# Small Slope Implies Low Speed for McGeer's Passive Walking Machines


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**Abstract**

We consider passive dynamic walking machines of the kind originally studied by McGeer. For passive walking on arbitrarily small slopes, we show that any existing gaits must be correspondingly slow. The argument is first presented for nonsingular mass distributions, where it is shown that small slopes preclude long steps and that small steps imply low speeds. The argument is then extended to singular walkers (viewed as physically meaningful limiting cases of nonsingular walkers). A design for a different passive machine that might walk on flat ground is discussed briefly. The discussion in this paper lends insight into biped walking theory and may help to inspire designs for efficient bipedal robots.


## 1 Introduction

Passive machines that walk on shallow slopes were first studied by McGeer (1990a, 1990b), inspired by the pendulum models of a leg proposed by Mochon and McMahon (1980). These machines consist of hinged rigid bodies that make collisional and rolling contact with a rigid sloped surface. Passive bipedal walking machines can be *straight-legged* (with no knees) or *kneed* (with hinged knees).

In this paper, we will prove that for McGeer-like two-dimensional passive walking machines, if steady walking motions exist at vanishingly small slopes, then they must be at vanishingly small speeds.

It might at first *seem* intuitively obvious that passive walking machines with intermittent plastic collisions must dissipate nonzero energy while moving at nonzero speeds. Thus, finite nonzero speeds while walking down arbitrarily small slopes would intuitively seem impossible.

However, it is not *a priori* obvious that (say) a kneed walker could not possibly have special passive gaits with gentle, non-dissipative foot placements. Controlled walkers can certainly have gaits with such gentle foot placements (Blajer and Schiehlen (1992)). As argued by Winter (1987), gentle foot placements during walking may also be a human strategy to minimize energy loss and body stress at heelstrike. It is known that running machines with massless springs on the feet can have dissipation-free running motions (McGeer (1992)). A system of two blocks connected vertically by a spring, where the lower block has perfectly plastic collisions with the ground, can have special motions where it hops up and down indefinitely without losing any energy through collisions (Reddy and Pratap (1999)). There are models of brachiation (Bertram et al. (1999)) that allow dissipation-free support transfer under reasonable idealizations.

So it seems worthwhile to prove that for McGeer-like machines, small slopes do imply slow speeds. This result is of interest in the study of walking dynamics because it guides designs of highly efficient walking machines that walk at very small slopes, and also provides basic limitations on the efficiency of such walking machines, as discussed in more detail in a companion paper (Garcia et al. (1999)).


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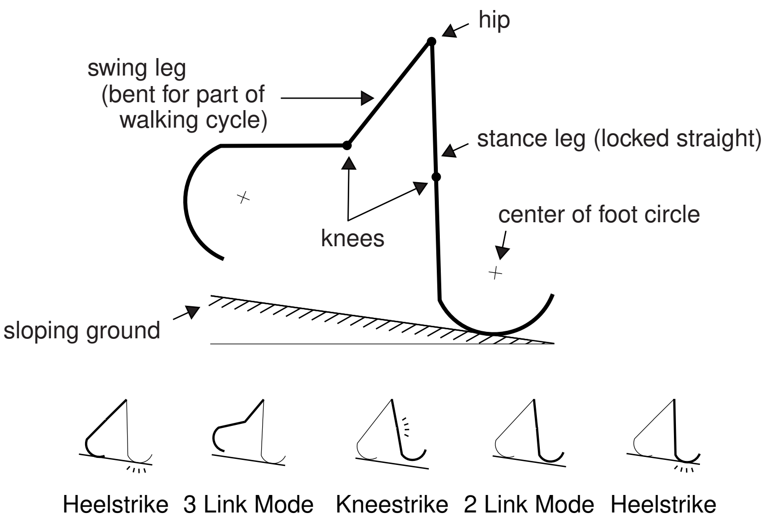


Figure 1: McGeer's kneed walking machine.

## 2 Description of Passive Dynamic Walking

Readers familiar with McGeer's walking machines (e.g., McGeer (1990a, 1990b) and Garcia et al. (1999)) may skip this section.

A kneed walking machine is shown schematically in Figure 1 (top). It consists of a *swing* leg (not touching the ground) and a *stance* leg (touching the ground), connected by a frictionless hinge at the *hip*. Each leg has a rigid thigh and shank. The stance knee is locked. The swing knee is a frictionless hinge with a knee-stop preventing hyperextension. The two legs have identical dimensions and mass distributions.

A steady walking motion or *gait cycle* is now described (for details see Garcia et al. (1999)).

At the start of the cycle, just after *heelstrike*, both legs are straight and touch the ground. Then the trailing leg, or swing leg (Figure 1, bottom), lifts off the ground and swings forward, and the swing knee flexes (*3 link mode*). The leading leg, or stance leg, maintains contact with the ground, remains straight, and rolls forward. *Kneestrike* occurs when the swing leg straightens out; the swing knee locks instantaneously, and the swing leg remains straight thereafter (*2 link mode*). At the end of the cycle, the swing foot has a collision with the ground at *heelstrike* (instantaneous and plastic). At the instant of double-support (i.e., contact at both feet), a collisional impulse acts at the new contact point on the new stance foot, and the legs exchange stance and swing roles. Then a new walking cycle begins.

Straight-legged walkers have no knees and so their walking cycle is the same as above except that the swing leg never flexes and there is no kneestrike. Dynamically-unimportant physical measures must be taken to allow for swing-foot clearance in these cases (see Garcia et al. (1999) or McGeer (1990a)).

## 3 Assumptions

Since there are passive devices with nonsingular mass distributions and intermittent plastic collisions that can nevertheless have special energy-conserving periodic motions (e.g., the two-block system mentioned above), we need to restrict the class of systems we will consider.

