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Slip Instability and State Variable Friction Laws

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The dependence of the friction force on slip history is described by an experimentally motivated constitutive law where the friction force is dependent on slip rate and state variables. The state variables are defined macroscopically by evolution equations for their rates of change in terms of their present values and slip rate. Experiments may strongly suggest that one state variable is adequate or prove that one is inadequate. Analysis of steady slip governed by a single state variable in a spring and (massless) slider predict oscillations at a critical spring stiffness $k = k_{crit}$. The critical stiffness k_{crit} is given by a simple formula and steady slip is stable for $k > k_{crit}$ and unstable for $k < k_{crit}$. State variable friction laws may superficially appear as a simple slip rate dependence, slip distance dependence, or time dependent static friction, depending on experiment and testing machinery. Truly complicated motion is possible in a spring-slider model if more than one state variable is used. Further consequences of state variable friction laws can include creep waves and apparent rate independence for some phenomena.

INTRODUCTION

Since the proposal of *Brace and Byerlee* [1966] that stick-slip instabilities in laboratory friction experiments might be analogous to earthquake rupture, a fair amount of experimental work has been done to determine the nature of these instabilities and the conditions under which they occur. Within this view, laboratory experiments are thought of as models of possible fault motions in the earth. Experiments have been performed with many rock types, with and without various fault gouge layers, at a range of slip rates (often reported as "strain rates"), confining pressures, pore pressures, temperatures, and in machines with different geometries and compliances.

A slightly different approach to friction experiments has been to use them as a means to discover a constitutive description of surface slip from which earthquake or laboratory instabilities can be predicted through modeling. Such modeling of elastic systems reveals that instabilities in frictional slip depend on a reduction of the friction force during some part of the sliding (slip weakening). For this reason, much discussion of friction has emphasized the characterization of slip weakening phenomena associated with slip. For example, *Byerlee* [1970] proposed that the friction coefficient varies from point to point on slip surfaces and that instabilities are associated with decreases in the friction force from peak values as sliding proceeds. Alternatively, *Dieterich* [1972] proposed that slip weakening occurred after a time dependent healing during stationary contact. Also, a friction force that is a decreasing function of the instantaneous slip rate also leads to slip weakening (during accelerating slip) and can lead to slip instabilities. All of these mechanisms have been proposed previously as a basis for slip instabilities, primarily in metals [e.g., *Jenkin and Ewing*, 1877; *Bowden and Leban*, 1938; *Rabinowicz*, 1958, 1959; *Kosterin and Krageliskii*, 1960].

More recently, a deeper understanding of friction laws (at least in some materials and under some conditions) has come from work of *Dieterich* [1978] and *Rabinowicz* [1958, 1959] further developed by *Dieterich* [1979*a*, 1980, 1981], *Johnson* [1981], and *Ruina* [1980]. This work leads to a general class of friction laws that may be described by using state variables.

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These friction laws have features resembling a description with simple displacement dependence, slip rate dependence, or time dependent static friction. It is hoped that understanding of these laws, and the stability of slip in systems governed by these laws, will lead to better understanding of fault dynamics.

The aim of this paper is (1) to describe these state variable laws with examples, as motivated by general considerations and experimental results; (2) to present some simple results about the stability of steady sliding, neglecting inertia, with a friction law based on one state variable; and (3) to mention some further consequences of state variable friction laws. Geophysical applications will not be addressed directly in this paper.

Before getting involved in the details of the friction laws, some general features of certain experiments are mentioned now. These have mostly been presented by *Dieterich* [1978, 1979*a*, 1980, 1981]. Figure 1 shows the variation in the friction stress τ in an idealized experiment in a stiff machine with no inertia in which the slip rate V is changed suddenly from one value to another greater value with constant normal stress σ. The basic features of the curve in Figure 1, also observed approximately in real experiments, follow:

1. A steady state friction stress τ^{ss} associated with any slip rate V, (either τ_1^{ss} with V_1 or τ_2^{ss} with V_2 in Figure 1*b*).
2. A positive instantaneous slip rate dependence, visible as the positive jump in τ when the slip rate is suddenly increased and the negative jump in τ when V is suddenly decreased.
3. A long-term decrease in friction stress τ following the positive jump in slip rate V. A long-term increase in τ occurs following a negative jump in V. The long-term decrease in τ following an increase in V may, or may not, be larger than the instantaneous increase in τ; that is, $d\tau^{ss}/dV$ may be negative or positive.

Another feature that may have less generality but appears to be common to the limited recent observations is that

4. The decay of stress value after the step change in slip rate has characteristic length(s) that are independent of slip rate.

More specific features, but also roughly approximated by recent observations (at slip rates on the order of 1 μm/s), are that

5. The instantaneous rate dependence of the friction force (the jump in τ Figure 1*b*) is approximately proportional to ln (V_2/V_1).

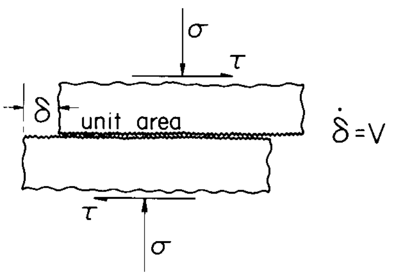

Fig. 1*a*. A unit of slip area. The friction stress τ is determined by the normal stress σ, the slip rate $\dot{\delta} = V$, and the surface state.

6. The steady state frictional stress τ is approximately logarithmically dependent on V (i.e., $\tau^{ss}(V) \cong$ constant + C ln (V) where the constant C may be positive or negative in different materials or environments).

We take curves like those in Figure 1*b* to be the basis of discussion here. They are examples of the evolution of friction force with a particular slip history. A set of such examples is then used to construct a constitutive law. This contrasts slightly with the work of *Dieterich* [1978, 1979*a*] in which the concept of static friction plays a central role in the development of the constitutive law.

Relation to Observations

The discussion in this paper is aimed at an idealized mathematical characterization of the basic experimentally observed features just mentioned, especially the first three: (1) fading memory and steady state; (2) positive instantaneous slip rate dependence; (3) negative dependence on recent past slip rates.

The fading memory and steady state feature is implicitly assumed in almost all experiments or discussion of metallic friction. Few rock friction experiments provide solid verification of this idea since they are apparently dominated by transients associated with first sliding. However, many of the experiments of, for example, *Summers and Byerlee* [1977] show a trend toward the leveling of load τ versus slip δ or repeatable stick-slip events, both of which are indicative of fading memory. A superposition of a long-term displacement dependence on a fading memory law (with possibly slowly changing parameters) is a possible correction in the cases where steady state is not observed. Thus, although occasionally observed oscillations (discussed later) and slip preceding instability in the tests by Summers and Byerlee may seem indicative of the laws we propose here, these experiments do not offer direct support.

Experiments of *Johnson* [1981] in a servo-controlled triaxial testing machine show evidence of a steady state, both in the leveling of τ versus δ curves and repeatability of experiments. Additionally, his experiments substantiate the idea of a direct velocity dependence competing with a memory to the same extent as the earlier work of *Dieterich* [1979*a*]. Servo-controlled experiments by *Dieterich* [1980] with fault gauge also give results in support of his earlier work. Because of the artificial stiffness that servo control provides, this later work by Dieterich shows clearly the distinctness of the essentially instantaneous direct velocity dependence from the memory dependence. However, some of his results (unpublished) show a more complex relaxation to steady state.

Experiments [*Ruina*, 1980] in the "sandwich" shear apparatus of Dieterich using servo control on displacement u, measured close to the slip surface, show some features consistent with Dieterich's and Johnson's results as well as some new features. The sample in these experiments was quartzite ground with 90 grit abrasive and loaded at about 3 MPa normal stress. The range of slip rates was 10^{-8} to 2×10^{-6} m/s. Typical results are shown in Figure 2. The effective machine stiffness is high enough (dashed line) so that the curves may be viewed as τ versus δ. These experiments clearly show the existence of a steady state, independent of recent history. In Figure 2*a*, two experiments are shown for step changes in load point velocity (like the first third of Figure 1*b*). In both cases, τ approaches the same level despite the difference in previous velocities. A step change in control rate V leads to a jump in τ as described before. Often, electrical noise would cause an unwanted servo "correction" causing a sudden transient jump in τ and V. These very short disturbances (0.1 s → 0.1 μm) caused little or no memory effects. The new features are these:

1. The curve for step changes from 0.1 μm/s to 1 μm/s does not retrace the curve for step change from 0.01 μm/s to 1 μm/s.
2. The curves retrace almost exactly an exponential decay after an initial transient decay that also is roughly exponential.

Not directly verifiable from Figure 2 but clearly observed [*Ruina*, 1980] was the independence of the characteristic distances of the exponential decays from slip rate. Further, all the jumps in τ and ultimate relaxations were roughly proportional to the log of the ratio of the velocity after the jump to the velocity before. Figure 2*b* shows the result of simulating the experiments with a state variable constitutive law to be introduced later.

