamics studies.

“Even now, after we’ve been building them for 100 years, it’s very difficult to understand just why a bicycle works - it’s even difficult to formulate it as a mathematical problem.” — Freeman Dyson interviewed by Stewart Brand in Wired News, February 1998.

This appendix builds on Hand (1988) and is the source of the brief literature review in the main body of the paper. We categorize the literature on bicycle dynamics as:

- a) Attempts to explain stability and self-stability without using equations of motion.
- b) Dynamical analyses of simplified models. Typically these use point masses, neglect gyroscopic terms, or use a vertical steer axis. Further some of these models suppress steering dynamics by specifying the steering angle as if by a rider. These simplest models are incapable of self-stability, so controlled steering is essential to maintain balance. Because velocipedes (primitive bicycles

with vertical steer axis, no trail, and front-assembly mass symmetry) were not self-stable, it is natural that the very first crop of mathematical analyses, before the late 1890's, were all in this controlled-steering class.

- c) Equations of motion of a model with enough generality to predict hands-free self stability. These more complex analyses began in about 1900 and were based on the modern safety bicycle, with its tilted steer axis and front-fork offset.

The historical discussion below is in chronological order within each of the three categories above. Within category *c*, we have concentrated on linear treatments. The nonlinear literature is further reviewed in Basu-Mandal & Chatterjee (2006).

(a) *Discussions of balance and stability without equations of motion*

The self-stability of a bicycle involves several complex dynamic phenomena that seem to us to be beyond precise description without appeal to correct governing equations. Thus it is not surprising that equations-free discussions of self-stability below tend to be somewhat vague or incomplete. In contrast, basic features of balance by means of controlled steering are accessible without detailed equations, and are reasonably described in some of these papers.

1866 Lallement's velocipede-improvement U.S. Patent, which is on the addition of front-wheel pedals (as opposed to pushing the feet on the ground), includes a concise explanation of balancing by steering: "If the carriage is inclined to lean to the right, turn the wheel [to the right], which throws the carriage over to the left...".

1869 W. J. Macquorn Rankine, mechanic and thermodynamics theorist, presents semi-quantitative observations on lean and steer of a velocipede. This seems to be the first description of 'countersteering' — briefly turning to the left to generate the rightward lean necessary for a steady rightward turn. Rankine discusses steer only by means of rider control and seems unaware of the possibility of self-stability.

1896 Archibald Sharp, an engineering lecturer at what was to become Imperial College, publishes his book covering nearly all technical aspects of bicycle theory and practice. (Sharp later authored the classic 1910 11th edition Britannica entry on bicycles.) In calculating the handlebar torque required to maintain a steady turn, Sharp's equation 6 is wrong, first by the typographical error of a sign change in the second parentheses, and second by neglecting the centrifugal force on the mass center of the front assembly. He also neglects precessional torque on the front wheel. However he explicitly recognizes the mechanical trail and implicitly recognizes the quantity we call S_A .

Sharp developed his equation to investigate no-hands riding, but fails to recognize the possibility of self-stability.

Sharp concludes, correctly in part (see Jones 1970 below), that the no-hands rider probably exercises control of steering through upper-body lean causing frame lean and in turn causing gyroscopic precession of the front wheel. A

rider can thus control this precession and make corrective turns much like he or she would with direct handle-bar torques.

- 1896** Appell, in his dynamics textbook, summarizes Bourlet's analysis (see category *b* below) of balancing and steering a velocipede. Surprisingly, this master of the differential equations governing dynamics includes none in his discussion of bicycles.
- 1929** Wallace's long technical paper on motorcycle design contains thoughtful qualitative discussions on the likely handling characteristics of various motorcycle designs (pp. 177–184). He examines steer torque, including the contribution of toroidal tires and gyroscopic torques. Wallace's analysis of non-linear geometric effects (pp. 185–212) erroneously assumes no pitch of the rear frame due to steering.
- 1970** David E. H. Jones Physics Today article is perhaps the single best-known paper on bicycle stability. With simple experiments Jones showed that, for the bicycles he tried, both front-wheel spin momentum and positive mechanical trail were needed for self-stability. Jones observed that a rider can easily balance most any bicycle that was not self-stable by turning the handlebars appropriately. But when riding no-hands, Jones had difficulty stabilizing a bicycle whose front-wheel gyroscopic terms were canceled by an added, counter-spinning wheel. And Jones was unable to master no-hands balance of a bicycle with negative trail. Jones experimental observations indicate correct trends, but do not represent precise boundaries on what is or can be stable or controlled.

On the theoretical side Jones wanted to counter the widely-quoted simple gyroscopic explanation of no-hands bicycle stability, presented for example in Sharp above, that a rider leans the frame making the gyroscopic front wheel precess and turn towards the lean, exactly as a hands-on rider would turn it. His experiments with a variety of bicycles pointed to mechanical trail as another important factor in bicycle stability.

Jones did no dynamical modeling, and focused only on trail's effect on steer torque as a function of lean. His thought was that the "static" torque would define the steering tendency for a leaned bicycle, and thereby explain self-stability. In effect Jones explored only the gravitational-potential part of one entry in the stiffness matrix, while ignoring the velocity-dependent centrifugal and gyroscopic terms. His approach and theoretical conclusions concerning mechanical trail are thus erroneous. Unfortunately a variety of subsequent investigators have built on Jones' weakly-founded potential-energy treatment.

- 1942-98** Various other qualitative discussions, none making use of already published governing dynamics equations, were authored by Arthur Jones (1942), Den Hartog (1948), Higbie (1974), Kirshner (1980), Le Hénaff (1987), Patterson (1993) and Cox (1998). A paper by Maunsell (1946) seems to be the best of this lot. These papers, somewhat like Jones (1970), generally describe one or another term in the dynamics equations (e.g., centripetal forces or gyroscopic terms), often correctly, but then incorrectly make general statements about bicycle stability which, to be accurate, would require a more complete

dynamic description. None of these papers make a reasonable statement about the conditions necessary for self-stability.

1988 Olsen & Papadopoulos' qualitative article discusses aspects of dynamic modeling based on the full correct uncontrolled bicycle equations in Papadopoulos (1987).

2004 Wilson's *Bicycling Science* includes a chapter by Papadopoulos which qualitatively discusses bicycle stability.

2005 Åström, Klein & Lennartsson present a three part paper on bicycle dynamics. Part three summarizes more than two decades of experimental work by Klein and students (more recently carried by Åström and students) that is much in the spirit of Jones (1970). Parts 1 and 2 of this paper are discussed below.

