Some Simple Non-Holonomic Mechanical Systems Are Too Symmetric to Have Exponential Stability

Also, Examples

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- A quick way to see that some systems cannot be stable.
- An explanation (history of science) for why people have been slow to see the utility of non-holonomic mechanics.
Mechanical Systems

- Particles and rigid bodies
- Only conservative forces
- No explicit $t$ dependence
- Non-holonomic or Holonomic Constraints imposed in a workless way.
- Smooth

Particles & Rigid bodies connected to the world and each other with hinges, frictionless sliding, rolling, skates, ...

\[ KE + PE = \text{Constant} \]

"Conservative" Systems
Geometric Constraints

$q_1, q_2, q_3 \ldots$ are configuration variables.
[eg.: $x, y, \theta, \phi$ etc. of particles or rigid bodies.]

Holonomic: "Integrable"

\[ F(q_1, q_2 \ldots) = 0 \quad \text{or} \quad f_1(q) \dot{q}_1 + f_2(q) \dot{q}_2 + \ldots = 0 \]

eg. \[ x^2 + y^2 - 7 = 0 \]
\[ N = M \]

e. g. \[ xy + yx = 0 \]

Non-Holonomic: "Non-Integrable"

\[ \cancel{F(q_1, q_2 \ldots) = 0} \]

\[ \cancel{\text{no such thing}} \]

\[ f_1(q) \dot{q}_1 + f_2(q) \dot{q}_2 + \ldots = 0 \]

eg. \[ x \cos \theta - y \sin \theta = 0 \]
\[ N < M \]

NO SUCH: \[ f(x, y, \theta) = 0 \]
Holonomic:
(N = m)

- hinge: 4 = 4
- particle on a surface: 2 = 2
- particle on a wire: 1 = 1

N = dim. of vel. space
M = dim of config. space

Non-Holonomic:
(N < m)

- sliding: rigid bodies in sliding contact: 5 = 5
- disk rolling on a plane:

\[ V_c = 0 \]
3 < 5

- skate on plane:

\[ x \sin \theta - y \cos \theta = 0 \]

- \[ \approx \] Wings, Sails, Keels

You can get to a space of configs bigger than the space of vels.

"parallel parking"
Exponential Stability

For all solutions near a reference solution $z_i$, say $z_\tau$, goes exponentially to zero while no $z_i$ or $\dot{z}_i$ goes exponentially to $\infty$.

Note: Assume reasonable variables.

"Thm": Hamiltonian systems cannot have exponential stability

\[
\dot{z} = \frac{\partial H}{\partial p} \\
\dot{p} = -\frac{\partial H}{\partial z} \\
\nabla \cdot \nabla = \frac{\partial z}{\partial p} + \frac{\partial p}{\partial z} = \frac{\partial^2 H}{\partial p \partial z} - \frac{\partial^2 H}{\partial z \partial p} = 0
\]

[To conserve Vol., if $z_i$'s die, something else has to blow up.]
pend. \[ \theta \to t \]

inv. pend. \[ \Theta \to t \]

\[ \phi \to \phi \]

not stable for this talk.

exp. decay \[ \gamma \to t \]

not stable \[ \phi \to t \]

particle sliding in a bowl

particle sliding in a trough

e tc
Recall: Ideal top not exponentially stable.
Often, but real top is stable.

**Non-holonomy does it?**

- Non-holonomic rolling contact

**Non-holonomy does it?**

Real rolling disk observed to be stable.

**Euler disk does it?**
Bike: 3D

4 linked rigid bodies

Conservative

\[ \dot{\psi} + \psi = 0 \]
\[ \dot{x} + x = 0 \]

For realizable \( A, B, \ldots, K \) there are solns. where \( \psi \) and \( x \) are exponentially stable!

\( \psi, x \) → \( t \)
Skateboard: 3D

(Monte Hubbard 1978)

Conservative

Rigid rider welded to skateboard

Wheels steer proportional to lean

Lean vs. t

For some parameters, exponential stability
Rattleback, Celt

Walker '95 (well, 1895 actually)

rigid body (1), rolling on plane

for some mass dist. & contact shapes & spin rates

some var.

$t$
SO......

1) All classical (Conservative & Hamiltonian) systems do not have exponential stability.

2) Some obscure, erratically known, conservative non-holonomic systems do have exponential stability.

[ bike, skateboard, celt, trike, trailers, ... ]

"...[nothing]..." - Niemark & FuFaev in NONHOLONOMIC MECHANICS

1) How much can non-holonomic contact explain stability?

2) Why doesn't everyone know about it?
Rolling Disk:

(e.g. Goldstein)

\[ x^2 = \frac{1}{2J+1} \]

\[ J = I/mR^2 \]

\[
(1+\lambda^2)\ddot{\psi} - 2\cos(\psi)\dot{\psi}\dot{\theta} + (1+\lambda^2)\sin\psi \dot{\phi}^2 - 2\lambda^2 \sin\psi = 0
\]

\[
\cos\psi \dddot{\phi} + 2\dot{\psi} \dot{\phi} = 0
\]

\[
\ddot{\theta} + \sin\psi \ddot{\phi} + (1+\lambda^2)\cos\psi \dot{\phi} \dot{\psi} = 0
\]

Linearize about, say,
\[ \phi = \text{const}, \ \psi = 0, \ \dot{\theta} = \text{big enough} \]

\[ \psi \]

\[ t \]

imaginary eigenvalues, \[ \Rightarrow \]

NO EXponential Stability!
Chaplygin sleigh: v1.0

Planar rigid body moving in plane.
Skate at C.O.M.

\[ m = \text{mass} \]
\[ I = \text{mom. of inertia} \]

\[ \dot{v} = 0 \]
\[ \dot{\omega} = 0, \quad \theta = \omega t \]
\[ \dot{x} = v \cos(\omega t) \]
\[ \dot{y} = v \sin(\omega t) \]

⇒ All motions are steady circles.