State Variable Friction Laws

Constitutive Assumptions

The description presented here is totally macroscopic in that a surface is characterized by its mechanical behavior, the relation between friction force and slip displacement rather than the microscopic mechanisms. These mechanisms only affect the stability of slip through their effect on the mechanical constitutive description. Thus they may be neglected for some purposes. Of course, extrapolation of the results, beyond environments or materials directly tested, would be better justified by understanding the microscopic mechanism of the constitutive laws. The mechanism(s) are not discussed much in this paper due to lack of directly relevant results at this point.

The friction law is described at a "point" on the surface. Such a point is defined as a unit of area of surface on the

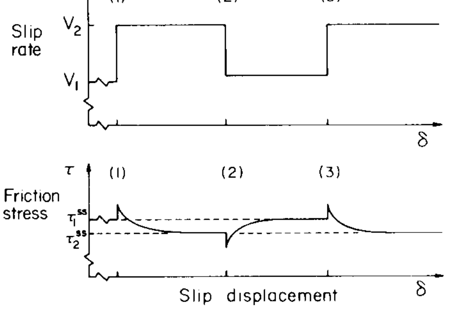

Fig. 1*b*. Idealized evolution of friction stress τ, at constant normal stress σ, when the slip rate $\dot{\delta} = V$ is changed. At location 1 the slip rate is suddenly increased from V_1 to V_2, τ jumps up from τ_1^{ss} and subsequently decays to τ_2^{ss}. At location 2, a sudden drop in slip rate back to V_1 causes a sudden drop in τ followed by a slow recovery to τ_1^{ss}. At location 3 the picture at location 1 is repeated.

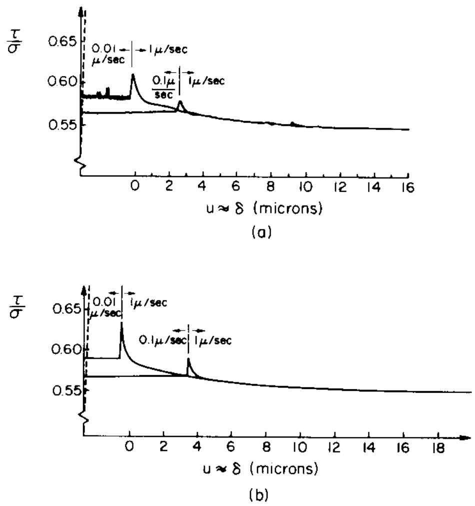

Fig. 2. Friction force variation due to step change in slip rate: (*a*) Quartzite polished with #90 grit abrasive and about 30 bars normal stress. Experimental results from two tests in which the slip rate was changed from 0.01 μm/s and 0.1 μm/s to 1 μm/s; (*b*) numerical simulation of the two tests using equation (19) (figure from *Ruina* [1980]).

boundary of a solid and is isolated as in Figure 1*a*. The surface is mechanically coupled to its surroundings by the positive slip displacement δ, the separation distance of the surfaces (dilation), the shear stress τ, and the normal stress σ. *Dieterich* [1978] suggested that shear strain γ in a deforming layer may replace the slip δ as the primitive deformation variable. However, theory [*Ruina*, 1980] and experiment [*Dieterich*, 1981] indicate that homogeneous shear straining localizes to concentrated slip if instabilities are possible. The surface may also be coupled to its surroundings through the conduction of heat, chemical diffusion, or pore fluid flow. Since our ultimate interest is the slip on the surface, primary interest is on the slip displacement δ and the driving shear stress τ. No experiments relating the dilation and the normal stress σ to the effects discussed here have been conducted. The role of σ and dilation will thus only be discussed peripherally.

The first basic assumption is that the surface (or surface region) has, at any instant in time, a state. Also, the friction stress depends only on the slip rate V the normal stress σ and the state. That is,

$$\tau = F(\sigma, V, \text{state}) \tag{1}$$

At a given point on the surface the state is assumed to vary as a continuous function of time for finite V (continuous δ). The rate of change of state is assumed to depend only on the instantaneous state at the point, the normal stress σ, and the slip rate V:

$$d(\text{state})/dt = G(\sigma, V, \text{state}) \tag{2}$$

In fact, one expects that the surface state should change if the external temperature, fluid pore pressure, or chemical environment changes. The effects of these environmental changes during slip are, for simplicity, excluded here. The function G in equation (2) may be different for different fixed environments, though, and thus the effects of temperature, pressure and chemical environment are not excluded (only excluded are changes during slip). The temperature distribution, pore pressure, and chemical environment near the surface which directly interacts with the surface (as opposed to the external environment) may be included in the state. (The state may include, for example, pore fluid suction due to dilation of pores or the temperature field due to frictional heating. If these quantities alter the properties of the adjoining solid, mechanical modeling will be complicated, however). A further exclusion implicit in (2) is that the state at one point is unaffected by the state at other points on the surface (i.e., no diffusion of state along the surface is included).

The general description here, which follows *Ruina* [1980], is very close to that commonly promoted as an approach to constitutive laws for metal deformation [*Onat*, 1981]. The principal difference is the inclusion of V (corresponding to deformation rate) in equation (1). This inclusion is motivated by the experiments which show effectively instantaneous step change in the friction force for step changes in slip rate (see Figure 2). This has been observed by *Ruina* [1980], *Dieterich* [1980], *Teufel* [1981], and was postulated by *Dieterich* [1979*a*] earlier. Experiments of *Johnson* [1981] also indicate this property.

In this paper we define δ as the inelastic deformation on the slip surface. The vagueness of the term "inelastic" can be disregarded since recent experiments [*Ruina*, 1980] indicate no noticable elastic contribution to δ if it is measured very close to the slip surface. Earlier experiments of *Dieterich* [1978] do not show the instantaneous rate dependence (the V in equation (1) nor does the earlier work with metals [*Rabinowicz*, 1958; *Sampson et al.*, 1943].

For the concept of state to be useful, it must have a definite realization. The approach here is to assume the state can be characterized by a collection of variables θ_i or collectively $\boldsymbol{\theta}$. In terms of these variables, equations (1) and (2) are

$$\tau = F(\sigma, V, \theta_1, \theta_2, \cdots) \tag{3a}$$

$$d\theta_i/dt = G_i(\sigma, V, \theta_1, \theta_2, \cdots) \qquad i = 1, 2, \cdots \tag{3b}$$

One hopes that for practical purposes the numbers of state variables θ_i required is small. The variables θ_i then represent some kind of average of an undoubtedly complicated surface state. *Dieterich* [1978, 1979*a*] suggested that the state be described by a single variable representing the average time of asparity contacts. *Rabinowicz* [1958] proposed average recent slip rate as the governing state variable. *Tolstoi* [1958] suggests the dilation (surface separation) is the governing state variable. The temperature of the surface could be taken as a single state variable if the heat flow is idealized as being dependent on only the temperature of the surface and the temperature of an external constant temperature reservoir. The usefulness of the state variable concept does not depend on physical interpretation of the state variables (like temperature or entropy in thermodynamics) though discovery of such interpretation would add tremendously to the credence and usefulness of the theory.

The friction stress τ is well known to be roughly proportional to the normal stress σ. Limiting attention to constant normal stress histories we assume that

$$\tau = \sigma F(\boldsymbol{\theta}, V) \tag{4a}$$

$$d\theta_i/dt = G_i(\boldsymbol{\theta}, V) \qquad \boldsymbol{\theta} = \theta_1, \theta_2, \cdots \tag{4b}$$

where F (slightly different in meaning than in (3)) and G are approximately independent of σ. Since nothing is known about the effects of changes of normal stress we will use equation (4) assuming σ = constant.

Fading Memory

The state variables θ_i represent the surface memory of previous sliding. Experiments of Dieterich and Ruina indicate that after sufficient sliding on virgin samples the memory appears to be only short term. More specifically, if a slip history $\delta(t)$ is imposed on a surface and the response $\tau_1(t)$ is observed and then if, subsequent to arbitrary intervening sliding, the slip history $\delta(t)$ is repeated (time origin offset) then the response $\tau_2(t)$ will approach $\tau_1(t)$ once sufficient time or displacement has elapsed. In other words, reproducible results may be obtained with a single surface merely by repeating displacement history. This is illustrated, for example, by the similarity of the first and third step changes in Figure 1*b*. A fading memory implies the existence of a steady state corresponding to constant velocity sliding, since in steady sliding the same slip history is being reapplied continuously and continuously repeatable results must be obtained. In terms of the friction law (3) or (4) the existence of a steady state is interpreted to mean that for any value of V there are corresponding values of state and shear stress denoted $\theta_i^{ss}(V)$ and $\tau^{ss}(V)$ to which θ_i and τ must approach closely after sufficient time or displacement at a constant slip rate; θ_i^{ss} solve $G_i(\theta_1^{ss}, \theta_2^{ss}, \ldots, V) = 0$ for $i = 1, 2, \cdots$.