(b) *Simplified dynamic analyses that are incapable of explaining self-stability*

Simplified dynamic models have appeared from the mid 1890's to the present day. In these papers one or both of the following simplifications is central:

i) Simplified geometry and/or mass distribution. In these models some collection of the following assumptions are made:

- inertia axes of rear frame are vertical/horizontal
- inertia axes of front frame are vertical/horizontal or aligned with steer axis
- no spin angular momentum of wheels
- point masses for the frames and/or wheels
- massless front frame
- massless wheels
- vertical steer axis
- zero trail
- vanishing wheel radii

Such simplified models are most often incapable of self-stability.

ii) Steer angle fully controlled by the rider; no steer dynamics. In these models balance is effected entirely as a result of rider-controlled steering angle, and the steer angle δ has no uncontrolled dynamics. Appropriately controlled steer angle is indeed the only way to stabilize many simplified bikes. Controlled-steer treatments cannot illuminate a bicycle's self stability because, in the small-angle regime, a bicycle with locked steering has no self stability.

A common geometric issue. Many of these simplified-dynamics analyses include some non-linear terms (e.g., $\sin \phi$ instead of ϕ). However, all purportedly nonlinear simplified-bike treatments of which we are aware, from Bourlet (1894) to Getz & Marsden (1995), do not actually write correct non-linear equations for any simplified model of a bicycle. In these treatments wheel base, trail, frame pitch,

path curvature and other such quantities are treated as being independent of the lean angle for any given non-zero steer angle. That these quantities all vary with lean angle for an ideal bicycle is demonstrated by considering a small leftward steer angle. As the lean angle goes to -90 degrees, with the bicycle almost lying on its left side, the front contact point moves forward around the front wheel approximately by 90 degrees, while the rear contact point moves backward around the rear wheel the same amount. This alters the wheel base length, the angle between ground traces of the two wheels, and the trail. Depending on the frame geometry, it also places the front contact well outside the rear frame's symmetry plane, and introduces substantial pitch of the rear frame about the rear axle, relative to the ground trace of the rear wheel. Even the simplest bicycle models (with vertical steering axis, zero trail, and vanishing front-wheel radius) are subject to at least an alteration of the path-curvature due to lean. Treating trail, path curvature and other such quantities as independent of lean is not accurate for any bicycle, for large lean. In effect these models represent a bicycle as an inverted pendulum mounted on a tricycle and then write full non-linear equations for that tricycle.

To be fair, this pendulum-on-a-tricycle model of a bicycle agrees with a similarly simple bicycle model in the linear (near upright, small steer angle) regime and may catch some of the non-linear features of a controlled bicycle for large angles. Further, in the case where the simplified bicycle model has a vertical steer axis, no trail and zero wheel radius and where, further, steer is fully controlled, any control for the tricycle model maps exactly onto an equivalent control of the bicycle with a correction for the relation between steer and path curvature.

1894-1899 Mathematician Carlo Bourlet devotes several papers and multiple editions of his book to lateral balance of a steer controlled velocipede (vertical steer axis and no trail). The mass and geometry are somewhat general, although spin angular momentum of the wheels is neglected. The treatment is largely nonlinear, but has the shifting-contact error described above. Bourlet considers steering moves that can eliminate a lean, or follow a path. His final and most technical paper (1899) is awarded the Prix Fourneyron prize. Bourlet claims to have outlined the design factors leading to self-stability in his book, but he does not address them analytically in his papers.

1899 Physicist Joseph Boussinesq wrote 6 papers on velocipede balance and control, very similar in approach and content to prior work by Bourlet (but slightly anticipating Bourlet's most sophisticated dynamical modeling). Boussinesq also notes that the system's center of mass can usually be displaced sideways by upper-body lean relative to the frame (This is the means by which an inverted double pendulum can be balanced by actuation of the connecting hinge). Self-stability was not addressed.

1899 G.R.R. Routh (son of renowned dynamicist E.J. Routh) considers steering strategies for roll stability and path following of a slightly more general model of a velocipede than was considered by Bourlet (1899) and Boussinesq (1899).

1910 Bouasse, in his dynamics textbook, reviews and simplifies the velocipede analyses of Bourlet (1899) and Boussinesq (1899).

- 1915** Bower investigates the stability of an uncontrolled velocipede via linearized equations that are incorrect, presumably due to their *ad hoc* derivations. However, the central result, that such a bicycle has no self-stability, happens to be correct. Comparable incorrect treatments are also presented in Pearsall (1922, citing Bower), Lowell & McKell (1982, citing Pearsall), and Fajans (2000, citing Lowell & McKell). Typically, one or several terms are missing from each equation of motion. Authors sometimes write of ‘approximating’ by intentional neglect of supposedly small contributions. Unfortunately, one can always find cases where the neglected terms or approximations are significant in effect.
- 1934** Loïcjanskiï & Lur’e, in their textbook, study an uncontrolled velocipede which is cited by Neïmark & Fufaev (1967) and revisited in Lobas (1978).
- 1948** Timoshenko & Young’s well-known dynamics text presents the analysis of Bouasse (1910).
- 1955** Haag independently derives bicycle equations of motion in his book, but simplifies by inconsistently ignoring various terms involving trail, spin momentum, front assembly mass, cross terms in the potential energy, etc. The resulting incorrect differential equations of a simplified bicycle model lead him to incorrectly conclude that bicycle self-stability is never possible.
- 1967** Neïmark & Fufaev, in their classic text on non-holonomic dynamics, consider a point-mass uncontrolled velocipede (separately from their detailed bicycle model discussed in category *c* below). They neglect wheel inertias so they can compare to the model of Loïcjanskiï & Lur’e (1934). They mistakenly omit the mass from the second term in equation 2.65 (English edition p. 354), leaving a dimensionally incorrect quantity ν to propagate through to equations 2.67 and 2.68. They only find self-stability for finite viscous friction in the steer column.
- 1995** Getz & Marsden consider the possibility of following an arbitrary path without falling over, when not only steering but also forward speed may be controlled. Their simplified nonlinear model incorporates no wheel radius nor wheel inertias. Like many others before them (e.g. Bourlet 1894) this paper makes geometric assumptions that are equivalent to modeling a bicycle as a pendulum mounted on a tricycle (see subsection above on a “geometric issue”).
- 2005** Åström (control-system specialist), Klein & Lennartsson present, in section one of their paper, a correct linearized roll equation for a highly simplified point-mass bicycle with no wheel spin momentum, front assembly mass, steer axis tilt, or trail. Then steer axis tilt and fork offset are added in *ad hoc* fashion, so that the resulting roll equation is missing most of the contributions of trail. The steer equation neglects all derivative contributions from steer and lean, which are caused by trail. Despite their statement that this model gives “a correct explanation for self-stabilization”, the simplified model presented is actually incapable of self-stabilization. Sections 2 and 3 of this paper are discussed above and below.

(c) *Equations of motion for a general bicycle*

Here we discuss literature on linear equations of motion for general bicycle models with uncontrolled steering, similar to the one in this paper. Papers in which e.g., toroidal wheels, tire-slip models, frame or rider elastic deformation, rider steering inputs, or rider-controlled torso lean, were difficult to remove from the analysis are not discussed.