**Assumption 1** *The machines under consideration follow the description of walking given above.*

Our discussion does not apply to machines with other types of motions with intermittent contact, such as (say) a machine with springs on the feet that might allow dissipation-free support transfer; or a machine with two legs *and* a trunk (extra degree of freedom); or one that has springs at the knees that let it reach heelstrike at a flexed-knee configuration without falling down; or one which rises into the air for a portion of the gait cycle, only to land gently some time later; or one with three or more legs that allow gentle, dissipation-free transfer of support between legs, and so on.

**Assumption 2** *Each link in the machine has nonzero mass and nonzero central moment of inertia, i.e., nonsingular mass distribution.*

**Assumption 3** *The foot radius $R$ of the walking machines is strictly smaller than the leg length $l$, where $l$ is the distance from the hip to the farthest point on the foot (for kneed walkers, this distance is measured when the leg is straightened).*

McGeer (1989) also studied "silly wheels" with $R = l$, for which non-dissipative walking motions are possible on perfectly flat ground.

Note that for $R = l$, the system is geometrically singular in that the ground-contact points for the two feet coincide, in the instant of double support (straightened legs, both touching ground), no matter what the angle between the legs. Moreover, during the entire walking cycle, the swing foot slides along but never lifts off the ground.

If $R > l$, then the contact point of the trailing foot is ahead of the hip while the contact point of the leading foot is behind the hip (we disallow this case as well).

**Remark 1** *If the walker has point-feet the following discussion still applies, with $R = 0$.*

**Assumption 4** *The walking machine has no mechanism for energy dissipation except through the kneestrike and heelstrike collisions.*

Assumption 4 is adopted largely for convenience, since it seems unlikely that walking machines with secondary dissipation mechanisms in addition to collisions can walk fast on small slopes if such walking is not possible *without* these secondary dissipation mechanisms. However, attempting to prove this rigorously would be long and tedious at best, since there are many different reasonable energy dissipation mechanisms (dry or viscous friction in the bearings, rolling resistance at the feet, air drag on the walker, ...).

Under Assumptions 1 through 4, we will prove in the next two sections that if passive walking motions exist on arbitrarily small slopes, then such motions must be correspondingly slow as well.

## 4 Small Slope Precludes Long Steps

We now show that steady passive walking on infinitesimally small slopes must necessarily occur with infinitesimally small steps, if it occurs at all. We say, in short, *small slope precludes long steps.*

**Proposition 1** *For a straight-legged walker, small slope precludes long steps.*

**Proof:** We argue by contradiction. Taking the limit of small slope, we assume that there is a walking cycle on flat ground with nonzero step length.

Any heelstrike collision with nonzero foot velocity must dissipate energy, precluding a walking motion. Thus, heelstrike must occur with zero foot velocity. We now examine a small portion of the whole solution, close to the instant of heelstrike.

With rolling contact at the old stance foot, the system has two degrees of freedom. As shown in Figure 2, we use the $x$ and $y$ coordinates of the point of impending contact on the foot, measured from the corresponding point on the ground, as generalized coordinates that describe the system. Note that for nonzero step length, the coordinate system is nonsingular.

Heelstrike occurs when $x = y = 0$. At that instant, we must have $\dot{x} = \dot{y} = 0$ also, by assumption of zero foot velocity.

Using coordinates $\mathbf{q} = \{x, y\}^T$, we write equations of motion for the walker that will be valid in some neighborhood of $\mathbf{q} = \mathbf{0}$ (temporarily ignoring the new contact with the ground). The system is conservative with time-invariant constraints. Its Lagrangian is therefore of the form

$$L = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} - V(\mathbf{q}),$$

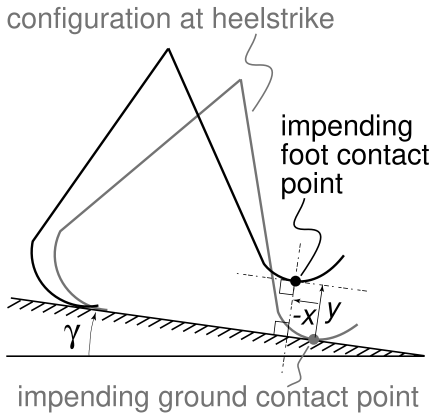


Figure 2: Close to heelstrike at nonzero step length, the $x$ and $y$ coordinates of any point on the foot relative to any reference point on the ground can be used as generalized coordinates to describe the configuration of the walker. Assuming a gait cycle exists, these two points are chosen to be the points of impending foot contact.

where $V(\mathbf{q})$ represents the potential energy of the system, and $\mathbf{A}(\mathbf{q})$ is a configuration-dependent, symmetric, positive definite matrix.

The equations of motion are

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q}) + v.d.t. = \mathbf{0}, \tag{1}$$

where $\mathbf{f}$ represents potential forces acting on the system, and $v.d.t.$ stands for "velocity dependent terms," which only include terms that are of degree two in $\dot{x}$ and $\dot{y}$ (i.e., quadratic terms).

Since the system is autonomous, we let the instant of heelstrike occur at $t = 0$. At $t = 0$, we have $\mathbf{q} = \mathbf{0}$ and $\dot{\mathbf{q}} = \mathbf{0}$. Assuming no new contact with the ground, Eq. 1 continues to hold. If we define $\mathbf{q}_1(t) = \mathbf{q}(-t)$, then $\mathbf{q}_1(t)$ satisfies the equations of motion, as well as the initial conditions. Therefore, as time moves forward, the solution is $\mathbf{q}_1(t) = \mathbf{q}(-t)$ and the old swing foot lifts off the ground.

Recall that a walking motion requires the old swing foot to stay in contact with the ground, while the old stance foot lifts off. By contradiction, a walking solution cannot exist (end of proof).

**Corollary 1** *For a kneed walker, small slope precludes long steps.*

**Proof:** For kneed walkers, at some time prior to heelstrike, the system switches to two-link mode. The walker will therefore retrace its path, by the previous time-symmetry argument, provided a locked-straight knee does not suddenly begin flexing. But the walker will not only retrace its path, but also retrace the history of constraint forces and moments at the knee, because at each position, the constraint forces depend on the configuration (the same), external forces from gravity (the same), quadratic velocity terms (negative-squared, and hence the same) and accelerations (the same). Therefore, positive knee-locking torques will stay positive, and unflexed knees will not suddenly flex (end of proof).