If, as is apparently consistent with the experiments of *Ruina* [1980], the variables θ_i can be chosen such that the evolution of any one does not depend on the other, we have that $\dot{\theta}_i = G_i(\theta_i, V)$. Assuming the inequality

$$\infty > G_i(\theta, V)/[\theta^{ss}(V) - \theta_i] > 0 \qquad \text{for all } \theta_i \tag{5}$$

ensures that θ_i approach unique steady state values $\theta_i^{ss}(V)$ if slip proceeds at constant V after an arbitrary slip history. Inequality (5) applied for θ_i near $\theta_i^{ss}(V)$ implies that the final approach to steady state is exponential.

Characteristic Distances

As noted, equations (4*a*) and (4*b*) have, as one solution, the steady state solution, when $\dot{\delta} = V =$ constant:

$$\tau = \sigma F(\boldsymbol{\theta}^{ss}(V), V) = \tau^{ss}(V) \tag{6a}$$

$$\dot{\theta}_i = G_i(\theta^{ss}(V), V) = 0 \tag{6b}$$

Linearizing equations (4*a*) and (4*b*) with respect to θ_i near this steady state we obtain

$$\tau = \tau^{ss}(V) + \sigma \Sigma F_{\theta_i} \theta_i^* \tag{7a}$$

$$\dot{\theta}_i^* = (\partial G_i/\partial \theta_i)\theta_i^* \tag{7b}$$

where $\theta_i^* = \theta_i - \theta_i^{ss}(V)$ and $F_{\theta_i} \equiv \partial F/\partial \theta_i$. Equations (7*a*) and (7*b*) describe the friction force near the steady state for an unvarying V and have the full solution

$$\tau = \tau^{ss}(V) + \sum c_i \exp\left((\partial G_i/\partial \theta_i)\delta/V\right) \tag{8}$$

where t has been replaced by δ/V and c_i are arbitrary constants. Since $\partial G_i/\partial \theta_i$ is negative (by inequalities (5)), the final approach to steady state is thus a sum of exponentials with characteristic distances equal to $d_i \equiv -V/(\partial G_i/\partial \theta_i)$. Experiments indicate that these distances are constants independent of V [*Dieterich*, 1978, 1979*a*, 1980; *Ruina*, 1980]. For sudden small changes in slip rate the number of exponentials needed to approximate well the approach to steady state is not greater than the number of internal variables needed to describe the friction law and is equal if

$$\partial F/\partial \theta_i \neq 0 \qquad \text{for all} \qquad \theta_i \tag{9a}$$

and

$$\partial G_i/\partial \theta_i \neq \partial G_j/\partial \theta_j \qquad i \neq j \tag{9b}$$

If, in a given slip history, the slip rate V only changes slightly over the characteristic distance $d_i \equiv -V/(\partial G_i/\partial \theta_i)$ associated with a given variable θ_i, then that variable may be considered as always having its steady state value $\theta_i^{ss}(V)$. For slip histories with characteristic wavelengths much shorter than the characteristic distance associated with a given θ_i, that variable may be regarded as constant in some circumstances (e.g., if the velocity fluctuations are not so large that θ_i changes even with slip displacements much less than d_i). Thus, even though a full description of the friction may require several internal variables, only those with characteristic distances on the same scale as the slip histories of interest may need to be treated carefully.

In the following section we give some examples of state variables.

Examples of State Variables

Here we consider some examples of state variables. The state variables are assumed to evolve independently so that each one satisfies an evolution law of the form

$$\dot{\theta} = G(\theta, V) \tag{10}$$

Rabinowicz [1958] proposed that the average of the slip speed over a characteristic distance d_c (an asperity size) should be the governing variable, i.e.,

$$\theta = (1/d_c) \int_{\delta - d_c}^{\delta} V(\delta')\, d\delta' \tag{11}$$

Unfortunately, θ as described in (11) cannot be described exactly by a single differential equation of the form of equation (10). More generally, one may think that θ should be the weighted average of some function f of the recent slip speed. Thus,

$$\theta = \int_{-\infty}^{\delta} w(\delta - \delta') f[V(\delta')/V_c]\, d\delta' \tag{12}$$

where V_c is a constant introduced for dimensional consistancy and w is a weight function in the average.

If the weight function w is a decaying exponential, (i.e., $w(x) = e^{-x/d_c}$) equation (12) can be written in the form of (10). (Or, if the weight function can be written as a sum of n exponentials, then the variable θ in equation (12) is the sum of several θ_i each of which obeys an equation of the type (10).) In particular, differentiation of the integral in equation (12) using a decaying exponential weight, shows that

$$\theta = (1/d_c) \int_{-\infty}^{\delta} e^{-(\delta - \delta')/d_c} f[V(\delta')/V_c]\, d\delta' \tag{13a}$$

$$= e^{-(\delta - \delta_0)/d_c} \theta(\delta_0) + (1/d_c) \int_{\delta_0}^{\delta} e^{-(\delta - \delta')/d_c} f[V(\delta')/V_c]\, d\delta' \tag{13b}$$

is equivalent to

$$d\theta/dt = G(\theta, V) = V[f(V/V_c) - \theta]/d_c \tag{13c}$$

where $f(V/V_c)$ is the function of slip rate that is averaged by equation (13*a*). Here, $f(V/V_c)$ is also the steady state value of θ for slip at constant V (i.e., $\theta^{ss}(V) = f(V/V_c)$). One might for convenience also use $\phi(\theta)$, a function of θ, as the state variable and alter $G(\theta, V)$ accordingly.

Some examples of θ which all happen to fit the description in equation (13) will be discussed briefly. *Dieterich* [1979*a*] has suggested that friction is primarily a function of a variable he called "the average time of asperity contact." His rationale, similar to *Rabinowicz* [1958], was the "static" friction in his experiments increased as a function of time in about the same way friction decreases as a function of steady sliding speed. These two facts are regarded as different aspects of the same phenomena. That is, slip at speed V over an asperity of dimension d_c involves contact for a time d_c/V and thus leads to a friction force corresponding to a stationary time of contact $\phi = d_c/V$. This reasoning does not, however, admit explicit rate dependence (observed in experiments as the jumps in Figures 1*b* and 2 with jumps in slip rate). Dieterich defined the "average time of contact" ϕ by describing how it changes from $\phi_1 = d_c/V_1$ to $\phi_2 = d_c/V_2$ when the slip rate changes suddenly from V_1 to V_2. Dieterich's definition of "time of contact" ϕ is equivalent to any of the four descriptions below [*Ruina*, 1980] as can be verified by differentiation and use of the chain rule (equation (14*c*) was published in *Kosloff* and *Liu* [1980].)

$$\theta = e^{-(\delta-\delta_0)/d_c}\theta(\delta_0) - (1/d_c)\int_{\delta_0}^{\delta} e^{-(\delta-\delta')/d_c} \ln[V(\delta')/V_c]\, d\delta' \qquad \phi = d_c e^{\theta}/V_c \tag{14a}$$

$$d\theta/dt = V[-\ln(V/V_c) - \theta]/d_c \qquad \phi = d_c e^{\theta}/V_c \tag{14b}$$

$$d\theta/d\delta = [-\ln(V/V_c) - \theta]/d_c \qquad \phi = d_c e^{\theta}/V_c \tag{14c}$$

$$d\phi/dt = -(\phi V/d_c)\ln(\phi V/d_c) \tag{14d}$$

From equation (14*a*) one can say θ is the weighted average of $-\ln(V/V_c)$ over the distance d_c. From equation (14*b*) one can say θ chases its steady state value $-\ln(V/V_c)$. That θ approaches θ^{ss} with characteristic distance d_c is apparent in (14*c*). At constant V the approach is exponential, even for large changes in slip rate. A flaw in Dieterich's reasoning is apparent from (14*d*). If the slip V jumps to zero, the expression in (14*d*) yields $d\phi/dt = d\theta/dt = 0$ (i.e., $x \ln x \to 0$ as $x \to 0$). Thus, ϕ, as defined by *Dieterich* [1979*a*], does not change during stationary contact and cannot be interpreted as "average contact time." The form (14) seems to be quite useful, however, despite this complication in interpretation [*Ruina*, 1980; *Rice* and *Ruina*, 1982; *Gu et al.*, 1983; *Mavko*, 1980].