1899 Whipple undertakes the first serious analysis of the self-stability of a bicycle.

Whipple was a Cambridge University undergraduate at the time, and was a Second Wrangler in the Tripos mathematics exam. He later had a long career in mathematical meteorology. Whipple's parameters are fully general and his model can be compared to the one presented here. His paper was awarded Honorable Mention for the prestigious Smith's Prize. Whipple undertook the difficult task of a fully nonlinear analysis, which was flawed by an incorrect expression of the front-wheel ground-contact vertical constraint. However, when linearized this error is irrelevant, and Whipple's linearized equations are correct, except for a few typographical errors. Whipple's results include scaling rules, the dynamic modes (nowadays known as weave and capsize), rider control inputs via torso lean, etc. Whipple also recognized that the exponential decay of lean and steer perturbations is not inconsistent with energy conservation.

1900 Carvalho was already an accomplished applied mathematician and mechanic when he was awarded the Prix Fourneyron (like Bourlet just before him), for a two-part 300-page monograph on the dynamics of uncontrolled monocycles (single wheels surrounding a rider) and bicycles. Although Carvalho neglects the mass and inertia of the handlebar assembly, his equations for his model are correct. Carvalho identified the four standard eigenmodes, and presented equations for the upper (capsize) and lower (weave) limiting velocities for hands-free stability. Carvalho mentions the use of Grassman's geometric calculus, and innovative stability calculations similar to Routh-Hurwitz. According to Carvalho, bicycle constructors of his time recommended that the steer axis be designed to pass under the front axle, half way between axle and ground, a feature approximately maintained to the present day. Both Whipple and Carvalho laid solid foundations for future work. But despite being publicized by citation in the 11th edition *Encyclopedia Britannica* (Gyroscope article), and also Grammel's 1920 Gyroscope textbook, their achievements languished for decades. Their work may have guided Noether's 1910 analysis which was elaborated by Döhring (1953) and in turn by Singh & Goel (1971).

1910 Klein & Sommerfeld's 4th volume on gyroscopes appears with an extensive chapter on bicycles written by Fritz Noether (brother of mathematician Emmy). These governing equations for a slightly simplified bicycle model, derived by techniques used for other gyroscopic systems, are equivalent to those in Carvalho (1900) and fully correct. While Noether claims to have compared his equations with Whipple as well as Carvalho, he makes an erroneous statement about Whipple's derivation, and acknowledges neither the more general model nor the typographical errors. The illuminating discussion of gyroscopic contributions is particularly valuable.

The analysis of bicycles or motorcycles subsequently fell into a lull. Apart from Döhring (1953), we are not aware of any publication in the open literature of correct equations of motion until 1970. The most likely candidates, which we have not been able to check, are papers or presentations by Kondo in Japan, 1948-1964 including Kondo's Report on bicycle stability research at the meeting of JSME in November, 1948, unpublished (no preprints). Another possibility is the work of Ge (1966) in Taiwan. Lastly, another candidate is A.M. Letov (1959).

1951 Manning, in a proprietary technical report of the British Road Research Laboratory, appears to provide correct nonlinear configuration geometry, and possibly correct linearised equations of motion for a general model. (However, we have not yet checked these.) Manning acknowledges Carvallo's work but did not compare equations in detail.

1953-1955 Döhring, University of Technology Braunschweig, Germany, writes a PhD thesis on the stability of a straight ahead running motorcycle. He builds on the model by Noether (Klein & Sommerfeld, 1910) to make the mass distributions fully general. Döhring's equations agree with ours in detail. They are the first perfectly correct equations of a general model presented in the open literature. Döhring also did experiments on a motor-scooter and two different motorcycles (1954) in an attempt to validate his results .

1963-1964 University of Wisconsin dissertations by Collins (1963) and Singh (1964) were both too hard to follow in enough detail to evaluate. Their multi-page equations employed chained parameter definitions. Collins relied on Wallace's (1929) incorrect nonlinear geometry, but this should not affect the correctness of his linearization. Although we were not able to compare Collins' equations in detail, we noted a missing term and Psiaki (1979) found computational disagreement. Singh's subsequent conference and journal publications were based on Döhring's (1953) equations rather than his own.

1967 Neĭmark & Fufaev, in their authoritative book on nonholonomic dynamics, present an exceptionally clear and thorough derivation of the equations of motion for a general bicycle (based on our reading of the 1972 English translation). Unfortunately their treatment has several typographical errors, and also a fatal error in the potential energy: equation 2.30 ignores downward pitch of the frame due to steering. This flaw was later corrected by Dikarev, Dikareva & Fufaev (1981) and independently by Hand (1988).

The year 1970 marks a sudden jump in the quality and quantity of single-track vehicle research, perhaps because of the advent of digital computers and compact instrumentation, increased popularity of large motorcycles (and attendant accidents), and a surge in bicycle popularity. Most authors were eager to incorporate tire models which simplifies the equation formulation by avoiding having to implement geometric constraints. But tire models add empirical parameters and complicate the resulting equations and their interpretation. One major concentration of single-track research was at CALSPAN (then the Cornell Aeronautical Laboratory), funded by the US government, Schwinn Bicycles and Harley Davidson. Over the period 1970-1978 CLASPAN generated about 20 bicycle reports and papers. The CALSPAN

program included hand calculations (involving linearized equations and algebraic performance indices for a somewhat simplified model), nonlinear computer models (including high-order rider control inputs), and a comprehensive experimental program (including tire measurements and comparisons to real-world performance).

1970 Rice & Roland, in a CALSPAN report sponsored by the National Commission on Product Safety, included an appendix on nonlinear equations (except linearized for small steer angles), where compliant, side-slipping tires avoid the need to apply lateral or vertical contact constraints. Rider lean relative to the frame is included. Thus the governing system includes all six velocities of a rigid body, plus the two extra degrees of freedom (steer and rider lean). The tabulated 8×8 first order system is forbiddingly complex, and terms such as wheel vertical force require a host of subsidiary equations to be defined. This report seems to have the first use of the term ‘mechanical trail’ to describe the moment arm of the front contact forces about the steer axis.

1971 Roland & Massing, commissioned by the Schwinn bicycle company, write a CALSPAN report on the modeling and experimental validation of an uncontrolled bicycle. The mix of modeling, measuring, and testing is unusually thorough. After correcting an expression for tire slip, then linearizing and imposing constraints their equations agree with the equations here.

1971 Robin Sharp considers a model with tire slip, and front-assembly inertia tensor aligned with the steering axis. His partly-nonlinear model treats rear-frame pitch as zero, with a constant force acting upward on the front wheel. When he linearizes and takes the limit of infinite lateral tire stiffness, he introduces minor algebraic and typographical errors (see Hand 1988). This, Sharp’s first of many bicycle and motorcycle dynamics papers, has had a lasting influence. It includes his original naming of the two major rigid-body eigenmodes as ‘weave’ and ‘capsize’.