# 5 Small Steps on Small Slopes Imply Small Speeds

As shown above, for walking on a small slope $\gamma$, the step length must be also small. We now show that if the step length is small, the overall walking speed must be small also.

**Proposition 2** *A straight-legged walker, walking on an infinitesimal slope with infinitesimal steps, cannot walk at a finite (nonzero) speed.*

**Proof:** Let the slope be $0 < \gamma \ll 1$, and the step length be $0 < \varepsilon \ll 1$. By Proposition 1 we know that $\varepsilon \to 0$ as $\gamma \to 0$.

We argue by contradiction. If the average forward walking speed is $\mathcal{O}(1)$, then the time period of each step must be $\mathcal{O}(\varepsilon)$.

The gravitational potential energy dissipated per walking step is *weight* $\times$ *step length* $\times$ *slope* which is $\mathcal{O}(\varepsilon\gamma) \ll 1$.

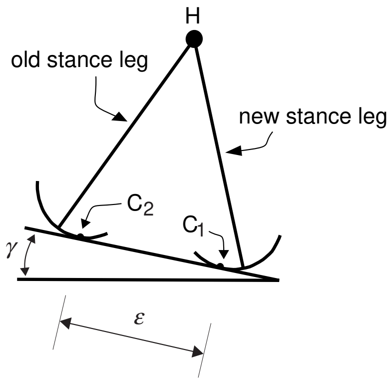


Figure 3: Impending heelstrike. The step length $\varepsilon$ and slope $\gamma$ are exaggerated; knees, if present, are locked straight and not shown. The old contact point is $C_2$, and the new contact point (where an impulse acts) is $C_1$.

See Figure 3. The new contact point at heelstrike is labeled $\mathrm{C}_1$, and the old contact point on the old stance leg is labeled $\mathrm{C}_2$. The slope $\gamma$ and the step length $\varepsilon$ are shown exaggerated, but understood to be infinitesimal.

Since the energy dissipated in foot collisions cannot exceed the infinitesimal gravitational potential energy available, the pre-heelstrike velocity of $\mathrm{C}_1$ must be $o(1)$ or infinitesimal. Consequently, the heelstrike-induced velocity changes of all points on the walker must be infinitesimal as well.

The pre-heelstrike velocity of $\mathrm{C}_2$ is zero, so its post-heelstrike velocity is infinitesimal. However, its average forward speed, by assumption, is $\mathcal{O}(1)$. So at some instant during the walking cycle, its velocity must be $\mathcal{O}(1)$. Thus, a velocity change of $\mathcal{O}(1)$ needs to occur over an $\mathcal{O}(\varepsilon)$ time interval, requiring an unbounded acceleration during the smooth phase of the motion (i.e., without collisions). This cannot occur for a finite-dimensional passive system with finite nonzero inertia moving at bounded speeds, giving a contradiction (end of proof).

Readers familiar with the *rimless wheel* of McGeer (1990a) and Coleman et al. (1997) will observe that in that case, a wheel with many spokes can move at $\mathcal{O}(1)$ speeds on a small slope because, although the time interval between successive spoke collisions is small, the time interval between lifting of one spoke off the ground and the next collision of that *same* spoke is not small, but $\mathcal{O}(1)$.

**Proposition 3** *A kneed walker, walking on an infinitesimal slope with infinitesimal steps, cannot walk at a finite (nonzero) speed.*

**Proof:** The proof for the kneed walker case also rests on showing that finite (nonzero) speeds require unbounded accelerations. The one difference here is that for kneed walkers there are, in fact, unbounded accelerations during *kneestrike*. So, in addition to the arguments for the preceding case, we need to also check that kneestrike-induced velocity changes are infinitesimal.

As before, assume that the walker walks at an $\mathcal{O}(1)$ speed.

See again Figure 3, now taken to depict a kneed walker in the straight-leg configuration at heelstrike. The points $\mathrm{C}_1$ and $\mathrm{C}_2$ are defined as before.

As argued for Proposition 2, due to the small potential energy budget, the velocity of the heelstrike-contact point $\mathrm{C}_1$ at the instant prior to heelstrike must be infinitesimal; and heelstrike-induced instantaneous velocity changes of all points on the walker must be infinitesimal as well.

Now consider the old stance leg. The pre-heelstrike angular velocities of the stance thigh and stance shank are identical. Due to heelstrike, the difference induced between them is infinitesimal. Since accelerations are bounded and the step period is infinitesimal, the *change* in this difference before kneestrike is also infinitesimal. Therefore the kneestrike collision impulses are infinitesimal; and the velocity changes caused by kneestrike are infinitesimal as well.

Now we can use the arguments used for Proposition 2. The post- heelstrike velocity of $C_2$ is infinitesimal; kneestrike induced changes in it are infinitesimal; the overall time period is infinitesimal; so its average forward speed cannot be $\mathcal{O}(1)$ without unbounded accelerations (end of proof).

We have thus shown that for McGeer-like walking machines with nonsingular mass distributions, small slopes imply small speeds. We now turn our attention to singular mass distributions.

# 6 Singular Walkers

## 6.1 Motivation

In Garcia et al. (1998), a straight-legged machine was considered with legs whose mass distribution was taken to be equivalent to a finite, nonzero mass at the hip, and an infinitesimal point mass at the foot. Thus, the radius of gyration of each leg was infinitesimal when compared to the leg length, i.e., the walker was singular in some sense. That walker violates Assumption 2 of this paper. The study of that walker was, nevertheless, useful for two reasons. First, it substantially simplified the dynamics. Second, it predicted scaling rules for walking which are apparently obeyed by even nonsingular walkers over intermediate slopes (see Garcia et al. (1999)).