As another example consider θ as an average of a power of $V, f(V/V_c) = (V/V_c)^n$, so

$$\theta = \theta(\delta_0)e^{-(\delta-\delta_0)/d_c} + (1/dc) \cdot \int_{\delta_0}^{\delta} e^{-(\delta-\delta')/d_c}[V(\delta')/V_c]^n\, d\delta' \tag{15a}$$

or, equivalently,

$$d\theta/dt = V[V/V_c)^n - \theta]/d_c \tag{15b}$$

or

$$d\phi/dt = -[(V\phi)^{n+1}/V_c{}^n - V\phi]/nd_c \qquad \theta = \phi^{-n} \tag{15c}$$

The value $n = -1$ is the only way equation (15) leads to a finite, nonzero rate of change of state for zero slip rate ($\infty > \dot{\theta} > 0$ for $V \to 0$). So, a state variable that has a simple interpretation as time dependent is one that averages the slowness $1/V$ ($n = -1$). In this case, equation (15) can be written as (with (15*c*) changed by a constant),

$$\theta = (1/d_c)\int_{-\infty}^{\delta} e^{-(\delta-\delta')/d_c}[V_c/V(\delta')]\, d\delta' \tag{16a}$$

or

$$\theta = (V_c/d_c)\int_{-\infty}^{t} e^{-(\delta-\delta(t'))/d_c}\, dt' \tag{16b}$$

$$d\theta/dt = (V_c - \theta V)/d_c \tag{16c}$$

$$d\phi/dt = 1 - \phi V/d_c \qquad \phi = d_c\theta/V_c \tag{16d}$$

where equation (16*d*) shows that $d\phi/dt = 1$ when $V = 0$. At steady state, ϕ has the value d_c/V consistent with *Dieterich*'s [1978, 1979*a*] original reasoning.

Another example to look at is that of *Rabinowicz* [1958]. His "state variable," as defined in equation (11), is similar to $n = 1$ in equation (15). In either case one can see that average recent slip rate leads to no change of state at zero slip rate and thus does not coincide with a simple concept of time-dependent static friction. Many other state variables consistent with equation (4*b*) or (10) may undoubtedly be used profitably. Those mentioned in this section, being a subset of (13), include all those used thus far for modeling, however.

Sufficiency of One Internal Variable

Here we focus our attention on a friction law adequately described by only one internal variable, for example,

$$\tau = \sigma F(\theta, V) \tag{17a}$$

$$\dot{\theta} = G(\theta, V) \tag{17b}$$

where equation (5) is still a restriction on (17*b*). This form includes all the laws proposed by *Dieterich* [1979*a*, 1980, 1981]. First it should be noted that the expression (17) is not unique for a given material. No general aspects of the theory would be changed if the variable θ could be replaced by $\phi(\theta)$ where ϕ is any monotonic function. New functions $\bar{F}(\phi, V)$ and $\bar{G}(\phi, V)$ can then be constructed from (17) that represent the same friction law.

We now address two questions: (1) How, in principle, can one deduce that a form like (17) applies to any given surface? (2) Knowing a form (17) does apply, how can one deduce the functions F and G? The difficulty in answering these questions arises because θ cannot be measured directly.

One simple way to justify the sufficiency of (17) is by exhaustive testing of some given $F(\theta, V)$ and $G(\theta, V)$ with affirmative results. That is, if an exhaustive set of V versus time experiments all have τ versus time curves identical to that given by integration of (17) then, within the domain of the experiments equation (17) is correct. If the experiments differ from the integration of equation (17), then at least one of the functions F and G is not correct. A more general approach can however test equation (17) without testing particular functions F and G.

The following two statements are implied by (17), independent of the functions F and G (assumed to be monotonic in their arguments). Violation of either of them implies that no single state variable law of the form (17) can apply.

1. For given initial conditions, (τ_0, V_0), very fast changes (corresponding to changes at a fixed state) must lead to a unique relationship between τ and V independent of the nature of the fast change.

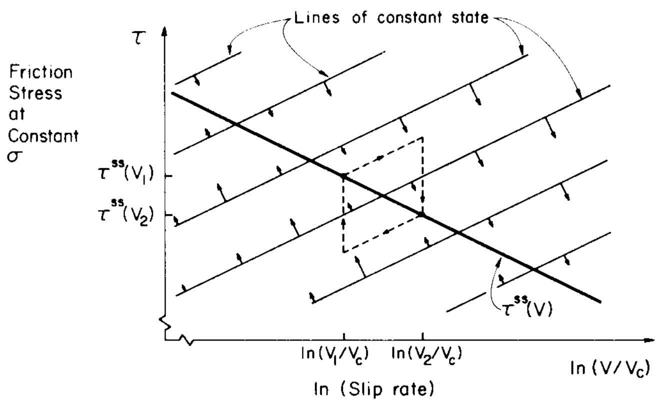

Fig. 3. Simplified Dieterich Friction Law, equation (18). The heavy solid line shows the relation between friction stress τ slip rate V at steady sliding, illustrated for $B > A$. The light solid lines are lines of constant state. The arrows indicate the rate of change of state, increasing below $\tau^{ss}(V)$ and decreasing above $\tau^{22}(V)$. The dotted line shows the experiment of Figure 1. τ increases along a line of constant state from $\tau^{ss}(V_1)$ when V jumps from V_1 to V_2 and then decreases gradually to $\tau^{ss}(V_2)$. τ then decreases along a line of constant state when V jumps back to V_1 and subsequently rises back to $\tau^{ss}(V_1)$.

2. The value of $\tau(t_1 > 0)$ is uniquely determined by $\tau(0)$ and $V(t)$, $0 < t < t_1$. That is, all slip velocity histories equal to $V(t)$ beginning at an arbitrary time = 0 (but possibly different previously) and causing a given value of τ at time = 0, cause the same values of $\tau(t)$ for all subsequent time. In other words, the state θ is uniquely determined by the instantaneous values of τ and V.

Statement (1) follows, assuming G is bounded, because θ is continuous in time from (17b). Thus for any sudden changes in V, τ is a function of V alone, from (17a). Statement (2) follows from solving (17a) for $\theta(\tau, V)$. Thus $\tau(0)$, $V(0)$ determine the state $\theta(0)$, and $\tau(t)$ is found from the unique integration of (17) with given initial conditions $\tau(0)$, $V(0)$ and "forcing" function $V(t)$.

Conditions 1 and 2 if verified by exhaustive experiments also nearly imply that equation (17) applies. The condition 1 leads to a family of curves on the τ, V plane that, by condition 2, cannot intersect. Numbers assigned to these curves in an arbitrary but monotonic manner can be identified as values of θ. Thus, $\theta = \theta(\tau, V)$, which inverted is (17a). For fast changes in V, θ is constant; thus, $\dot{\theta}$ can only depend on V and higher derivatives in a manner that gives no singularities in $\dot{\theta}$ for singular $\dot{V}$, $\ddot{V}$, etc. Assuming, then, that $\dot{\theta}$ does not depend at all on $\dot{V}$, $\ddot{V}$ etc., we have that $\dot{\theta}$ can only depend on θ and V. Thus, $\dot{\theta} = G(\theta, V)$.

Experiments tending to confirm conditions 1 and 2 above thus tend to confirm the validity of a single state variable description (17a) and (17b). Experiments that violate these conditions, like experiments that violate any predictions of (17a) and (17b), demonstrate that one internal variable is not sufficient to describe the friction law.

If (17) is known to apply but the functions F and G are not known they could, in principle, be found by the following experiments. Assign to θ the value of a parameter which is the single parameter in a family of slip histories, each of which is long enough to uniquely determine the state. For example, if the single parameter family of histories is slip at constant slip rate (for sufficient distance to clear the memory), θ may be taken as some function of the speed of slip. Equation (17) is then deduced by imposing a range of slip histories $\delta(t)$ corresponding to a range of θ, each of which is followed by a range of slip velocities V. The function $\sigma F(\theta, V)$ of (17a) is just the value of τ immediately following initiation of slip speed V. The function $G(\theta, V)$ is found by solving (17a) for $\theta(\tau, V)$ and the time derivative of (17a) for $\dot{\theta}(\theta, \dot{\tau}, V, \dot{V})$. So both θ and $\dot{\theta}$ are known in terms of $(\tau, \dot{\tau}, V, \dot{V})$ and (17b) can be determined from measurement of $(\tau, \dot{\tau}, V, \dot{V})$ at a number of points for various slip histories.

On the other hand, the sets $(\tau, \dot{\tau}, V, \dot{V})$ are overdetermined if (17b) exists, since then one could solve, say, for $\dot{\tau}(\tau, V, \dot{V})$. So, if for given τ, V, $\dot{V}$ the variable $\dot{\tau}$ (or the evolution of τ) also depends on previous slip history then no representation of the form (17) exists. This last result will be used later in the analysis of experiments. The discussion is simplified if one restricts attention on the evolution of θ to constant V in which case $\dot{V}$ drops out of all the equations above, and the changes of τ with slip are directly due to change of θ.