1971 Singh & Goel use Döhring’s (1955) correct equations.

1972 Roland & Lynch, commissioned by the Schwinn bicycle company, write a CALSPAN report on a rider control model for path tracking, bicycle tire testing, experimental tests to determine the effect of design parameters on the stability and manoeuvrability of the bicycle, and the development of computer graphics for display purposes. For the bicycle model the equations from from Roland & Massing (1971) are used.

1972 Weir’s PhD thesis explicitly compares his correct equations with the previous slightly incorrect and slightly specialized results of Sharp (1971). He appears to be the first to perform such a check.

1973 Eaton presents governing equations without derivation. He explains that he reconciled his own derivation with (corrected) Sharp (1971) and Weir (1972), in his own notation with some embellishment of the tire models.

1973 Roland reports, in the open literature, basically the same equations as in Roland & Massing (1971). Apparently, few if any typos were corrected and some further typos were introduced.

- 1974** Rice at CALSPAN uses linearized analysis to develop steady-state and transient performance indices. He investigates the stiffness matrix (with rider lean included, statically equivalent to a lean moment), which requires only point-mass bicycle parameters. A lot of the complication depends on tire parameters. As in Carvalho (1900) and Whipple (1899), formulae are given for capsize speed and upright turning speed.
- 1975** Van Zytveld's MSc thesis on a robot bicycle controller develops equations that are correct except for some of the terms involving 'rider lean'. According to Van Zytveld his advisor, David Breakwell, had developed independent equations of motion, without a rider-robot, that matched his when simplified to remove rider lean (see also Breakwell 1982).
- 1975** Singh & Goel elaborate the standard 'general' model to allow deviations from left-right symmetry and incorporate more sophisticated tire models, leading to a very high order system of governing equations. The derivation appears to follow Sharp (1971) but we have not checked the results in detail.
- 1976** Rice writes a CALSPAN report on simplified dynamic stability analysis. He assumes all inertia tensors to have a vertical principal axis. This report explicitly identifies a frequently-occurring combination of terms which we call S_A .
- 1978** Weir & Zellner present Weir's equations but introduce a significant error in the mistaken belief they are making a correction, and commit several typographical errors (see Hand 1988). Weir's thesis (1972), not this paper, should be used for correct equations.
- 1978** Lobas (in translation misspelled into Gobas) extends the treatment by Neĭmark & Fufaev (1967) to add forward acceleration. When we set acceleration to zero, it appears that the static lean contribution to Lobas's steer equation is in error.
- 1979** Psiaki writes a dense Princeton undergraduate honors thesis on bicycle dynamics. Starting from a fully nonlinear analysis, he developed linearized equations for both his upright configuration and his hands-free turns. The analysis details were impenetrable to us, but his numerical results match ours to plotting accuracy.

The early 1980's essentially mark an end to the development of sound equations for the geometrically and inertially general bicycle. Equations from Sharp, Weir, or Eaton are widely recited as valid, even though explicit comparisons are hard to find. Subsequent high-quality work tends to focus on elaborations necessary for modeling tire and frame deformations.

- 1985** Sharp presents a very comprehensive review of extended motorcycle dynamics equations, with an emphasis on capturing weave motions that seemingly depend on tire and frame flex. He has some errors in his description of the pre-1970 literature. Sharp (1985) identifies Sharp (1971) as 'confirmed', which, but for minor errors, is correct.

- 1987** Papadopoulos focused on achieving a compact notation and simple derivation of the equations of motion, using Hand’s (1988) results as a check.
- 1988** Hand’s Cornell MSc thesis compares a variety of publications and settles on a compact, transparent notation. He shows that several approaches (Döhring 1955, Neïmark & Fufaev 1967, Sharp 1971, and Weir 1972) all led to the same governing equations when errors were corrected.
- 1988** Mears verified Weir’s (1972) thesis and noted Weir’s later (1978) errors. Mears also checked against Hand (1988).
- 1990** Franke, Suhr & F. Rieß derive non-linear equations of a bicycle, with neglect of some dynamic terms. This paper was the topic of an entertaining lead editorial in *Nature* by John Maddox (1990). We did not check the derivation in detail. The authors did not find agreement between integration of their differential equations for small angles and the integration of the Papadopoulos (1987) equations (private communication). However, recently Lutz and Aderhold (private communication) applied our benchmark bicycle parameters to an updated form of the Franke, Suhr & Rieß non-linear model and obtained agreement of eigenvalues in an approximately upright configuration, within plotting accuracy.
- 2004** Meijaard in preparing for this publication, makes an independent derivation of the linearized equations of motion that agrees with the equations here.
- 2004** Schwab, Meijaard & Papadopoulos write a draft of the present paper and present it at a conference.
- 2005** Åström, Klein & Lennartsson in part 2 of their paper, present Lennartsson’s simulations from a general purpose rigid-body dynamics code and get agreement with the benchmark equations in Schwab (2004), although not with enough precision to convincingly demonstrate correctness. They also present various results from Schwab (2004) and Papadopoulos (1987).

In summary, substantially correct and general linearized bicycle equations of motion were initially provided by Whipple (1899) (typographical errors), Carvallo (1900) (slightly simplified model), and Noether (Klein & Sommerfeld 1910) (also slightly simplified). The first ‘perfect’ and general equations were provided by Döhring (1953). After 1970, most analyses included tire models. Sharp (1971) presented equations with just a slight error and slight specialization. Weir (1972) presented the second ‘perfect’ and general set. Eaton (1973), Van Zytveld (1975), probably Psiaki (1979), Papadopoulos (1987), Hand (1988) and Mears (1988) also presented accurate results for the model we use here.

2. SPACAR model

The SPACAR model for the benchmark bicycle is sketched in figure 5 and the input file for the SPACAR program describing this model is presented in §2 a.

Because the SPACAR program is based on finite-element methods (FEM), the input file shows an FEM structure. SPACAR is designed to minimize the information needed in the input file. The SPACAR input file is roughly divided into four

parts: element connectivity, nodal data, boundary conditions, and some additional data like masses, inertias, applied forces and simulation settings.

In the first section of input in §2 a the elements are listed, they are given a type, a unique element number followed by a list of node numbers and an initial rotation axis. These element statements implicitly define the associated nodes. The nodes are either translational or rotational. A **hinge** element allows large relative rotation between two rotational nodes. A **wheel** element allows rolling contact at the contact point node. A **pinbody** element generates a node within a rigid body by which another finite element can be connected. Within this finite element approach a rigid body can be defined two ways: either as a deformable element with all deformation modes set to zero or as a body with one three-degree-of-freedom translational node and one three-degree-of-freedom rotational node.

In the second section of the input file the nodes, which are placed at the centre of mass of the rigid bodies, are given their reference-configuration coordinates. Translational nodes have three coordinates (x, y, z) in a global reference frame whereas rotational nodes are parameterized by four Euler parameters. These parameters are set to $(1, 0, 0, 0)$, the unit transformation, in the reference configuration.