⇒ No exponential stability!
The two most famous examples of non-holonomic systems (disk, skate) don't have exponential stability.

[Maybe that's why the possible exponential stability of conservative systems is/was not so well recognized.]
Chaplygin Sleigh: v2.0

\[ m = \text{mass} \]
\[ I = \text{moment of inertia about com} \]

\[ \mathbf{v} = l \omega^2 \]
\[ \omega = -\frac{ml}{I+ml^2} \mathbf{v} \mathbf{w} \]

\[ \begin{bmatrix} \dot{\theta} = \omega \\ \dot{x} = \mathbf{v} \cos \theta \\ \dot{y} = \mathbf{v} \sin \theta \end{bmatrix} \]

Linearize about \( \mathbf{v} = \mathbf{v}^*, \omega = 0 \)

\[ \Rightarrow \quad \dot{\mathbf{v}} = 0, \quad \dot{\omega} = \left[ -\frac{ml}{I+ml^2} \mathbf{v}^* \right] \mathbf{w} \]

exponentially stable.
Disk: V2
(Coleman 1997)

crooked mass attached to disk
frictionless slip
rolling contact

bank rate

Exponential stability

\( b \)
What is the difference between the non-holonomic systems that have exponential stability and those which do not?

Symmetry

2 Kinds
1) Time reflection
2) Space reflection

Put them together and you get a new view of the same solution.
Time reflection/reversal

If \( q(t) \) solves equations so does \( q(-t) \).

A movie run through a projector backwards is a legitimate motion of the system.

TRUE for collection of particles & rigid bodies interacting w/ workless constraints, holonomic or non-holonomic, and forces dependent only on \( q \) (don't have to be conservative even).

E.g., bike, skate board, particle on wire, disk 1, disk 2, Sleigh 1, Sleigh 2, celt, every example in this talk, all conservative systems, etc.

Why? \( q(t) \Rightarrow q(-t) \) has same \( \text{accel} \) at every \( t \Rightarrow \text{ma} \) balanced by same forces and same const. forces
Note: no time reversal for, say, damped oscillator.

\[ m\ddot{x} + c\dot{x} + kx = 0 \]

say \( \hat{x}(t) = x(-t) \)

\[ m\ddot{x} + c\dot{x} + k\hat{x} = 0 \]

\[ m(-\ddot{x}) + c(\dot{x}) + kx = 0 \]

\[ 2c\dot{x} = 0 \]

only if \( c = 0 \)

Dashpot force depends on \( \dot{x} \) not just \( x \).
TIME REVERSAL SYMMETRY

If a steady motion is exponentially stable, the backwards steady motion is exponentially unstable.

stable

unstable
Spatial Symmetry:

Say the steady solution of interest is associated with rotation \( \Theta \) about \( O \).

[Diagram: A system rotated by \( \Theta \) about \( O \)]

ie., as system moves it traverses states which are equivalent to rotating each material point about \( O \).

\[
\text{Spatial Symmetry} \equiv \text{Replace every point on body w/ point at } -\Theta \text{ is same system}
\]
A system w/ spatial symmetry w/ reflect to reflection in a symmetry direction of motion

IS

governed by the same equations when moving backwards (in reflected coordinates) as the original system moving forwards.

↓

If an exponentially decaying soln. exists for forward motion, so does it for backward motion.
Compose the two symmetries

Say a system has an exponentially decaying soln.

Time reversal $\Rightarrow$ growing soln. when moving backwards

Space symmetry $\Rightarrow$ growing soln. when moving forward

$\Rightarrow$ Exponential stability is not possible.

(for these systems)
Examples of systems which by inspection can't be stable.

- exponentially disk or torus

Ball or disk in:
- disk in trough

Skateboard rider in middle of board

Ball or disk in surf. of revolution (e.g. gravity well)

Golf ball in hole, basketball on rim, symmetric celt.
Summary of Facts

- Conservative Holonomic Systems
  (Subset of Hamiltonian Systems)
  Sometimes: √√√√ Never: √√√√√

- Conservative, non-holonomic Systems
  (not Hamiltonian)
  ∗ Symmetric: Sometimes: Never: √√√√√
  Sometimes: Never: √√√√√

∗ Not Symmetric
  Sometimes: Never: √√√√√
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<td>* Hamiltonian;</td>
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<td>* Can be asymptotically stable (though damping is not always stabilizing);</td>
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<td>* Simple example(s): simple pendulum, spring-mass.</td>
<td>* Simple example(s): Damped oscillators.</td>
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<td>* Integrable kinematic constraints (# of degrees of freedom equal to dimension of the configuration space);</td>
<td>* Non-integrable kinematic constraints (dimension of the instantaneously accessible velocity space less than the dimension of the configuration space);</td>
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<td>Piecewise holonomic</td>
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<th>Stability?</th>
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<td>* Simple example(s): piecewise skate?</td>
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<td>Example(s): 2D, 3D rimless wheel, McGeer’s 2D walking machines, Tinkertoy Walker</td>
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