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Some of the material cut from Olsen & Papadopoulos' 12/88 article in Bike Tech.

— explaining the terms in lean + steer equ'ns
— describing Routh-Hurwitz stability criteria

EXPLAINING THE COEFFICIENTS

The M, C, K coefficients look like simple, single quantities, but that's an intentional illusion --- most are fairly complicated combinations of basic measurable parameters! To show you just how information-packed these coefficients can be, their physical significance is described below.

M_{LL} and M_{SL}: assume the bicycle is given a 'lean' angular acceleration $\ddot{L}$, with the steering held straight. The necessary leaning moment is given by M_{LL}, which is the rotary inertia of the rigid bicycle plus rider about the ground contact line. M_{SL} gives the steer moment needed to hold the handlebars straight. In part it is due to masses such as the handlebar bag, which tend to get left behind and thus turn the handlebars; but even the rider contributes --- the fact that he or she is being pushed sideways because of $\ddot{L}$ means that both tires would like to skid in the opposite direction. This tends to turn the front wheel if there is any mechanical trail.

C_{LL} and C_{SL}: these concern the bicycle leaning or rolling at a steady rate $\dot{L}$, with the steering held straight. The first, C_{LL}, is always <u>zero</u> --- simply rolling the bicycle at a constant rate requires no effort. However $C_{\delta L}$ <u>is</u> significant --- because of gyroscopic effects (which are proportional to bicycle speed), the tilting of each wheel requires a turning or pivoting effort. That for the front wheel is felt directly through the handlebars, while in addition the overall pivoting effort on the bicycle required to tilt both wheels leads to equal and opposite tire side forces, of which the front acts on the mechanical-trail lever arm.

K_{LL} and $K_{\delta L}$: Imagine that the bicycle is held tipped somewhat (L = constant) by badly installed training wheels, with the steering held straight by the rider. K_{LL} reflects the support required of the training wheels, due to the unbalanced weight of the bicycle plus rider; it is a large number. $K_{\delta L}$ gives the tendency of the handlebars to turn --- in part because of that handlebar bag forward of the steering axis, and more importantly because the upwards force of the ground on the front wheel (arising from the weight distribution) acts on the lever arm of the mechanical trail.

Okay, now everyone stand up and stretch, then we can cover the other six coefficients. These all have to do with the frame being held perfectly upright (L=0), with some activity going on in the steering department .... essentially like a tricycle.
Remember that turning the handlebars causes pivoting of the frame

about a vertical line (unless the mechanical trail is zero).

M_{LS} and M_{SS}: In this case imagine the steering being accelerated from rest ($\ddot{S}$) with the frame held upright. M_{LS} gives the support moment needed to hold the frame vertical --- since both the trusty handlebar bag and the rider are being thrust sideways to some extent. (Guess what --- $M_{LS} = M_{SL}$, exactly!) M_{SS} relates to the steering effort required to get every element of mass moving in its pivoting motion. (Remember that the frame and rider rotate about a vertical line through the rear contact; and if you took measurements of the front wheel's motion you would find that it is actually pivoting about a 'fixed' line through the front contact and nearly parallel to the steering axis.)

C_{LS} and C_{SS} (two hard ones): now we get to a fixed steering <u>rate</u> $\dot{S}$ with the moving bicycle held upright. C_{LS} represents the support required to prevent leaning, and is proportional to the bicycle's speed. It is partly due to gyroscopic effects, in that the pivoting rate of the front and rear wheels requires a leaning moment. In addition, increasing the steer angle of a moving bicycle whips the front end around in a faster and faster pivoting motion of the whole bicycle, so you (and your water bottle) tend to be left behind, causing a tipping tendency. Finally, the pivoting of the frame if there is any mechanical trail, combined with forward motion, causes it to travel in a circle, so centrifugal force also tends to tip the bicycle. C_{SS} gives the steering effort required to turn the handlebars at a constant rate, and is also proportional to velocity. Whipping the front end around as described above tends to leave behind the handlebars, handlebar bag, etc. which exert their resisting reactions at points offset from the steering axis. Also, the front end can only accelerate sideways in this fashion if the ground (assumed non-icy) pushes it, so the rotary inertial resistance of the whole bike about a vertical axis is felt through the lever arm of the mechanical trail. Finally, the centrifugal force of the abovementioned circular path has two steering-related effects: (A) the centrifugal force acting on the whole bike is carried by side forces at the wheels, and the forward one opposes the turning of the front wheel because of mechanical trail; and (B) centrifugal force on the handlebars also resists the turning.