The approach in establishing a bicycle model is to consider it in a reference configuration: upright, orientated along the x-axis, and with the rear contact at the origin. This configuration is used to define nodal positions and rigid body orientations. Relative to this reference configuration it is easy to set an initial lean or steer angle and set the rates as initial conditions. To do a simulation from an arbitrary configuration, however, you have to drive it there by specifying a path from the initial configuration to the desired initial state.

Any consistent set of units may be used, SI units give a bicycle that is sized to a real bicycle.

In the third section the boundary conditions are set, the implicit definition is that all nodes are free and all elements are rigid. A node's position or orientation in space can be fixed by the **fix** command otherwise it is free to move in space. An element can be allowed to "deform" e.g. a hinge element is allowed to rotate, by the **rlse** command. A non-zero prescribed "deformation" mode is specified by **inpute**, e.g. the forward motion of the bicycle in this example. For generating linearized equations of motion the **line** command identifies a degree of freedom to be used. The **enhc** command ties a nonholonomic constraint to a configuration space coordinate so as to identify those configuration coordinates for which the time derivative is not a velocity degree of freedom.

In the last section mass and inertia are added to the nodes, one value for translational nodes and six values for rotational nodes (the terms in the upper triangle portion of the inertia matrix in the initial configuration). Finally applied (constant) forces are added and some initial conditions and simulation settings are made.

When the program is run, for each output time step, all system variables (coordinates, deformations, speeds, accelerations, nodal forces, element forces, etc.) are written to standard files which later can be read by other software for plotting or analysis. At every time step the coefficients of the SPACAR systematic linearization are also written to standard files.

(a) *SPACAR Input file*

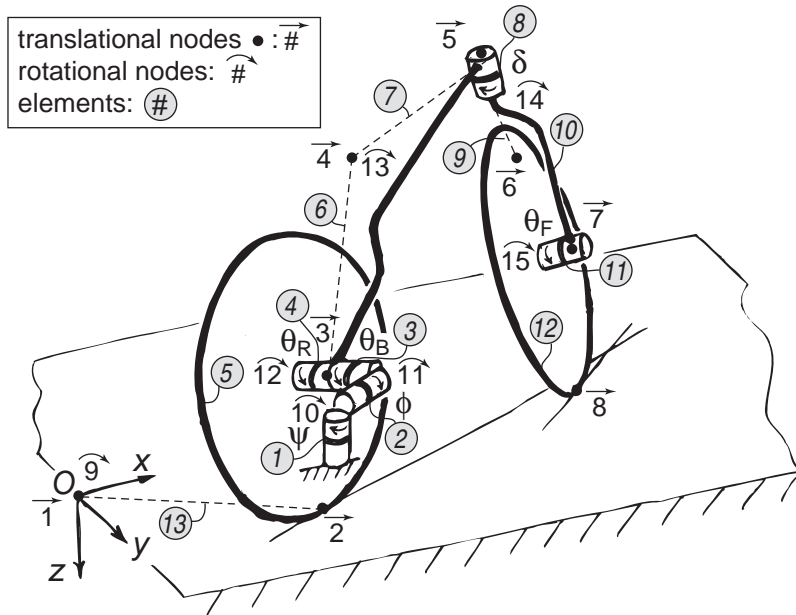


Figure 5. Sketch of the bicycle model for SPACAR input together with node numbers (straight arrows for translations 1...8, curved arrows for rotations 9...15) and element numbers encircled.

The sketch of this model is shown in figure 5.

```
% SPACAR input file for bicycle benchmark I
% SECTION 1, ELEMENT DECLARATION AND CONNECTIVITY:
% type number nodes      rotation axis vector
hinge  1   9 10          0 0 1 % yaw angle rear frame between node 9(ground) and 10
hinge  2  10 11          1 0 0 % lean angle rear frame between node 10 and 11
hinge  3  11 13          0 1 0 % pitch angle rear frame between node 11 and 13(frame)
hinge  4  13 12          0 1 0 % rear wheel rotation between 13(frame) and 12(wheel)
wheel  5   3 12  2       0 1 0 % rear wheel, cm nodes 3, 12, contact pnt 2
pinbody 6  4 13  3       % node 3(rear hub) in rigid body nodes 4, 13(frame)
pinbody 7  4 13  5       % node 5(head) in rigid body nodes 4, 13(frame)
hinge  8  13 14          0.32491969623291  0  1.0 % steering angle between 13 and 14
pinbody 9  5 14  6       % node 6(cm fork) in rigid body nodes 5, 14(front frame)
pinbody 10 5 14  7       % node 7(front hub) in rigid body nodes 5, 14(front frame)
hinge 11 14 15          0 1 0 % front wheel rotation between 14 and 15(wheel)
wheel 12  7 15  8       0 1 0 % front wheel, cm nodes 7, 15, contact pnt 8
pinbody 13  1 9  2       % node 2(rear contact pnt) in rigid body nodes 1, 9
% SECTION 2, NODAL DATA:
% node initial coordinates,      all rotational nodes are initialized:(1,0,0,0)
x  1  0  0  0  0 % fixed origin
x  2  0  0  0  0 % rear contact point
x  3  0  0 -0.3 % rear hub
x  4  0.3 0 -0.9 % cm rear frame + rigid rider
x  5  0.80757227339038 0 -0.9 % steering head
x  6  0.9 0 -0.7 % cm front fork + handle bars
x  7  1.02 0 -0.35 % front hub
x  8  1.02 0 0 % front contact point
% SECTION 3, BOUNDARY CONDITIONS:
% type number components
fix  1  1 2 3 % fix all(1,2,3) translations node 1(ground)
fix  9  1 2 3 4 % fix all(1,2,3,4) rotations node 9(ground)
```

```

rlse 1 1 % release rotation(1) hinge 1: yaw
rlse 3 1 % release rotation(1) hinge 3: pitch
rlse 11 1 % release rotation(1) hinge 11: front wheel rotation
rlse 13 1 2 3 % release all relative displacements(1,2,3) in pinbody 13
inpute 4 1 % rotation(1) hinge 4 is prescribed motion for forward speed
line 2 1 % generate linearized eqns for rotation(1) hinge 2: lean
line 8 1 % generate linearized eqns for rotation(1) hinge 8: steering
% tie a non-holonomic constraint to a configuration space coordinate
%type lmnt mode lmnt mode (lmnt means element number)
enhc 5 4 13 1 % wheel 5 4=long slip tied to pinbody 13 1=x-disp node 2
enhc 5 5 13 2 % wheel 5 5=lat slip tied to pinbody 13 2=y-disp node 2
enhc 12 4 1 1 % wheel 12 4=long slip tied to hinge 1 1=yaw rear frame
enhc 12 5 11 1 % wheel 12 5=lat slip tied to hinge 11 1=front wheel rot
% SECTION 4, ADDITIONAL DATA: MASS, INERTIA, APPLIED FORCES, AND SIMULATION SETTINGS
% node mass:(m) or mass moment of inertia:(Ixx,Ixy,Ixz,Iyy,Iyz,Izz)
mass 3 2.0 % mass rear wheel
mass 12 0.0603 0 0 0.12 0 0.0603 % inertia rear wheel
mass 4 85.0 % mass rear frame + rider
mass 13 9.2 0 2.4 11.0 0 2.8 % inertia rear frame + rider
mass 6 4.0 % mass front frame + handle bars
mass 14 0.05892 0 -0.00756 0.06 0 0.00708 % inertia front frame + handle bars
mass 7 3.0 % mass front wheel
mass 15 0.1405 0 0 0.28 0 0.1405 % inertia front wheel
% node applied force vector (gravity used g = 9.81)
force 3 0 0 19.62 % gravity force rear wheel
force 4 0 0 833.85 % gravity force rear frame + rider
force 6 0 0 39.24 % gravity force front frame + handle bars
force 7 0 0 29.43 % gravity force front wheel
% initial conditions
ed 4 1 -3.33333333 % angular velocity in hinge 4(forward speed) set to -3.33333333
% simulation settings
epskin 1e-6 % set max constraint error for Newton-Raphson iteration
epsint 1e-5 % set max numerical integration error on coordinates
epsind 1e-5 % set max numerical integration error on speeds
timestep 100 2.0 % set number of output timesteps and simulation time
hmax 0.01 % set max step size numerical integration
end % end of run
eof % end of file

