Realizability of Arbitrary Local Mass Matrices in Single-Point Rigid Body Collisions

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ABSTRACT The 3 × 3 local mass matrix fully characterizes a pair of colliding "rigid" bodies for many purposes. We prove here that arbitrary 3 × 3, symmetric positive definite matrices have physical realizations as local mass matrices for collisions of two bodies with finite dimensions and inertia. General collision models thus must be able to handle all such matrices.

1 Introduction

If two rigid bodies (labeled 1 and 2) collide at a point C, the impulse $\mathbf{P}$ exerted by body 1 on body 2 is related to the net change in $\mathbf{V}_c$, the velocity of the contact point on body 2 relative to the contact point on body 1, by

$$\mathbf{P} = \mathbf{M} \cdot \Delta \mathbf{V}_c \tag{0.1}$$

where $\mathbf{M}$ is a well known tensor, here called the "local mass tensor" [1, 6]. $\mathbf{M}$ can be interpreted as the anisotropic inertia experienced by a force pushing apart the contact points on the two bodies. If equal and opposite forces $\mathbf{F}$ and -$\mathbf{F}$ act on the two bodies at C, then the contact points' relative acceleration $\mathbf{a}_c$ satisfies (neglecting the effects of external forces and centripetal acceleration terms)

$$\mathbf{F} = \mathbf{M} \cdot \mathbf{a}_c \tag{0.2}$$

For two unconstrained rigid bodies $\mathbf{M}$ is a symmetric positive definite (SPD) second order tensor given by

$$\mathbf{M}^{-1} = \sum_{i=1}^{2} \left(\frac{1}{m_i} \mathbf{I} + \mathbf{S}^T(\mathbf{r}_{C/cm_i}) \mathbf{J}_i^{-1} \mathbf{S}(\mathbf{r}_{C/cm_i}) \right) \tag{0.3}$$

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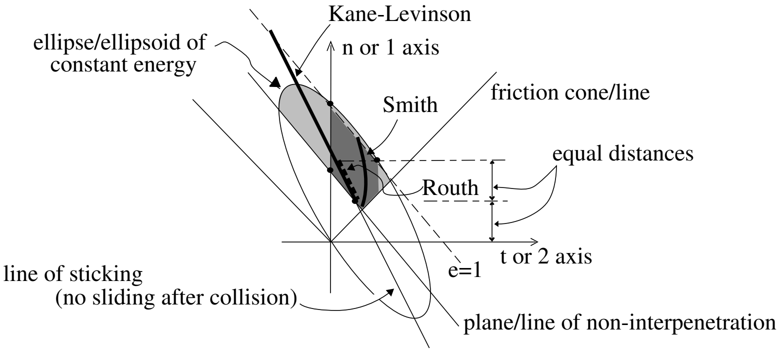
Kane-Levinson
ellipse/ellipsoid of constant energy
n or 1 axis
friction cone/line
Smith
equal distances
Routh
t or 2 axis
e=1
line of sticking (no sliding after collision)
plane/line of non-interpenetration

FIGURE 0.1. The region in impulse space accessible to generic "reasonable" collision. Also shown is the region accessible to some algebraic collision laws.

where $\mathbf{I}$ is the identity tensor, m_i is the mass of body i, $\mathbf{J}_i$ is the moment of inertia tensor of body i about its center of mass (c.m.), $\mathbf{r}_{C/cm_i}$ is the position vector of the contact point C with respect to the c.m. of body i, and the superscript T denotes transpose. $\mathbf{S}(\mathbf{r})$ is the cross product tensor of $\mathbf{r}$ satisfying $\mathbf{S}(\mathbf{r}) \cdot \mathbf{v} = \mathbf{r} \times \mathbf{v}$, for all $\mathbf{v}$. For calculations, we use the 3 × 3 matrix of components, $[\mathbf{M}]$, called the *local mass matrix.*

$\mathbf{M}$ fully characterizes the net dynamic interaction between the contacting rigid bodies. So, collision laws need only consider the family of all possible $\mathbf{M}$, and not the family of all pairs of possible contacting bodies or mechanisms. The set of all possible "reasonable" collisional impulses can be determined from $\mathbf{M}$ and the approach velocity of the about-to-contact points [1, 2]. These reasonable impulses are in a region of impulse space bounded by a non-interpenetration plane, a friction cone, and an energy-conserving ellipsoid, shown shaded light gray in the 2-D schematic Fig. 0.1. The coordinates are chosen with the 1 direction normal to the common contact-surface normal.

A geometric comparison is also presented in Fig. 0.1 between the range of possible predictions from Routh's law [4] (heavy dashed line), from Kane and Levinson's law [3] (heavy straight line), from Smith's law [6] (heavy curve) and a law proposed recently [2] (dark gray region). The geometric approach conveniently displays several features of the collision laws under study, such as the possibility of energy "creation" by Kane and Levinson's law, the impossibility here of a sliding collision as per Routh's law, etc.