EXAMPLES OF STATE VARIABLES FRICTION LAWS

Simplified Dieterich's Law

A law with only one internal variable that has all of the features named in the introduction is one based on the state variable in (14) [*Ruina*, 1980].

$$\tau = \sigma[\mu_0 + \theta + A\ ln\ (V/V_c)] \qquad (18a)$$

$$\dot{\theta} = (-V/d_c)[\theta + B \ln (V/V_c)] \qquad (18b)$$

This law is close to that proposed by *Dieterich* [1979*a*] for a large range of slip rates and shares the apparent defect of no-healing (no change of θ) for zero slip rate. This law can be illustrated graphically as in Figure 3.

Lines of constant state, θ, are light solid lines and show the instantaneous positive dependence of τ on slip rate V. The heavy line is the steady state friction law and is a decreasing function of slip rate in the example of Figure 3 ($B > A$). As governed by (18b), θ decreases above the steady state line, below it θ increases. Any slip corresponds to a pen motion on a plot of Figure 3 and is the simultaneous solution of the friction law and any constraints imposed by the loading mechanism. The arrows indicate the component of this motion perpendicular to the lines of constant θ. The imposition of constant slip rate, for example, constrains the motion to a vertical line on Figure 2 and θ approaches the steady state value (solid line) for the given V, as dictated by the arrows.

The experiment in Figure 1 is shown by the dotted line in Figure 3. At constant slip rate V_1, $\tau = \tau^{ss}(V_1)$. When the slip rate jumps to V_2, the friction stress jumps up along a line of constant state and subsequently slowly decays to $\tau = \tau^{ss}(V_2)$. The return to V_1 closes the loop in a similar manner.

Two State Variable Friction Law

The experiments of *Ruina* [1980] shown in Figure 2a cannot be described by a single state variable law of the form (17). This is because the two curves, both ending at $V = 1$ $\mu m/s = 10^{-6}$m/s do not trace each other. This violates condition (2) in the discussion of the sufficiency of a single state variable. Since the decay to steady state seems to be the sum of two exponentials, one may postulate a two state variable friction law. The experiments are well described by a friction law of the form

$$\tau = \sigma[\mu_0 + \theta_1 + \theta_2 + A \ln (V/V_c)] \qquad (19a)$$

$$\dot{\theta} = -(V/d_1)(\theta_1 + B_1 \ln (V/V_c)) \qquad (19b)$$

$$\dot{\theta}_2 = -(V/d_2)(\theta_2 + B_2 \ln (V/V_c)) \qquad (19c)$$

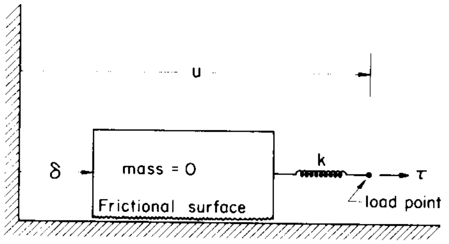

Fig. 4. Spring and slider model. A slider of negligible mass slides a distance δ. A load point moves a distance u stretching the spring with stiffness k (dimensions: (stress/distance)) and causing the driving stress τ.

two trajectories of which are plotted in Figure 2b as computed numerically with the spring block model of Figure 4 with a very stiff spring. Equation (19) provides a good approximation of the experiments in *Ruina* [1980] for velocity drops as well as jumps. Since the memory terms θ_1 and θ_2 are each identical to the memory term in the simplified Dieterich Law (equations (14b) and (18b), they do not give "time dependence" in the limit of zero slip velocity (e.g., $\dot{\theta}_1$ and $\dot{\theta}_2$ go to zero as V goes to zero). The appearance of the ln function in (17) obviously limits the general validity of the form since it implies negative τ if V is very large and nearly constant or if V is suddenly very small. However, these paradoxes occur at such extremes of V (since A, B_1, and B_2 are of the order of 0.01 in experiments) that other limitations apply sooner. Experiments of *Dieterich* [1979a] and *Teufel* [1981], for example, indicate that $\tau^{ss}(V)$ is not a decreasing function of V at higher V.

In the particular experiments of Figure 2 the direct rate dependence is very nearly negated by the short-term relaxation of θ_1. In this case the two terms may be thought of as canceling for very slow changes in slip rate and together appear as a resistance to rate changes for slightly larger changes in rate [*Ruina*, 1980].

In numerical spring-block models the two internal variable law (19) yields results that differ in some details from use of a one variable law, as will be mentioned. However, for simplicity we will discuss the stability of steady sliding governed by the general single state variable law.

STABILITY OF STEADY FRICTIONAL SLIDING

Constant Force Loading

The constitutive laws thus far have been written for τ in terms of slip history. However, assuming the necessary invertability they can be re-expressed to solve for slip from the friction force. For example, the simplified Dieterich law equation (18) can be expressed

$$V = V_c \exp\left[(\tau/\sigma - \mu_0 - \theta)/A\right] \tag{20a}$$

$$d\theta/d\delta = (B/A - 1)[\theta - \theta^{ss}(\tau)/d_c \tag{20b}$$

$$\theta^{ss}(\tau) = -B(\tau/\sigma - \mu_0)/(A - B) \tag{20c}$$

For constant τ loading this system can be solved for V in terms of δ as

$$V = V^{ss}(\tau)(V_0/V^{ss}(\tau))^{\exp((B/A)-1)\delta/d_c} \tag{21}$$

where $V^{ss}(\tau) = V_c[\exp(\tau/\sigma - \mu_0)/(A - B)]$ is the steady state value of V that corresponds to τ, and V_0 is the slip velocity at $\delta = 0$. For $B < A$ the solution decays to the steady state solution. For $B > A$, corresponding to a friction law that has τ decrease with steady state slip speed (e.g., $d\tau^{ss}/dV < 0$), steady sliding is extremely unstable. If $V_0 > V^{ss}(\tau)$ infinite velocities and displacements are reached in finite time since in (21) $V(\delta)$ is of greater order than δ(i.e., $\int dt = \int_0^{\infty} (1/V(\delta))d\delta$ = finite). This strong instability was observed in the numerical work of *Kosloff and Liu* [1980] using a slightly different friction law. If $V_0 < V^{ss}(\tau)$ slip continuously slows.

Constant force loading constrains the motion to horizontal lines on Figure 3. The arrows governing $\dot{\theta}$ indicate the subsequent velocities for given initial conditions. The stability conditions from the last paragraph are apparent with brief inspection.

Figures 3 can be used to show qualitative stability response for other loading. For example, as noted in *Kosloff and Liu* [1980] and *Dieterich* [1979a], sliding occurs, with this style law, before instabilities. In particular, unless sufficient force is applied to bring the slip speed up to the steady state value corresponding to that state, accelerating slip instabilities cannot occur. If the friction force is an increasing function of the steady state slip velocity, slip rates may grow quickly during and subsequent to quickly growing imposed load τ, but they remain finite and approach the steady state velocity corresponding to τ. Further stability analysis using the simplified Dieterich law is carried out in *Gu et al.* [1983].

Not surprisingly, stability of sliding at constant τ for any law of the form (17) is critically dependent on $d\tau^{ss}/dV$, as was the case in the last example. We examine the stability by looking for solutions near a steady state solution at slip speed V^{ss} and $\tau = \tau^{ss}(V^{ss})$, $\theta = \theta^{ss}(V^{ss})$, and $\dot{\theta} = 0$. Linearizing (17) near this solution one obtains

$$\tau = \tau^{ss} + \tau^* = \tau^{ss} + \sigma F_\theta \theta^* + \sigma F_V V^* \tag{22a}$$

$$\dot{\theta}^* = \dot{\theta} = G_\theta \theta^* + G_V V^* \tag{22b}$$

where $\theta^* = \theta - \theta^{ss}$, $\tau^* = \tau - \tau^{ss}$, $V^* = V - V^{ss}$, and subscripts denote partial differentiation.

For fixed τ loading, $\tau^* = 0$. Solving for V^* in (22a) and applying this to (22b) we have

$$\dot{\theta}^* = (G_\theta/F_V)[F_V - F_\theta G_V/G_\theta]\theta^* \tag{23}$$

The term in square brackets is $(1/\sigma)d\tau^{ss}/dV$. This can be seen by applying $d\theta^{ss}/dV = -G_V/G_\theta$ (from implicit differentiation of $G(\theta, V) = 0$) to the "total" derivative of (17a) with respect to V at steady state. The first term in (23) may be re-expressed by use of the association of G_θ with $-V/d_c$ made in the discussion following equation (9). So equations (22) and (23) reduce to

$$\dot{\theta}^* = -(V/(d_c \sigma F_V))(d\tau^{ss}/dV)\theta^* \tag{24a}$$

$$\dot{V}^* = -(V/d_c \sigma F_V))(d\tau^{ss}/dV)V^* \tag{24b}$$

where (24b) is found by differentiating (22a) with respect to time.