```

3. AutoSim model

The AutoSim input file used for the bicycle model is listed below. The generalized coordinates and velocities are the same as those in the SPACAR model. Two massless intermediate reference frames have been introduced: a yawing frame describing the horizontal translation and yawing of the rear frame and a rolling frame describing the lean of the rear frame with respect to the yawing frame. These additional frames allow a better control over the choice of the generalized coordinates by the program. The holonomic constraint at the rear wheel is automatically satisfied. The holonomic constraint at the front wheel and the four non-holonomic constraints are explicitly defined in the input file. For more details on the syntax used see the AutoSim documentation.

(a) AutoSim Input file

```

;;; This is the file fietsap2.lsp, with the benchmark1 model.
;; Set up preliminaries:

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(reset)
(si)
(add-gravity :direction [nz] :gees g)
(set-names g "Acceleration of gravity" )
(set-defaults g 9.81) ; this value is used in the benchmark.
;; The name of the model is set to the string "fiets".
(setsym *multibody-system-name* "fiets")
;; Introduce a massless moving reference frame. This frame has x and y
;; translational degrees of freedoms and a yaw rotational degree of freedom.
( add-body yawframe :name "moving yawing reference frame"
  :parent n :translate (x y) :body-rotation-axes z
  :parent-rotation-axis z :reference-axis x :mass 0
  :inertia-matrix 0 )
;; Introduce another massless moving reference frame. This frame has a rolling
;; (rotational about a longitudinal axis) degree of freedom.
( add-body rollframe :name "moving rolling reference frame" :parent yawframe
  :body-rotation-axes (x) :parent-rotation-axis x :reference-axis y :mass 0
  :inertia-matrix 0 )
;; Add the rear frame of the bicycle. The rear frame has a pitching (rotation
;; about the local lateral y-axis of the frame) degree of freedom.
( add-body rear :name "rear frame" :parent rollframe
  :joint-coordinates (0 0 "-Rrw") :body-rotation-axes y
  :parent-rotation-axis y :reference-axis z :cm-coordinates (bb 0 "Rrw-hh")
  :mass Mr :inertia-matrix ((Irxx 0 Irxz) (0 Iryy 0) (Irxz 0 Irzz)) )
( set-names
  Rrw "Rear wheel radius"
  bb "Longitudinal distance to the c.o.m. of the rear frame"
  hh "Height of the centre of mass of the rear frame"
  Mr "Mass of the rear frame"
  Irxx "Longitudinal moment of inertia of the rear frame"
  Irxz "Minus product of inertia of the rear frame"
  Iryy "Transversal moment of inertia of the rear frame"
  Irzz "Vertical moment of inertia of the rear frame" )
( set-defaults Rrw 0.30 bb 0.3 hh 0.9
  Mr 85.0 Irxx 9.2 Irxz 2.4 Iryy 11.0 Irzz 2.8 )
;; Add the rear wheel of the vehicle. This body rotates
;; about the y axis of its physical parent, the rear frame.
( add-body rw :name "rear wheel" :parent rear :body-rotation-axes y
  :parent-rotation-axis y :reference-axis z :joint-coordinates (0 0 0)
  :mass Mrw :inertia-matrix (irwx irwy irwx) )
( set-names
  Mrw "mass of the rear wheel"
  irwx "rear wheel in-plane moment of inertia"
  irwy "rear wheel axial moment of inertia" )
(set-defaults Mrw 2.0 irwx 0.0603 irwy 0.12)
;; Now we proceed with the front frame.
;; Define the steering and reference axes of the front frame.
;; Add in the front frame: define some points.
( add-point head :name "steering head point B" :body n
  :coordinates (xcohead 0 zcohead) )
( add-point frontcpoint :name "c.o.m. of the front frame" :body n
  :coordinates (xfcm 0 zfcm) )

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( set-names
  epsilon "steering head angle"
  xcohead "x coordinate of the steering head point B"
  zcohead "z coordinate of the steering head point B"
  xfcm "x coordinate of the c.o.m. of the front frame"
  zfcm "z coordinate of the c.o.m. of the front frame" )
( set-defaults epsilon 0.314159265358979316
  xcohead 1.10 zcohead 0.0 xfcm 0.90 zfcm -0.70 )
( add-body front :name "front frame" :parent rear :body-rotation-axes z
  :parent-rotation-axis "sin(epsilon)*[rearx]+cos(epsilon)*[rearz]"
  :reference-axis "cos(epsilon)*[rearx]-sin(epsilon)*[rearz]"
  :joint-coordinates head :cm-coordinates frontcmpoint :mass Mf
  :inertia-matrix ((Ifxx 0 Ifxz) (0 Ifyy 0) (Ifxz 0 Ifzz))
  :inertia-matrix-coordinate-system n )
( set-names
  Mf "Mass of the front frame assembly"
  Ifxx "Longitudinal moment of inertia of the front frame"
  Ifxz "Minus product of inertia of the front frame"
  Ifyy "Transversal moment of inertia of the front frame"
  Ifzz "Vertical moment of inertia of the front frame" )
( set-defaults Mf 4.0
  Ifxx 0.05892 Ifxz -0.00756 Ifyy 0.06 Ifzz 0.00708 )
;; Add in the front wheel:
( add-point fw_centre :name "Front wheel centre point" :body n
  :coordinates (ll 0 "-Rfw") )
( add-body fw :name "front wheel" :parent front :body-rotation-axes y
  :parent-rotation-axis y :reference-axis "[nz]"
  :joint-coordinates fw_centre :mass Mfw :inertia-matrix (ifwx ifwy ifwx) )
( set-names
  ll "Wheel base"
  Rfw "Radius of the front wheel"
  Mfw "Mass of the front wheel"
  ifwx "In-plane moment of inertia of the front wheel"
  ifwy "Axial moment of inertia of the front wheel" )
(set-defaults ll 1.02 Rfw 0.35 Mfw 3.0 ifwx 0.1405 ifwy 0.28)
;; The system is complete, except for the contact constraints at the wheels.
;; The holonomic constraint at the rear wheel is automatically satisfied.
;; The rear wheel slip is zero.
( add-speed-constraint "dot(vel(yawframe0),[yawframex])+Rrw*(ru(rear)+ru(rw))"
  :u "tu(yawframe,1)" )
(add-speed-constraint "dot(vel(yawframe0),[yawframey])" :u "tu(yawframe,2)")
;; For the front wheel we have a holonomic constraint for the contact and two
;; non-holonomic slip constraints. The slip velocities are defined now.
(setsym singammafw "dot([fwy],[nz])")
(setsym cosgammafw "sqrt(1-@singammafw**2)")
(setsym fw_rad "([nz] - [fwy]*@singammafw)/@cosgammafw")
(setsym slipfw_long "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[nx])")
;; No longitudinal slip on front wheel;
;; eliminate rotational velocity about the axis
(add-speed-constraint "@slipfw_long" :u "ru(fw)")
;; normal constraint; eliminate the pitch angle
(setsym slipfw_n "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[nz])")

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(add-speed-constraint "@slipfw_n" :u "ru(rear)")
(add-position-constraint "dot(pos(fw0),[nz])+Rfw*@cosgammafw" :q "rq(rear)")
;; No lateral slip on front wheel; eliminate yaw rate of the yawing frame
(setsym slipfw_lat "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[ny])")
(add-speed-constraint "@slipfw_lat" :u "ru(yawframe)")
(dynamics)
(linear)
```