Although singular walkers are mathematical ideas impossible to realize in practice, studying their motions on small slopes is useful for a third reason. We would like to understand the fundamental mechanical limitations on the efficiency of walking machines. A reasonable way to study the efficiency of horizontal locomotion is to quantify *in*efficiency using one of several measures which are all roughly equivalent to the smallness of the minimum slope required to sustain the motion (see Garcia et al. (1999)). As such, qualitative features of walking motions on small slopes are of interest. For nonsingular McGeer-like machines, we have shown that small slopes imply small speeds. Now at a sufficiently small but fixed slope, is it possible to make the mass distribution "more and more singular" to obtain faster and faster walking? If the answer is yes, then one key to designing highly-efficient walking machines is to simply try and design the walkers to be as close to singular as possible. On the other hand, if the answer is no, then we conclude that such a route to high efficiency does not exist; and the design of highly efficient machines must be based on other ideas.

With this motivation, we now address the question of whether small slopes imply small speeds even for singular walkers. We first need to identify a useful class of singular walkers. This process is unfortunately somewhat long; we do not know of a way to shorten the argument. Once the class of singular walkers is identified, the rest of the proof is straightforward, and similar in spirit to the proof for nonsingular walkers.

## 6.2 Physically Meaningful Singular Walkers

We will abandon Assumption 2, and introduce in its place some other assumptions.

In studying a singular walker, we define its dynamics as a physically meaningful limiting case of some nonsingular, dynamically determinate, behavior. The singular limit should not involve wild behavior like infinite accelerations during non-collisional motions; or infinite velocity changes at some points on the walker due to collision-induced finite changes at other locations. We refer to this subclass of singular walkers as being *physically meaningful*. For instance, the "simplest walker" of Garcia et al. (1998) is the "minimal biped" of Alexander (1995) made physically meaningful by the addition of infinitesimal point masses at the feet.

**Example 1** *Some possible leg mass distributions are shown in Figure 4. (a) shows a straight-legged walker, not physically meaningful because the moment of inertia about the center of mass is exactly zero, and so the equations of motion are indeterminate. (b), (c) and (d) depict some limiting cases as $\delta \to 0$; of these three, only (c) is physically meaningful. For (b) and (d), the frequencies of oscillation (and hence, accelerations) when suspended from the hip are not bounded as $\delta \to 0$.*

We write down the foregoing ideas as explicit assumptions.

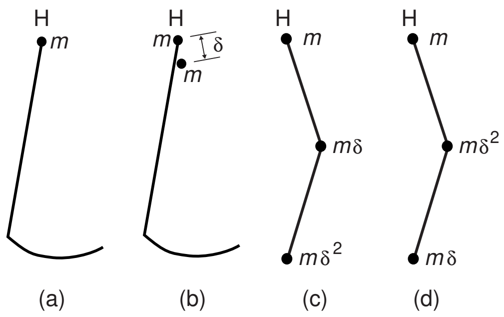


Figure 4: Some examples of mass distributions: (a) and (b) straight-legged walkers, (c) and (d) kneed walkers with point feet. The hip is marked H in each case. The straight lines represent massless links; the heavy dots represent point masses.

**Assumption 5** *Singular walkers are limiting cases of dynamically deterministic, nonsingular walkers.*

**Assumption 6** *For a singular walker with bounded initial conditions on velocities, under smooth (non-collisional motions), the accelerations remain finite.*

**Assumption 7** *Let $\Delta\mathbf{v}$ be an infinitesimal, instantaneous change in the velocity of an arbitrary point* C *on a singular walker, caused by an impulse at* C. *Then the resulting velocity changes at all other points on the walker are infinitesimal as well.*

Note that Assumption 7 is not quite independent of Assumption 6. For example, let Assumption 7 be violated, so that we can impart an infinitesimal change in the velocity of some point of the walker to obtain a finite (nonzero) change in the velocity of some other point. Then, passing to the limit, imparting no velocity change to the first point still causes a sudden, nonzero change in the velocity of the second point. This can only occur if the second point can have an infinite acceleration even in the absence of collisions, showing that violation of Assumption 7 implies violation of Assumption 6 as well. However, to avoid the complications of rigorously examining the various distinguished limits implicit in the previous informal argument, we explicitly adopt both Assumptions 6 and 7. In the same spirit, we adopt another assumption about the collisions of the walker.

**Assumption 8** *Let a kneed walker's swing thigh and swing shank have infinitesimally different angular velocities at the instant preceding kneestrike. Then kneestrike-induced velocity changes at all points on the walker are infinitesimal as well.*

Finally, we adopt an assumption that seems reasonable, and is further motivated later in the paper (Remark 6) after some preliminaries have been established.

**Assumption 9** *Let a kneed walker be placed upright on flat ground, with both legs straight, with feet touching the ground with any finite step length, and at rest. From these initial conditions, no walking solution exists. The walker either keeps standing, or falls down.*

Under Assumptions 1 through 9 but not including Assumption 2, we now set out to prove that small slopes imply small speeds. In the rest of this section, we use some general results of two-dimensional rigid body dynamics to identify the class of singular walkers allowed by our assumptions.

## 6.3 The Contact Inertia Matrix M For Collision Calculations

Consider a linkage consisting of several rigid objects connected by ideal, frictionless hinges (Figure 5). The point C on the linkage collides with a rigid, immovable floor. A collision impulse vector $\mathbf{P}$ acts at C. Let the pre-collision velocity vector of C be $\mathbf{v}_i$, and the post-collision velocity be $\mathbf{v}_f$.

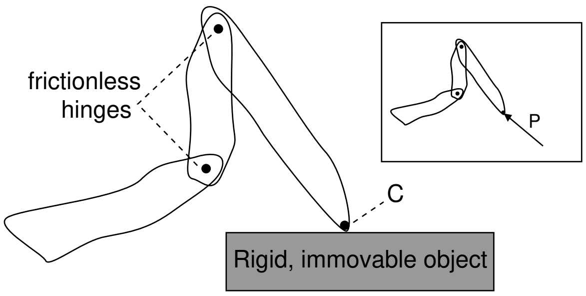


Figure 5: A free-floating linkage collides with a rigid immovable object (floor).