Assuming a positive direct velocity dependence ($F_V > 0$), equation (24) has solutions that grow or die exponentially depending only on whether $d\tau^{ss}/dV$ is less or greater than zero.

So, for constant force loading the stability criterion with a memory and slip rate dependent law is the same as with a strictly slip rate dependent law [e.g., *Rabinowicz*, 1958]. If force decreases with steady state velocity, steady slip is not stable. This similarity between the stability behaviors for the two laws break down, however, when a finite compliance is used in the loading. In this case, friction negatively dependent only on instantaneous rate is unstable for all stiffness, and the state variable law is conditionally stable.

Steady Sliding with a Spring-Block Model

A simple mechanical system in which to examine stability of sliding is the spring-block of Figure 4. A rigid slider with unit base area is held in frictional contact with another rigid surface by a normal force σ that may, in general, depend on the sliding displacement δ or the load point displacement u. The load point is connected to the block by a spring of stiffness k which transmits a stress τ in the direction of sliding. In this model all the inelastic shear deformation at a frictional surface region is modeled by δ, the slip displacement. The load point displacement u represents the displacement at the point at which the loading machine is controlled. The displacement u, no matter what it represents (hydraulic oil pumped, gears turned, or the shift of the base of a continental plate, for example) is measured in such a way that $\delta = u$ for constant force loading. All elastic compliance (e.g., elastic sample deformation) is incorporated in the spring constant k. In this discussion, k has dimension (stress/distance) and is defined $k \equiv -d\tau/d\delta$ at fixed u. The model does not include inelastic or nonlinear deformation away from the frictional interface, but some such effects can be added easily. If the friction stress depends on σ as in (4), even for not constant σ, the results can then be generalized if τ is replaced by μ and k by $-d(\tau/\sigma)/d\delta$ (fixed u) [*Ruina*, 1980]. For quasistatic (inertia neglected) analysis, the spring-block model is not much of an idealization so long as slip in the modeled system is nearly the same at all points on the slip surface. Most frictional instability analyses use the dynamic (lumped mass) spring-block model in which the block is endowed with a mass m. This is only an accurate model if the modeled system is dominated by a single mode of vibration. Our attention will focus on quasi-static motion and the onset of instabilities and so the block mass will be neglected and there will be no discussion of inertia governed slip times, stress drop overshoots, or other factors which depend on inertia or kinetic energy. (A more general linear analysis, including inertia, is presented in *Rice and Ruina* [1983].)

In the spring block model specification of the load point motion $u(t)$ approaches displacement control or force control in the high and low stiffness limits. The constitutive description is a limiting form of the model response.

We now look at the nature of nearly steady slip if the spring-block model (Figure 4) is used with a memory dependent law (equation (17)) with the load point moving at a constant speed V_0. Since the friction force must equal the force transmitted through the spring,

$$\tau = \sigma F(\theta, V) = k(u - \delta) \tag{25}$$

where for constant rate loading $u = V_0 t$.

We are interested in the stability of steady state sliding solution $\delta^{ss} = V_0 t - \tau^{ss}(V_0)/k$. To examine the solutions near the steady state solution equations (17) and (25) are linearized, as for constant force loading (equation (22)), to give

$$\tau^* = \sigma F_\theta \theta^* + \sigma F_V V^* = -k\delta^* \tag{26a}$$

$$\dot{\theta}^* = G_\theta \theta^* + G_V V^* \tag{26b}$$

$$\dot{\delta}^* = V^* \tag{26c}$$

where $\tau^* = \tau - \tau^{ss}(V_0)$, $\theta^* = \theta - \theta^{ss}(V_0)$, $\delta^* = \delta - (V_0 t - \tau^{ss}(V_0)/k)$. The coefficients of all the asterisked variables are subsequently regarded as constants. The linear constant coefficient equation (26) governs the motion of the block near steady state sliding with constant load point velocity.

The solutions of (26) have the form

$$\tau^* = Re\ A_1 e^{st} \qquad \theta^* = Re\ A_2 e^{st} \tag{27}$$

where $Re[\ \ ]$ denotes the real part of $[\ \ ]$. A_1, A_2 are constants, and s is a constant to be determined. Application of the solution (27) to the linearized equation (26) leads to the following quadratic equation and solutions for s:

$$s^2 + (Tk/\sigma F_V)s + Dk^2/(\sigma F_V)^2 = 0 \tag{28a}$$

$$s = (k/2\sigma F_V))[-T \pm (T^2 - 4D)^{1/2}] \tag{28b}$$

$$T = V(d\tau^{ss}/dV)/kd_c + 1 \tag{25c}$$

$$D = \sigma F_V V/kd_c \tag{28d}$$

The identification of d_c as $-V_0/G_\theta$ has been employed as well as the identity $d\tau^{ss}/dV = \sigma[F_V - G_V F_\theta/G_\theta]$. If, in either of the solutions of (28*b*) s has a positive real part, then perturbations near the steady state will grow exponentially by (27). Since small perturbations can always be assumed to exist, steady following of the load point by the block is impossible if $Re[s] > 0$.

From the solution to the quadratic equation (28*b*) we have instability when $T < 0$ or $D < 0$. Since the direct velocity dependence F_V is expected to always be positive D is always greater than 0. So the stability criterion reduces to

$$k < k_{crit} \Rightarrow \text{instability} \tag{29a}$$

$$k_{crit} = -V(d\tau^{ss}/dV)/d_c \tag{29b}$$

where k_{crit} is the value of stiffness at neutral stability.

This result may be applied to any single state variable law of the form (17). Applied to the simplified Dieterich law (18), (29) yields the simple result

$$k_{crit} = \sigma(B - A)/d_c \tag{30}$$

The dimensionless quantity T is negative when steady sliding is unstable and positive for possibly stable sliding and can serve as a measure for degree of stability. When s has an imaginary part the approach to, or growth away from the steady state solution will, from equations (27, and 28) be oscillatory. The condition for this, from (28*b*), is

$$T^2 = [V(d\tau^{ss}/dV)/kd_c + 1]^2 < 4\sigma F_V V/kd_c = 4D \tag{31}$$

At neutral stability, $T = 0$, and, from (28), s is given by

$$s = i(V/d_c)[-(d\tau^{ss}/dV)/(\sigma F_V)]^{1/2} \tag{32}$$

and the slip displacement wavelength of the associated persistant sine waves is

$$\text{wavelength} = 2\pi d_c[F_V/(-d\tau^{ss}/dV)]^{1/2} \tag{33}$$

The nature of the solutions to (22*b*) and (26) is described mostly by the eigenvalues s of (27) and not the "eigenvector" (A_1, A_2) which determines the relative phase and magnitude of τ and θ and shall not be discussed further (though it is relevant in the nonlinear stability analysis of *Gu et al.* [1983]).

In summary, steady state sliding with the spring block apparatus is unstable when $d\tau^{ss}/dV < 0$ only if k is sufficiently small (29). This is different from a strictly slip rate dependent law, for which steady sliding was unstable for any amount of rate weakening (the limit of $d_c \to 0$ in our results). Near the condition of neutral stability oscillatory solutions are expected that may grow or die depending on the sign of T (equations (27)–(32). The wavelength of the oscillations is on the order of $2\pi d_c$ (equation (33)).

A few complications may be added to the problem without

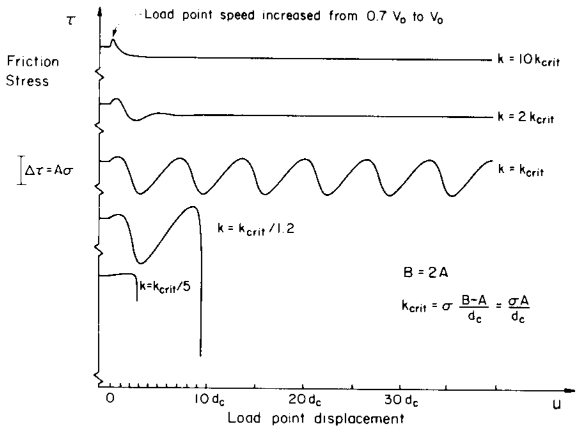

Fig. 5. Friction stress τ versus load point displacement u in the spring and slider model of Figure 4. The load point speed is suddenly increased from steady sliding at $0.7V_0$ to V_0. Subsequent motion for various k are shown, as calculated with equation (18), the simplified Dieterich Friction Law.

complicating the solution. If a viscous element is added, parallel to the spring in Figure 4, the solution is only modified by the addition of the dashpot constant to both F_V and $d\tau^{ss}/dV$. This further stabilizes the motion. Incidentally, one may note that for purely viscous loading (the spring replaced by a dashpot) qualitative solutions can be obtained graphically from Figure 3, since the loading could be represented as a single constraint curve (between V and τ) on the graph.