4. The first equation: $\dot{v} = 0$

Here we expand on the explanation in the main body of the text for why $\dot{v} = 0$ is one of the linearized equations of motion for a constant-energy unforced bicycle.

As explained in section X, some configuration variables do not show up in the equations of motion and so are not of central interest. These include position (x_P, y_P) on the plane, the yaw ψ , and the net wheel rotations θ_R and θ_F . Of interest is the evolution of the right lean ϕ , the right steer δ , and backwards rear wheel rotation rate $\dot{\theta}_R$. For conceptual and notational convenience define forward speed as

$$v \equiv -R_R \dot{\theta}_R \quad (\text{B1})$$

and use v instead of $\dot{\theta}_R$ in the discussion below.

We would now like to establish the forward motion governing equation in the special case when there is no applied thrust. As discussed in the text, the much more involved equations for lean ϕ and steer δ are derived using *ad hoc* linearization, and are verified by various indirect means. In contrast, this one equation is more-firmly derived as follows.

First, without writing explicit non-linear equations, we know they have this exact solution:

$$v(t) = v^*, \quad \phi(t) = 0 \quad \text{and} \quad \delta(t) = 0 \quad (\text{B2})$$

where v^* is an arbitrary constant.

The linearized equations are for small perturbations about this reference solution. For notational simplicity we take the lean and steer perturbations as merely ϕ and δ recognizing that we are only discussing infinitesimal values of these variables, a range in which the linear contributions to any governing equations dominate. For the forward motion take the perturbation to be \hat{v} .

For the argument below we only depend on the linearity of the equations, and not their detailed form. Take an arbitrary set of initial conditions to be $(\hat{v}_0, \phi_0, \delta_0)$. At some definite time later, say $t_d = 1$ s for definiteness, the values of the speed lean and steer at t_d must be given by

$$\begin{bmatrix} \hat{v}_d \\ \phi_d \\ \delta_d \end{bmatrix} = \mathbf{A} \begin{bmatrix} \hat{v}_0 \\ \phi_0 \\ \delta_0 \end{bmatrix} \quad (\text{B3})$$

for any possible combination of \hat{v}_0 , ϕ_0 , and δ_0 . The matrix

$$\mathbf{A} = \begin{bmatrix} A_{vv} & A_{v\phi} & A_{v\delta} \\ A_{\phi v} & A_{\phi\phi} & A_{\phi\delta} \\ A_{\delta v} & A_{\delta\phi} & A_{\delta\delta} \end{bmatrix} \quad (\text{B 4})$$

depends on which definite time t_d is chosen. Because the bicycle rolls on a flat horizontal isotropic plane and there is no time-dependent forcing, the constant matrix \mathbf{A} is dependent on the time interval t_d but independent of the starting time.

Now consider an initial condition 1 where only the lean is disturbed:

$$\begin{bmatrix} \hat{v}_0^1 \\ \phi_0^1 \\ \delta_0^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{B 5})$$

where we think of 1 as a small perturbation. This results in a perturbation a time t_d later of

$$\begin{bmatrix} \hat{v}_d^1 \\ \phi_d^1 \\ \delta_d^1 \end{bmatrix} = \begin{bmatrix} A_{v\phi} \\ A_{\phi\phi} \\ A_{\delta\phi} \end{bmatrix} \quad (\text{B 6})$$

where the right side is the middle column of \mathbf{A} .

Now consider the opposite perturbation 2 with

$$\begin{bmatrix} \hat{v}_0^2 \\ \phi_0^2 \\ \delta_0^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad (\text{B 7})$$

which results in a perturbation a time t_d later of

$$\begin{bmatrix} \hat{v}_d^2 \\ \phi_d^2 \\ \delta_d^2 \end{bmatrix} = \begin{bmatrix} -A_{v\phi} \\ -A_{\phi\phi} \\ -A_{\delta\phi} \end{bmatrix} \quad (\text{B 8})$$

where the right side is the negative of the middle column of \mathbf{A} . This is merely a statement that for a linear system negating the input negates the output.

Now we invoke lateral symmetry. Both the bicycle and the equations of motion are laterally (left-right) symmetric. Thus if knocking a bicycle to the left causes it to speed up, knocking it to the right must cause it to speed up equally. So

$$v_d^2 = v_d^1 \quad \Rightarrow \quad A_{v\phi} = -A_{v\phi} \quad \Rightarrow \quad A_{v\phi} = 0. \quad (\text{B 9})$$

Now we can similarly apply a rightwards perturbation to just the steer. On the one hand linearity requires a negative steer has to have the negative effect on forward speed. On the other hand, lateral symmetry requires that a rightwards steer perturbation have an equal effect as a leftwards perturbation. Thus, by the same reasoning as for lean we get $A_{v\delta} = 0$.