Using standard rigid body dynamics (e.g., Smith (1991)), it can be shown that the *change* in the velocity of C, i.e., the vector $\mathbf{v}_f - \mathbf{v}_i$, is linearly related to the impulse vector $\mathbf{P}$ by a symmetric, positive semidefinite tensor $\mathbf{M}$, i.e., the collisional interaction can be described by an equation of the form

$$\mathbf{P} = \mathbf{M} \cdot (\mathbf{v}_f - \mathbf{v}_i)\,. \tag{2}$$

The case of singular $\mathbf{M}$ is viewed as a limiting case of nonsingular $\mathbf{M}$, as some inertial properties of some components in the linkage are treated as "infinitesimal". $\mathbf{M}$ depends on the masses of the links, their moments of inertia, positions of pivot points, and the instantaneous configuration of the linkage. Assuming that a coordinate system (along normal and tangential directions at the contact surface) has been chosen and that we are considering two-dimensional systems, the tensor $\mathbf{M}$ is represented by a $2 \times 2$ matrix of components, which (with some abuse of notation) we also call $\mathbf{M}$, the *contact inertia matrix.*

The kinetic energy loss in a perfectly plastic (no-slip, no-bounce) collision, with $\mathbf{v}_f = 0$, is given by

$$\text{Energy loss } = \frac{1}{2}\mathbf{v}_i \cdot \mathbf{M} \cdot \mathbf{v}_i\,. \tag{3}$$

Also, for such a plastic collision, we have the impulse

$$\mathbf{P} = -\mathbf{M} \cdot \mathbf{v}_i\,. \tag{4}$$

**Definition 1** *For our purposes, the mass distribution of a walking machine at the heelstrike configuration is* singular *if* $\mathbf{M}$ *for the foot collision calculation is singular. Else, the mass distribution is nonsingular.*

## 6.4 Point Mass Representations of an Arbitrary Rigid Body

**Remark 2** *The next two propositions are trivially known facts from rigid body mechanics. We include them as numbered "propositions" merely to facilitate later reference to them.*

**Proposition 4** *In two dimensions with a spatially uniform body force field (such as constant gravity), any single rigid body with total mass $M$, center of mass a distance $r > 0$ from an arbitrarily chosen reference point* R*, and moment of inertia $J > 0$ about its center of mass, is dynamically equivalent to a pair of rigidly-connected point masses, one of which, say $m$, is at* R *and the other $(M - m)$ is a distance $s$ from* R*, where $m$ and $s$ are to be chosen appropriately.*

**Proof:** Observe that the total mass is the same, $m + (M - m) = M$; center of mass position and the moment of inertia will also be the same if $m$ and $s$ satisfy the simultaneous equations:

$$Mr = (M - m)s, \qquad \text{and} \qquad J + Mr^2 = (M - m)s^2, \tag{5}$$

where point R, the center of mass, and the mass $M - m$ are collinear. Equation 5 may be solved to obtain

$$m = \frac{J}{J + Mr^2}\,M, \qquad \text{and} \qquad s = \frac{Mr^2 + J}{Mr^2}\,r\,, \tag{6}$$

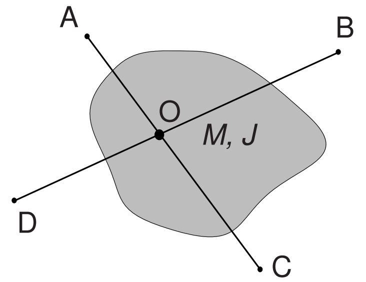


Figure 6: Two possible locations for two-point-mass representations of a single planar rigid body.

which concludes the proof.

To emphasize the arbitrariness with which the locations of the point masses may be chosen, consider Figure 6. A rigid body, with total mass $M$ and moment of inertia $J$ about center of mass O, may be represented using two appropriate point masses at points A and C, where the masses at A and C as well as the distance OC are determined by $M$, $J$ and the location of point A. The same body may, if desired, also be represented using two other point masses, at (say) B and D, with the masses at B and D as well as the distance OD being determined by $M$, $J$ and the location of point B.

Thus for 2-D straight-leg walking machines, treating each leg as having two point masses, one (say) at the hip and one at some other, arbitrary point, involves no loss of generality in that it covers all except the special case where the center of mass of the leg is exactly at the hip. In such cases, we can use a three point mass description as follows.

**Proposition 5** *In two dimensions with a spatially uniform body force field (such as constant gravity), any single rigid body with total mass $M$, center of mass at a point* R*, and moment of inertia $J > 0$ about its center of mass, is dynamically equivalent to three rigidly-connected collinear point masses, one of which, say $M - 2m$, is at* R *and the other two ($m$ each) are placed symmetrically about* R *at a distance $s$, where $s$ can be any distance greater than or equal to the radius of gyration $k = \sqrt{J/M}$, and $m$ depends on the choice of $s$.*

**Proof:** By inspection, the total mass as well as the center of mass are the same in both cases. Choosing

$$m = \frac{Mk^2}{2s^2}$$

ensures both that the moment of inertia is the same, and that $M - 2m \geq 0$ (end of proof).

**Remark 3** *If $J$ in Proposition 5 is non-infinitesimal (i.e., $k$ is non- infinitesimal), then we can pick $s$ to be $k$. In that case, $m = M/2$ and no point mass is needed at* R.

**Remark 4** *If $J$ in Proposition 5 is infinitesimal (i.e., $k$ is infinitesimal), then we can pick $s$ (for walking machines, say) to be the leg length. In that case, $m$ is infinitesimal as well, and essentially all the mass is at* R.