Experiments [e.g., *Johnson*, 1981] show memory effects indicative of the results idealized in Figure 1*b* but superposed on a long-term displacement hardening. If a term $f(\delta)$ is added to the friction law (17) the linearization can be carried out as before if the slope of f, f', is constant over a relevant slip distance. That is, a direct displacement dependence can be added to (17) if it can be modeled as having constant slope $f' = C$ (i.e., $\tau = \sigma F(\theta, V) + C\delta$). The corresponding solution is obtained by linearization about the steady state solution $V^{ss} = kV_0/(k + C)$ instead of $V^{ss} = V_0$. The results are identical but with k replaced by $k + C$ in results that depend on k. An added positive displacement dependence is stabilizing since it increases the effective stiffness of the spring.

I have not found a simple rationalization of the main result of this paper, equation (29), except in the somewhat special case where there is no explicit slip rate dependence, $F_V = 0$. Imagine the block sliding at the steady state solution corresponding to the load point speed. If the slip rate of the block were suddenly to change to a new, say, greater speed, the friction law would require that the friction force must begin changing. However, when the block speed changes, the spring begins to relax since the load point speed is constant. Equation (29) states that the spring force drops less quickly than the friction law force for imagined small jumps in slip rate and that force is thus imbalanced toward the direction of motion. A similar argument applies for imagined sudden small drops in velocity.

A surprising feature of the stability criterion (29) is that it does not contain explicit reference to the direct velocity dependence F_V neglected in the last paragraph. That is, only the amount of the steady state rate weakening, and the characteristic distance d_c, determinine whether or not stable sliding is possible in a given system, no matter how large the transient velocity strengthening. However, as can be seen from (28)–(32), the direct velocity dependence F_V does determine whether or not oscillations can occur, what their wavelength is, and, if sliding is unstable, at what rate instabilities grow. The lack of importance of F_V to stability does not generalize to friction dependent on more than one state variable [*Rice and Ruina*, 1983].

The solution of (28)–(32) for $k \to \infty$ implies that for a very stiff system, perturbations decay to steady state as in the constitutive law for constant slip velocity. As the stiffness goes to zero the constant force loading results are approached (compare (29) to (24)). At intermediate stiffness the characteristic lengths V/s in the exponential solutions are not d_c. For example, for compliances not quite large enough to cause oscillatory approach to steady state (equality in (31)) the characteristic distance of the approach to steady state $v/s = 2d_cF_V/(-d\tau^{ss}/dV)^{1/2} \neq d_c$. Thus, observed characteristic distances are "machine" dependent.

Simultaneous solution of the spring-block constraint (25) with the simplified Dieterich Law (not linearized) (18) has been carried out numerically and is shown in Figure 5. The steady state solution is perturbed by suddenly changing the load point velocity a small amount. In a very stiff machine the effect is small and gives the result that would be predicted by a sudden change in the rigidly controlled slip velocity. With more compliance decaying oscillations are observed. At neutral stability, equality in (29), oscillations persist with the wavelength of about $2\pi d_c$ as predicted by (32) with $B = 2A$ in (18). With still a more compliant machine oscillations grow, beyond the applicability of the previous linearizations, to a massive instability. This is indicated by the near vertical slope in force versus load point displacement implying rapidly increasing slip velocities. This indicates the failure of the quasistatic calculation and the onset of dynamic instability.

Oscillations like we discuss here were first noticed by *Scholz et al.* [1972] and were seen to occur in the transition from stable sliding to dynamic stick slip as the normal stress was increased. Assuming μ is independent of normal stress σ, as discussed, increases in normal stress are equivalent to decreases in stiffness k as emphasized by *Dieterich* [1978, 1979*a*]. In experiments where the stiffness was electronically controlled and gradually reduced [*Ruina*, 1980], there was a transition from steady state to decaying oscillations to large sustained oscillations bordering on dynamic stick slip. These large oscillations, incidentally, have an alternating amplitude as will be mentioned again.

Oscillations of this type are seen in some of the results of *Summers and Byerlee* [1976] as well as unpublished work of Teufel. That these works do not show these small quasistatic oscillations more frequently may be because in fault gouge layers instabilities of the type discussed thus far are mixed with instabilities associated with localization of deformation in the gouge layer [*Ruina*, 1980].

RELATION TO TIME, RATE, AND DISPLACEMENT DEPENDENCE

Time Dependence

The term "time dependent" is frequently used to describe any behavior that has a rate dependence and for which, consequently, results depend on experiment duration and/or rate. All the friction laws in this paper fall in this class. *Dieterich* [1972, 1978, 1979*a*, 1980] has used the words "time dependent" in a much more specific sense, however. He observed [*Dieterich*, 1972], following similar experiments with metals, that the force required to initiate slip rocks ("static friction") increases with the time of nominally stationary contact. He

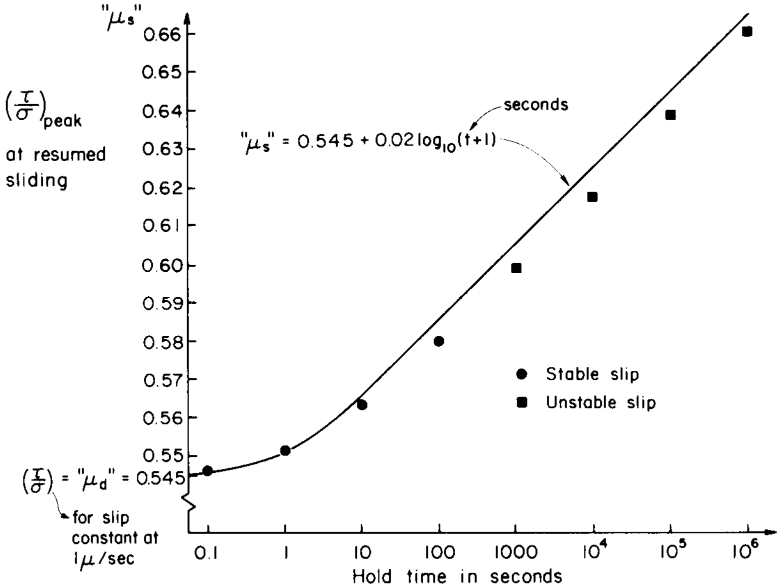

Fig. 6. Apparent time dependence of static friction. Equation (19) was used (with $\mu_0 = 0.545$, $A = 0.011$, $d_1 = 0.25$ μm, $d_2 = 5.2$ μm, $B_1 = 0.011$, $B_2 = 0.0092$, $\sigma = 10$ MPa, $V_c = 1$ μm/s.) in combination with the spring-slider model ($k = 3.3$ GPa/cm) to numerically simulate static friction experiments. The load point moved steadily at 1 μm/s, stopped for a time, and then moved again at 1 μm/s. The peak force with resumed sliding is denoted "μ_s." For the circles resumed sliding was stable, for the square resumed sliding led to infinite slip speed. The line shows *Dieterich*'s [1978] empirical fit to "static" friction data.

then [*Dieterich*, 1978, 1979*a*] went on to propose that any slip history can be characterized by the "effective time of contact." The variable he proposed is equivalent to ϕ defined by equation (14*d*). It has the flaw, as mentioned earlier, that it does not evolve with time if the slip rate is zero and thus can in no simple way be called "effective time of contact." More recently, *Dieterich* [1981] has been using a ϕ, defined by equation (16), that does allow an interpretation as effective time since $\dot{\phi} = 1$ when $V = 0$. Experimental work with metals [e.g., *Johannes et al.*, 1973] has revealed that the notion of "time dependence" of static friction was a consequence of a particular experimental design. Instead, they proposed that static friction depends on the rate of increase of the tangential loading at first slip.

In terms of the constitutive laws being promoted in this paper (equations (3), (4), (17), (18), or (19), however, static friction experiments should be viewed as slip histories. Static friction is then the peak friction force in a particular "static friction" slip history. If a static friction experiment consists of a load point, as in Figure 4, being stopped for some time and then moved as constant rate, then the friction force can be calculated through the interaction of the surface with the loading (e.g., (25) if mass can be neglected and only one state variable is used). The results of a numerical calculation of this kind are shown in Figure 6. The two state variable friction law, equation (19), was used in this numerical simulation. The load point was moved at constant rate and then stopped for successively longer times. Motion was then resumed at the original load point speed of 1 μm/s. The "static friction" is the peak force in the subsequent motion.