Next, consider perturbations to just the forward speed \hat{v} . By symmetry these can cause neither a left nor right lean or steer. So $A_{\phi v} = A_{\delta v} = 0$. Thus symmetry reduces the matrix \mathbf{A} to having zeros off the diagonal in both the first row and the first column.

Finally, we know the steady upright solution is an exact non-linear solution for any v^* . Assuming that the full non-linear equations have unique solutions for any given initial conditions, a perturbation in v^* just leads to a new constant speed solution at the perturbed v^* . Thus, $\hat{v}_d = \hat{v}_0$ and $A_{vv} = 1$.

Altogether this means that the linearized equations giving the perturbed values of the state at time t_d in terms of the initial perturbation are necessarily of the form of equation B 3 with \mathbf{A} having the form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{\phi\phi} & A_{\phi\delta} \\ 0 & A_{\delta\phi} & A_{\delta\delta} \end{bmatrix}. \quad (\text{B } 10)$$

This equation must hold for any t_d thus perturbations in lean ϕ and steer δ never have influence on the forward speed v and vice versa, perturbations in speed have no influence on lean and steer. A similar argument shows that lean and steer rates ($\dot{\phi}, \dot{\delta}$) are similarly decoupled from the forward motion. Further, because

$$\hat{v}_d = \hat{v}_0 \quad (\text{B } 11)$$

for all time, \hat{v} is a constant so

$$\dot{v} = 0. \quad (\text{B } 12)$$

For a constant energy bicycle the linearized equation of motion for forward motion is equation B 12.

Similar arguments show that forward forcing does not cause lean or steer and that lateral forcing does not cause changes in speed (to first order).

Thus a bicycle which is forced to go at exactly constant speed in a full non-linear analysis has the same lean and steer governing equations as for the bicycle that is free in forward motion. Such is confirmed by SPACAR numerical analysis where

1. For small deviations from upright constant energy and constant speed give the same solutions (to about 4 digits) and
2. Both constant speed and constant energy give the same values for the numerical coefficients in the linearized equations and which are the same as the values presented in the body of the paper here for our ad hoc linearization (to about 14 digits).

5. A simplified benchmark model

In a second benchmark various simplifications are made to permit comparison with less complete models. The design parameters are according to table 1 but with the following changes. Both wheels are planar, $I_{yy} = 2I_{xx}$, and identical with: $m_R = m_F = 3$ kg, $r_R = r_F = 0.35$ m, and $(I_{Rxx}, I_{Ryy}) = (I_{Fxx}, I_{Fyy}) = (0.14, 0.28)$ kgm². The mass of the rear frame and body assembly B is $m_B = 85$ kg located at $(x_B, z_B) = (0.3, -0.9)$ m, whereas the mass moment of inertia is zero, $\mathbf{I}_B = \mathbf{0}$. The front frame H has neither mass, $m_H = 0$, nor inertia, $\mathbf{I}_H = \mathbf{0}$.

Substitution of these values of design parameters for the simplified benchmark bicycle in the expressions from Appendix A results in the following values for the

v [m/s]	$\text{Re}(\lambda_{\text{weave}})$ [1/s]	$\text{Im}(\lambda_{\text{weave}})$ [1/s]
0	–	–
1	3.915 605 159 008 03	0.676 636 216 381 60
2	3.145 971 626 952 20	1.947 971 866 614 21
3	2.096 627 566 535 66	3.144 568 094 683 27
4	0.910 809 011 944 21	4.881 202 124 548 49
5	0.198 648 678 113 17	6.936 393 452 637 19
6	–0.245 683 866 155 55	8.903 125 360 683 31
7	–0.589 203 483 851 70	10.790 930 464 293 57
8	–0.883 875 624 871 00	12.628 966 109 587 14
9	–1.150 515 263 118 26	14.434 482 871 116 77
10	–1.399 313 952 184 76	16.217 648 368 548 84

Table 5. Complex eigenvalues λ_{weave} from the linearized stability analysis for the oscillatory weave motion for the simplified benchmark bicycle from §5 in the forward speed range of $0 \leq v \leq 10$ m/s.

entries in the mass matrix from (A 20),

$$\mathbf{M} = \begin{bmatrix} 69.865 & 1.868\,727\,853\,976\,56 \\ 1.868\,727\,853\,976\,56 & 0.239\,079\,887\,561\,38 \end{bmatrix},$$

the entries in the constant stiffness matrix from (A 22) which are to be multiplied by gravity g ,

$$\mathbf{K}_0 = \begin{bmatrix} -78.6 & -2.226\,580\,876\,684\,00 \\ -2.226\,580\,876\,684\,00 & -0.688\,051\,330\,245\,63 \end{bmatrix},$$

the coefficients of the stiffness matrix from (A 24) which are to be multiplied by the square of the forward speed v^2 ,

$$\mathbf{K}_2 = \begin{bmatrix} 0, & 74.779\,149\,614\,579\,71 \\ 0, & 2.306\,586\,620\,338\,71 \end{bmatrix},$$

and finally the coefficients of the “damping” matrix from (A 26) which are to be multiplied by the forward speed v ,

$$\mathbf{C}_1 = \begin{bmatrix} 0 & , & 29.140\,558\,140\,953\,37 \\ -0.880\,193\,481\,747\,67, & & 1.150\,360\,143\,808\,13 \end{bmatrix}.$$

To facilitate comparison with equations or results derived using different methods, eigenvalues are presented. These eigenvalues in the forward speed range show the same structure as those from the full benchmark bicycle, see figure 3, but with slightly different values. The precise eigenvalues for the simplified bicycle benchmark at some forward speeds are presented in table 5 and table 6. These results may differ from the fifteenth digit on due to the finite precision of the floating point arithmetic used. Even a reordering of term in the calculation of the intermediate expressions can have this effect.

v [m/s]	λ_{capsize} [1/s]	$\lambda_{\text{castering}}$ [1/s]
0	-3.321 334 354 955 67	-5.695 461 613 073 60
1	-3.339 571 399 042 72	-6.577 674 865 894 17
2	-3.122 857 194 829 05	-7.341 157 952 916 98
3	-2.196 003 785 406 69	-8.255 359 188 427 08
4	-0.787 290 747 535 25	-9.378 471 064 036 38
5	-0.161 936 233 356 19	-10.665 540 857 474 20
6	0.039 380 255 445 46	-12.064 228 204 659 15
7	0.114 168 685 341 41	-13.538 013 346 083 71
8	0.143 031 193 913 90	-15.063 567 519 538 39
9	0.152 632 341 109 21	-16.625 925 337 159 89
10	0.153 494 106 064 82	-18.215 225 670 903 33

Table 6. Real eigenvalues λ from the linearized stability analysis for the capsize motion and the castering motion for the simplified benchmark bicycle from §5 in the forward speed range of $0 \leq v \leq 10$ m/s.