## 6.5 Test For Singular M

**M** linearly relates any force **F** at the contact point C (Figure 5) to the resulting acceleration of point C, $\mathbf{a}_\mathrm{C}$, provided all bounded external forces and torques, all internal forces and torques except those arising from workless constraints, as well as centripetal terms (or "$\omega^2$ terms") are neglected:

$$\mathbf{F} = \mathbf{M} \cdot \mathbf{a}_\mathrm{C}\,. \tag{7}$$

In other words, if we imagine the system starts from rest (no $\omega^2$ terms), ignore springs (if any) at joints, and ignore all external forces except **F**, then **M** linearly relates the instantaneous acceleration of point C to the force **F**.

**Proposition 6** *Consider a linkage (e.g., Figure 5) at rest. Let the mass distribution of each link be given by two or more rigidly connected point masses (possibly infinitesimal) at distinct locations. Then* $\mathbf{M}$ *is singular if and only if it is possible to move the point C without moving any of the* non-infinitesimal *point masses.*

**Proof:** If the point C can move without setting any nonzero mass in motion, then C can be accelerated with zero force, since the only resistance to motion is inertial (no friction, springs, gravity, etc.). By Eq. 7, $\mathbf{M}$ must be singular. Conversely, if moving C means moving some mass, then the force needed is nonzero, so $\mathbf{M}$ is nonsingular (end of proof).

**Corollary 2** *If, in Figure 5, the last link (i.e., the link with the point* C*) has a mass distribution equivalent to two nonzero point masses separated by a nonzero distance, then* $\mathbf{M}$ *is nonsingular.*

**Proof:** Point C cannot be moved without moving at least one of the two nonzero point masses (end of proof).

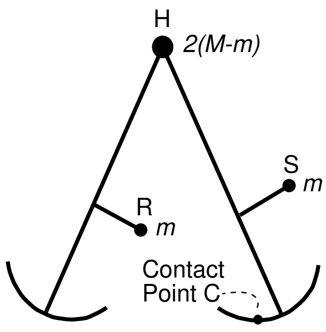


Figure 7: A straight-legged walker with the mass distribution of each leg expressed as an equivalent pair of point masses, with one point mass at the hip. The total mass of each leg is $M$.

**Example 2** *Consider Figure 7. If* $m = 0$*, then the point* C *can be moved by rotating the leg about* H*, and* $\mathbf{M}$ *is singular. If* R *and* S *do not coincide and* $m > 0$*, then* $\mathbf{M}$ *is nonsingular.*

**Corollary 3** *Consider one straightened leg of a walking machine. Let the center of mass of the leg be exactly at the hip. If the central moment of inertia* $J$ *of the leg as a single rigid body is non-infinitesimal, then* $\mathbf{M}$ *for the corresponding walker is nonsingular; while if* $J$ *is infinitesimal, then* $\mathbf{M}$ *is singular.*

**Proof:** If $J$ is non-infinitesimal, see Remark 3 and Corollary 2. If $J$ is infinitesimal, see Remark 4 and Proposition 6 (the leg can be rotated about the hip, without moving any non-infinitesimal mass). Note that these arguments apply to both straight-legged and kneed walkers at the heelstrike configuration, because for kneed walkers the new stance leg remains locked and behaves like a single rigid body (end of proof).

## 6.6 Singular Mass Distributions For Walkers

**Proposition 7** *A straight-legged walker with nonzero step length has singular mass distribution if and only if all its mass is concentrated at the hip* H.

**Proof:** If the center of mass of the leg is exactly at the hip, see Corollary 3 and Remark 4. Otherwise, express the mass distribution of each leg as a point mass at the hip and another point mass at some other location. See Example 2. For nonzero step length, points R and S do not coincide (end of proof).

**Remark 5** *A point mass on one two-dimensional link, located exactly on a hinge, can be "transferred" to another link hinged about the same point, with no change in the dynamics.*

(The contributions of the point mass to the system's Lagrangian are independent of which link it is attributed to.)

**Proposition 8** *A kneed walker with nonzero step length has singular mass distribution if and only if all its mass is concentrated at the hip* H *when the legs are straightened.*

**Proof:** Note that the new stance leg remains straight before *and* after heelstrike, so we treat it as a single rigid body.

If the center of mass of the leg is exactly at the hip, see Corollary 3 and Remark 4.

Otherwise, examine the instantaneous mass distribution for the straight-leg configuration just prior to heelstrike (see Figure 7). There are now 3 cases to consider.

*Case 1 (all mass at hip):* If $m$ is infinitesimal, then C can be moved without moving H, and so the walker is singular.

*Case 2 (some, but not all, mass at hip):* If both $m$ and $M - m$ are nonzero then, since the new stance leg is treated as rigid, the walker is nonsingular by Corollary 2.

*Case 3 (no mass at hip):* If $M - m$ is infinitesimal, it remains to prove that the walker is nonsingular. We examine this case below.

For straight-legged walkers, case 3 is nonsingular. For kneed walkers, we must account for the flexing of the old stance knee (the new swing knee) as a result of the collision.

Figure 8 shows just the trailing leg of the walker, along with the position of the knee, K. To emphasize that the knee need not lie on the line connecting the hip with the center of the foot's circle, the "actual" geometry is shown with heavy black lines, while the underlying gray shows the sketch from Figure 7.

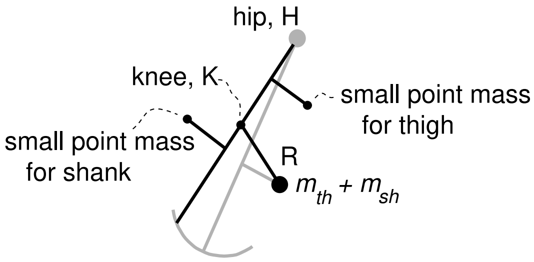


Figure 8: Trailing leg of a kneed walker with negligible equivalent hip-mass when in the straight-leg configuration. The figure is drawn to show the position of the knee.

We express the mass distribution of the shank using two equivalent point masses, but this time with one point mass $m_{sh}$ at point R of Figure 7, and another point mass at some other suitable point (which need not be the hip). Similarly, we express the mass distribution of the thigh using two equivalent point masses, with a point mass $m_{th}$ at point R and another point mass at some other suitable point.