K_{LS} and K_{SS} (at last) describe what happens to a tricycle in a steady turn (S = constant, L = 0). K_{LS}, which tells us the support moment required to prevent the frame from leaning, has three parts. (A) Imbalance is created because the handlebar bag is off to one side of the frame, while the front contact is off to the other side (the latter is why a rider held at the beginning of a time trial should keep his or her front wheel straight). (B) The bicycle travels in a curve so centrifugal force tends to tip it. (C) Both wheels pivot steadily because of the turn, so their gyroscopic reactions tend to tip the bike. (A) is independent of speed, while (B) and (C) are proportional to speed squared. K_{ss} is the steering torque required to hold our `tricycle'in a steady turn. It too has three parts: (A) If the head angle is less than ninety degrees, gravity tends to accentuate the steer angle even for a bicycle at rest, because turning the front wheel lowers the frame (an effect of mechanical trail) and also lowers the trusty handlebar bag. (B) As was the case for C_{ss}, centrifugal force due to the turn creates a steering moment due mostly to side force at the front wheel, but also from the handlebar bag. (C) The fact of a steady turn means that the front wheel has a gyroscopic reaction tending to tip the bike. If the head angle is not strictly vertical, a little of this is felt as a steer torque. Once again, (A) is independent of speed, while (B) and (C) are proportional to speed squared.

To summarize, the M terms (which are independent of bicycle velocity) give torque due to an angular acceleration, that is they are rotational inertias. The C terms (proportional to bicycle velocity) give torque due to an angular velocity, like a rotary damper or a gyroscope, but without any frictional or viscous effects (and sometimes even without actual gyroscopic effects). <u>The C terms are crucial to a bicycle's self-stability.</u> The K terms give torque per angle, sort of like a torsion spring, even though a conventional bicycle has no springs. And unlike traditional spring constants, these are often negative! The K terms are responsible for changes in bicycle behaviour as speed increases --- at low speeds gravity is more important, while at high speeds centrifugal and gyroscopic aspects predominate.

SKATEBOARDS, WHEEL FORCES, STABILITY

What's the payoff? An engineer could use these equations (perhaps slightly amended) for many purposes: Designing a bicycle-riding robot, or a tricycle with neutral handling, or a self-balancing skateboard. Calculating potentially wheel-damaging side forces in a rapid swerve, or the destabilizing aerodynamic effects of disk wheels and fairings. Understanding more complex problems (for example, the equations do not predict shimmy, which says that neglected factors such as frame flex are at work). But what about bicycle handling, or at least no-hands self-stability?

SOME HELP FROM ROUTH-HURWITZ (and just in time)

To go further than this, we must eliminate either L or S (it doesn't matter which) by combining the lean and steer equations into just one fourth-order equation that will look like this:

$$A_4\ddddot{L} + A_3\dddot{L} + A_2\ddot{L} + A_1\dot{L} + A_0L = 0.$$

These A coefficients are combinations of the M, C, and K terms discussed above, so you can see how involved they must be. However, we have finally come to an equation that we can use to talk about stability. One approach is just to solve the equation and note whether the bicycle falls over, however there is an easier method for predicting the stability of systems known to dynamicists as the Routh-Hurwitz tests. For the above fourth order case, these tests say that if <u>all the coefficients are greater than zero</u>, and also

$$(A_3 * A_2 * A_1) - (A_4 * A_1 * A_1) - (A_0 * A_3 * A_3)$$

is greater than zero, then the bicycle will be self-stable in the sense that even if it is knocked sideways, it will straighten up without rider intervention. (To put it another way, balance disturbances tend automatically to reduce to zero instead of growing.) If the bicycle fails any of these six tests, it must be unstable and

will require <u>some</u> degree of rider attention and control. Remembering that all of the A coefficients are comprised of the M, C, and K terms described above, and that many of these terms are functions of velocity, then the stability of a bicycle must also vary with velocity. In fact, a typical bicycle is unstable at low speeds, stable in some speed range, and slightly unstable at high speeds. Please note that a bicycle which is not self-stabilizing is often rideable! As an example, a unicycle is NEVER self-stabilizing and yet unicycles are ridden every day. Still, it is our hypothesis that a bicycle which assists in the balancing task will be easier to ride and require less attention than one which is doing its best to fall over.