Since all rigid body collision laws use $\mathbf{M}$ as part of their input (possibly implicitly) it is useful to know the set of all possible $\mathbf{M}$ (equivalently, all possible ellipsoids in impulse space, with some restrictions on their position [1, 2]) that need be considered as valid inputs for candidate collision laws.

As an analogy, consider the set of all possible rigid body moment of inertia tensors. These are SPD, but also have an additional restriction on

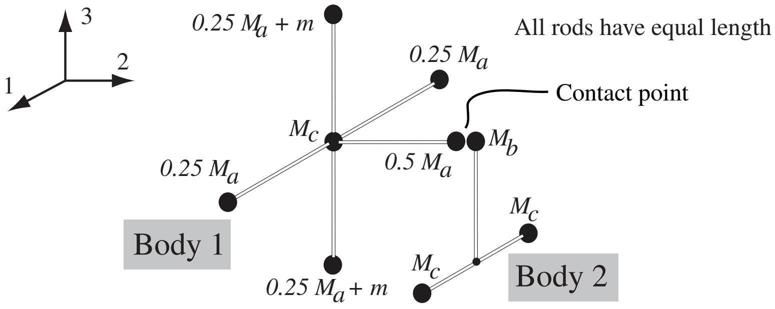

FIGURE 0.2. Physical realization of arbitrary mass matrices

the three eigenvalues: none of them is bigger than the sum of the other two. Are there any such restrictions on $\mathbf{M}$?

2 Physical Realizability

Claim: Any 3×3, SPD matrix has a physical realization as $[\mathbf{M}]$ for two unconstrained rigid bodies of finite size, mass, and moments of inertia.

Proof: In general the contact tangent plane is independent of mass distribution and can have *any* orientation depending on the *local* shape of the colliding objects, near the contact point. Thus, we only need to demonstrate the realizability of arbitrary diagonal $[\mathbf{M}]$ with non-negative elements (eigenvalues) $(\lambda_1, \lambda_2, \lambda_3)$. Let $\lambda_1 \leq \lambda_3$.

Consider the 2-body system in Fig. 0.2. Body 1 has a mass distribution that is equivalent to six point masses as shown. Body 2 has a mass distribution equivalent to three point masses as shown. The rigid, massless rods are of equal length. We will show the existence of masses (M_a, M_b, M_c, m) that yield the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$.

Observe that as $M_c \to \infty$, we obtain a constrained system: body 1 has a ball and socket joint at point mass M_c (on body 1), while body 2 is hinged about the axis through the point masses M_c (on body 2). Now, by Eq. 0.2, $\mathbf{M}$ will have the required eigenvalues if

$$M_a = \lambda_1\,, \quad M_b = \lambda_2\,, \text{ and } M_a + 2m = \lambda_3\,, \text{ and } m = (\lambda_3 - \lambda_1)/2\,. \quad (0.4)$$

Thus with infinite masses, or with hinges, arbitrary $\mathbf{M}$ are obtainable.

For finite M_c, the mass matrix in the coordinate system shown in Fig. 0.2 (found using Eq. 0.3) is diagonal. Its eigenvalues (M_{11}, M_{22}, M_{33}) are functions of M_a, M_b, M_c and m (not reproduced here). We define $\epsilon := 1/M_c$, and find (using computer algebra) that for small ϵ,

$$M_{11} = M_a - \frac{11}{4} M_a^2\,\epsilon + \mathcal{O}(\epsilon^2)\,, \qquad (0.5)$$

$$M_{22} = M_b - M_b^2\,\epsilon + \mathcal{O}(\epsilon^2)\,, \tag{0.6}$$

$$M_{33} = M_a + 2m - \left(\frac{3}{4}M_a^2 + 4M_a m + 6m^2\right)\epsilon + \mathcal{O}(\epsilon^2)\,. \tag{0.7}$$

Setting $\epsilon = 0$ gives Eq. 0.4 as expected.

From Eqs. 0.5 through 0.7, the matrix of partial derivatives of (M_{11}, M_{22}, M_{33}) with respect to (M_a, M_b, m) evaluated at $\epsilon = 0$ is invertible. By the implicit function theorem (see e.g., [5]), if $0 < \lambda_1 \leq \lambda_3$ and $0 < \lambda_2$, then there is a finite M_{c_0} such that for $M_c > M_{c_0}$ there are nonnegative, finite $M_a(M_c), M_b(M_c)$, $m(M_c)$ for which [**M**] has the specified eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. The demonstration presented here just shows one family of pairs of finite-inertia bodies associated with any possible positive definite **M**. In practice, the minimum value of M_{c_0} may not be very large compared to the other masses.

3 Relevance to Collision Modeling

The physical realizability of arbitrary [**M**] shows the completeness of the geometric view of single-point rigid body collisions. That is, in drawing Fig. 0.1, we are assured in advance that a pair of bodies can be found to match *any* such figure. We can examine the nature of collision laws in relation to arbitrary impulse-space ellipsoids with the foreknowledge that all such ellipsoids correspond to some pair of realizable colliding rigid bodies.

4 References

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