By assumption (for case 3) the moment of inertia about point R of the leg, in the straight-leg configuration, is infinitesimal. Thus the "other" point masses, mentioned above, are infinitesimal. The exact positions of these two new point masses for the shank and thigh are irrelevant, and not computed here.

We now proceed with the examination of case 3, which has three subcases.

*Subcase 3.1:* If the thigh-mass $m_{th}$ is finite (nonzero), then the shank mass is irrelevant. The geometry of Example 2 shows that $\mathbf{M}$ is nonsingular.

*Subcase 3.2:* If the thigh-mass $m_{th}$ is zero or infinitesimal, but the point R coincides with the knee K, then by Remark 5 we attribute the shank-mass $m_{sh}$ to the thigh, and obtain subcase 3.1. Thus, $\mathbf{M}$ is nonsingular again.

*Subcase 3.3:* If the thigh mass $m_{th}$ is zero or infinitesimal, and the point R does *not* coincide with the knee K, then point K can have infinite accelerations. This case violates our assumptions about being physically meaningful, and is excluded.

Thus, we have shown that if the mass in the straight leg configuration is concentrated at the hip, then the mass distribution is singular; and if the mass in the straight-leg configuration is *not* all concentrated at the hip, then the mass distribution is nonsingular. This concludes the proof of Proposition 8.

**Corollary 4** *For a walker at heelstrike with nonzero step length,* rank($\mathbf{M}$)$\geq 1$.

**Proof:** If $\mathbf{M}$ is singular, there is nonzero mass at the hip. In accelerating C towards the hip, nonzero force is needed. By Eq. 7, $\mathbf{M}$ is not the zero matrix (end of proof).

**Corollary 5** *For a singular walker at heelstrike with nonzero step length, the eigenvector of* $\mathbf{M}$ *corresponding to the nonzero eigenvalue is directed from heelstrike-contact point* C *towards hip* H*, i.e., line* CH*; and the eigenvector corresponding to the zero eigenvalue is perpendicular to line* CH*.*

**Proof:** All the mass of the old swing leg is at the hip. Accelerating C in a direction perpendicular to CH requires no force, showing that it is an eigenvector direction corresponding to a zero eigenvalue. $\mathbf{M}$ is symmetric and has rank one, so the zero eigenvalue has multiplicity one, and the zero-eigenvalue eigenvector is unique. Since $\mathbf{M}$ is symmetric, the eigenvectors are orthogonal and so the other eigenvector is along CH (end of proof).

# 7 Small Slope Implies Low Speed for Singular Walkers

Using the results of the previous section, we now discuss singular walkers walking on small slopes.

## 7.1 Small Slope Precludes Long Steps

**Proposition 9** *For a singular walker (straight-legged or kneed), small slope precludes long steps.*

**Proof:** As before, to obtain a contradiction, we assume that as the slope goes to zero, a walking motion exists whose step length does not go to zero. We then examine the limiting case of zero slope. There are two cases to consider.

*Case 1:* Let the velocity of the hip H at the instant preceding heelstrike be nonzero. Then, since the step length is nonzero, this velocity has a component along HC, i.e., towards the heelstrike contact point. This means that the contact point has a pre-collision velocity with a nonzero component along the nonsingular eigenvector, and some energy must be dissipated by Eq. 3 (a contradiction).

*Case 2:* Let the velocity of the hip H at the instant preceding heelstrike be exactly zero. Then it is clear that the foot collision involves no impulse and no energy dissipation. However, in this case the walker comes to a dead stop as a result of heelstrike, and either remains stationary thereafter, or falls down (by Assumption 9; see also Remark 6 below). Thus, walking does not continue (a contradiction).

**Remark 6** *A kneed walker will stand on flat ground in "double support" equilibrium on two straightened legs only if there is no flexing torque at either knee. Else, a knee will buckle. For a singular kneed walker standing in this way, the ground contact forces must be directed straight to the hip since the legs have no weight anywhere else. A simple sketch then shows that if the front knee does not buckle then the rear knee does not buckle either. So, either no knee buckles; or the front knee buckles, in which case it seems reasonable that the walker should fall down.*

Remark 6, though not a theorem, provides some *a posteriori* justification for Assumption 9.

## 7.2 Small Steps Imply Low Speeds

The following arguments are essentially identical to the ones used for nonsingular walkers, so they are kept brief.

**Proposition 10** *A straight-legged singular walker, walking on an infinitesimal slope with infinitesimal steps, cannot walk at a finite (nonzero) speed.*

**Proof:** Arguing by contradiction, we assume that a walking motion exists with an average $\mathcal{O}(1)$ forward velocity, on an infinitesimal slope with infinitesimal step length.

At small slope with small steps, the energy dissipated per step must (as for nonsingular walkers) be infinitesimal. This implies that the impulse magnitude is infinitesimal. It follows that the change in the velocity of the hip (where the mass is concentrated) is infinitesimal. The velocity changes experienced by the trailing leg (old stance leg) are caused by the instantaneous velocity change of the hip. Thus, velocity changes on the trailing leg are infinitesimal. It follows that just after heelstrike, the old contact point on the old stance leg has an infinitesimal velocity. As is the case for nonsingular walkers, to maintain an $\mathcal{O}(1)$ average forward velocity, unbounded accelerations are required during the swing phase, leading to a contradiction (end of proof).

**Proposition 11** *A kneed singular walker, walking on an infinitesimal slope with infinitesimal steps, cannot walk at a finite (nonzero) speed.*

**Proof:** Again, arguing by contradiction, we assume that a walking motion exists with an average $\mathcal{O}(1)$ forward velocity, on an infinitesimal slope with infinitesimal step length.