The constants in the friction law were chosen to fit the experiments of *Ruina* [1980]; the spring constant 3.3 GPa/cm and normal stress (10 MPa) are approximately those of *Dieterich* [1978]. Figure 6 clearly shows an apparent time dependence of static friction. In fact, after about 1 s the "static friction" coefficient increases by about 0.02 for each factor of 10 in nominally stationary contact time. Also plotted is the curve $\mu_s = 0.545 + 0.02 \log_{10}(t + 1)$ which is the empirical curve reported by *Dieterich* [1978]. The friction law (19) gives no change of state for $V = 0$, however. The apparent static friction is due entirely to small amounts of slip that take place while the load point is still, not time of contact. For the simulated experiment just described, roughly the same apparent time dependence is obtained for a broad range of stiffness (0.03 GPa/cm $< k <$ 30 GPa/cm). This is consistent with Dieterich's observation that time dependence does not depend on normal stress (since normal stress and stiffness are reciprocally related in our modeling). However, both $k \rightarrow 0$ (since the stopped load point is not sensed by the slider) and $k \rightarrow \infty$ (since in displacement control, with no slip rate, the state variables in (19) do not evolve) gives no apparent time dependence in the above simulation.

I do not want to claim that no healing is ever possible when there is no slip, but that static friction cannot necessarily be characterized by a single number like "time of stationary contact."

Similarly, the negative dependence of "static" friction on load rate proposed in *Johannes et al.* [1973] may also be seen as the manifestation of a state variable constitutive law in particular experiments. A necessary condition for an apparent negative dependence of "static" friction on load rate is that the state variable(s) evolve more quickly at a slip rate slightly below steady state than at a much slower slip rate. This is true of the state variables in equations (18) and (19).

Rate Dependence

The constitutive laws (equations (3), (4), (17)) have two kinds of rate dependence: (1) The instantaneous positive (viscous-like) rate dependence in the function F and (2) the long-term rate dependence after evolution to steady state. The short-term rate dependence seems always to be positive and the long-term rate dependence is often negative. For very slow changes in slip rate the approximation that the state variables

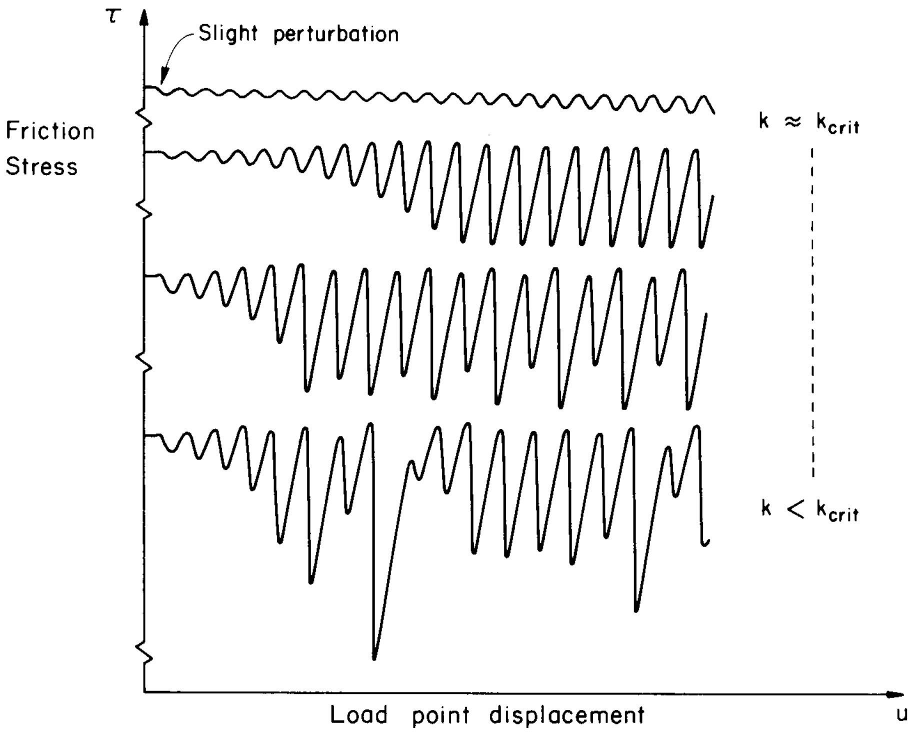

Fig. 7. Spring-slider with two state variable laws and constant load point rate. Top curve has k slightly less than k_{crit} for neutral linear stability. k diminishes for lower curves.

and thus τ have their steady state values, $\tau = \tau^{ss}$, $\theta_i = \theta_i^{ss}$ is accurate. The widespread applicability of this approximation is likely the reason that the memory effects discussed here were not well understood long ago.

Displacement Dependence

Though direct displacement dependence has been excluded (since δ does not appear explicitly in equations (3), (4), or (10)) and since a steady state has been postulated), the friction force does vary in a way that may appear as displacement dependent (e.g., the transients in Figure 1). *Dieterich* [1978] has utilized this approximation for his qualitative analysis of stability for stepwise constant rate motion in the spring block model of Figure 4. Additionally, a particular subset of state variable laws (including e.g., 19) satisfy a scaling rule that in particular experiments produces apparent rate independence [*Ruina*, 1980].

FURTHER IMPLICATIONS OF STATE VARIABLE FRICTION LAWS

We propose that state variables are a reasonable way to describe the constitutive laws for frictional slip. These constitutive laws than can be used in mechanical models as has been done by *Dieterich* [1979*b*, 1980], *Ruina* [1980], *Mavko* [1979], *Gu et al.* [1983], *Rice and Ruina* [1983], and *Kosloff and Liu* [1980]. Some results of such modeling besides that reported here are briefly described below.

Oscillations

The single state variable friction law coupled with a spring and massless slider has been shown here, in a linearized analysis, to lead to oscillations at a critical stiffness k_{crit}. Nonlinear analysis of *Gu et al.* [1983] based on equation (18) shows that these oscillations persist for a finite range of amplitudes. The linear analysis has been generalized in *Rice and Ruina* [1983] to include any friction law yielding a picture like Figure 1*b*, whether or not it has a simple state variable description and also including inertia. The results show the generallity of the existence of oscillations at a critical stiffness which is increased by inertia.

Creep Waves

An analysis parallel to that with the spring block model can be done with a deformable elastic solid [*Ruina*, 1980; *Rice and Ruina*, 1983]. The oscillations in the spring-block model become propagating creep waves. Whether or not such waves are seen or can be seen in nature is not yet known.

Rate Scaling

A particular subset of the state variable laws obey a rate scaling rule [*Ruina*, 1980] that predicts (1) *Johnson*'s [1981] experimental observation that apparent time dependence of static friction scales with previous slip rate; (2) *Scholz et al.* [1972] experimental observation that the displacement scale of slip preceding instabilities is independent of the applied load rate; (3) stability criteria for steady slip should be independent of rate (*Teufel* [1981] only observes this over a limited range of slip rates, however).

Complicated Motion

Some observed features do not seem to depend in a qualitative way on whether a figure like Figure 1*b* is generated by a friction law governed by one or by more than one state variable. Figure 1*b* could look nearly the same (with the shape of the relaxation curve slightly changed as in Figure 2) and small sinusoidal oscillations at a critical stiffness would also occur if there were a number of state variables [*Rice and Ruina*, 1983]. However, the friction law (4), coupled with the spring and slider, is a system of three (or more) first-order differential equations if two (or more) state variables are used. It is known that nonlinear systems of the order of 3 (or more) may exhibit complicated motions.

For example, the experiments of *Ruina* [1980] showed the period of oscillations doubling when the testing machine stiffness was reduced below that required for neutral stability. Numerical modeling of the friction law (19) with a constant speed load point also shows this period doubling as well as other motions shown in Figure 7 as stiffness is decreased. The nonlinear modeling with two state variables is extended in *Gu et al.* [1983].

The complicated motion depicted in Figure 7 suggests that even a simple system (slider block) with simple boundary conditions (constant load point displacement rate) can lead to very complicated motions. One might extrapolate that earthquakes' motion and cycles may be essentially complicated due at least in part to the dynamics of the process and not necessarily complicated boundary conditions and/or initial conditions.

Future Work

Future work of many kinds is possible. The constitutive description is not accurately known especially at the low slip rate limits. This is important to understand since this governs the amount of healing between slip events. The mechanism of the observed constitutive laws is also not understood. Nonlinear stability analysis should eventually bridge the gap between classical episodic "stick-slip' and the near steady sliding results presented here. Whether or not a nonlinear analysis of the creep waves can lead to a phenomenon analogous to shear fracture is also unknown.

CONCLUSION

The state variable approach to friction laws has been introduced as a generalization of the work of *Dieterich* [1978, 1979*a*, 1980, 1981*b*]. The primary assumed quality of the state variables, the existence of a steady state, then leads to characteristic distances and allows analysis of steady sliding. Some of the unobvious implications of state variable descriptions are oscillations of a massless system, creep waves, chaotic motion, and apparent time dependence of static friction. Since many geological systems are governed by friction that is possibly of the type described, analyses extending those presented here may have important applications.

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