Given the infinitesimal potential energy budget, the heelstrike impulse must be infinitesimal. The resulting change in the hip velocity (where at least the mass of the new stance leg, which remains straight, is concentrated) must be infinitesimal. The consequent velocity changes on the new swing leg (i.e., the old stance leg) must be infinitesimal. The relative angular velocity induced between the new swing thigh and shank must be infinitesimal. The change in this relative velocity over the infinitesimal time interval till kneestrike must be infinitesimal. Therefore kneestrike occurs with an infinitesimal relative velocity. Finally, the velocity changes caused by the kneestrike collision are therefore all infinitesimal as well.

Now we have the situation of Proposition 3. The post- heelstrike velocity of the old contact point (i.e., on the old stance leg) is infinitesimal; kneestrike induced changes in it are infinitesimal; the overall time period is infinitesimal; so its average forward speed cannot be $\mathcal{O}(1)$ without unbounded accelerations (end of proof).

# 8 Discussion

We have shown that, under reasonable assumptions, passive walking machines that follow McGeer's basic design cannot walk on infinitesimal slopes at other than infinitesimal speeds. We have also shown that somehow making the walkers closer to singular will not lead to dramatically more efficient walking (i.e. the conclusions are the same for physically-meaningful singular walkers as they are for nonsingular walkers).

Of course, a given walker may not have walking motions on arbitrarily small slopes in the first place. The main utility of this paper is that it guides the design of walking machines that *do* walk on near-zero slopes. This issue is discussed in detail in a companion paper (Garcia et al. (1999)), where scaling rules for small slope walking are obtained. Those rules shed light on the fundamental mechanical limits on the efficiency of McGeer-like walking machines.

Perhaps this paper might guide the design of other passive walkers that can, in fact, walk on flat ground at nonzero speed. Like the two-mass hopper (Reddy and Pratap (1999)), machines that do not follow McGeer's basic design might have special energy- conserving gaits at nonzero speeds on perfectly flat ground.

Let us briefly consider how a passive machine might walk without energy dissipation in spite of perfectly plastic foot collisions (ignoring other dissipative effects such as friction). For simplicity, we consider nonsingular and straight-legged walkers. Proposition 1 applies to all two-degree-of-freedom conservative time-invariant holonomic systems where the contacting point has two independent velocity components. In contrast, the two-mass hopper (Reddy and Pratap (1999)) does not fit the conditions of Proposition 1, because its contact point has one velocity component, while the system as a whole has two degrees of freedom. At the instant of gentle contact between the lower block and the ground, the upper block has a nonzero velocity. Thus, the forward-evolving solution of the system equations does *not* just retrace its path.

By Proposition 1, for a walker in two dimensions and with two straight legs, if a gentle foot collision is to occur without the foot lifting off the ground again, a necessary condition is that the system have a third degree of freedom with nonzero velocity at the instant of contact, in order to break the time-reversal symmetry in initial conditions. For walking machines, a reasonable choice is a spring-mounted trunk that

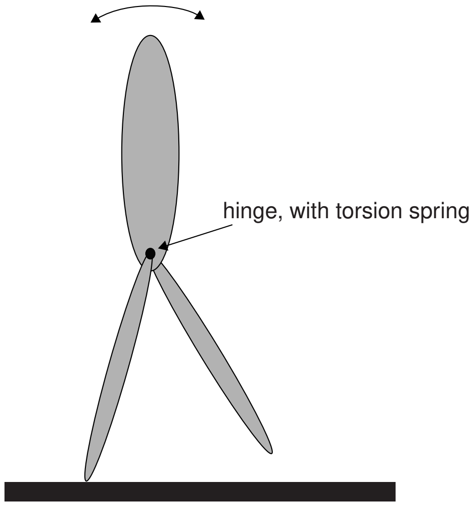


Figure 9: A walking machine that might walk passively on flat ground without collisional dissipation.

pivots around the hip, as shown schematically in Figure 9. The exact design, analysis and simulation of this system have not yet been carried out. We imagine that the hip will be stabilized in the near-vertical position using torsion springs; that there will be an extended period of double support (both feet touching the ground); and that at the instant of the gentle heelstrike, the trunk will necessarily have a nonzero angular velocity. As is the case for the two-mass hopper, we expect that linearized stability analysis of any walking motions will show neutral stability at best. Physically, we see that a passive machine on flat ground cannot gain energy under any circumstances. If a steady walking motion is slightly disturbed so that the net system energy decreases by a small amount, that lost energy cannot be recovered. Therefore, the system cannot return to the original walking motion (ruling out asymptotic stability). In actual construction of such a machine, an active controller will probably be used to stabilize the passive gait (such control requires little energy,in comparison to, say, actively controlled non-passive trajectories as in Blajer and Schiehlen (1992)).

Finally, for human walking, we speculate that one possible strategy to reduce impact losses and impulsive loads might be to use muscles and the trunk to approximately *mimic* a spring-mounted, third degree of freedom. As noted in a companion paper (Garcia et al. (1999)), the impact-dominated energy losses for McGeer's trunk-less walkers seem a little high when compared to human walking power consumption. Further theoretical study of machines of the type shown in Figure 9, along with comparisons with human data, are needed before this question can be answered.

# 9 Conclusions

For anthropomorphic walking machines without extra links, for which McGeer's machines provide a good model, collisional losses are inescapable and small slopes imply low speeds (equivalently, low power implies low speeds). The observation that small slopes imply small speeds leads to restrictions on the mass distributions on McGeer-like walkers that *do* walk on near zero slopes, as discussed in a companion paper (Garcia et al. (1999)).

For more anthropomorphic machines that can have additional degrees of freedom with significant energy storage (such as a trunk mounted on a spring), it may be possible to avoid collision losses through passive means. Thus, it may be possible to design passive-dynamics-based walking robots where the most significant energy losses are not in collisions, but in the hopefully smaller dissipation through various non-ideal elements like springs and hinges.

Together with a companion paper (Garcia et al. (1999)), the results and discussion in this paper provide some insights into mechanisms and strategies which biped robots and animals might use in order to lessen the effects of collisions and thus improve their walking efficiency.

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