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BICYCLE DYNAMICS

The Meaning Behind the Math

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Imagine that you lived in a world without single-track vehicles and wanted to build the first bicycle. Would you grab some tubing and a torch and begin a long cycle of experimentation with different combinations of frame geometry? Or would you get out your calculus and physics texts and arrive at a design mathematically?

Ever since the development of the bicycle, engineers, mathematicians, and physicists have been attempting to predict or model handling behavior using mathematics. In many areas of engineering, mathematical modeling is regarded as a valuable and practical design tool. Unfortunately, the mathematical description of bicycle handling is a challenging task, even if simplifying assumptions are made.

With some success, this task has been assumed by a number of researchers throughout cycling history. For instance, accurate equations describing bicycle motion were derived as early as 1899 by Whipple. However, no one has translated mathematical descriptions into practical design rules.

Then, in 1986, the Cornell Bicycle Research Project (CBRP) was created to apply modern scientific techniques to the engineering problems of the bicycle. The goal was to shed light on long-standing questions and develop engineering approaches and tools that would lead to better bikes.

One of the first projects was to model bicycle handling and make the resulting mathematics useful to the layman. As a first cut, CBRP attacked the problem at a level of complexity that included the important effects of steering geometry and mass placement, but avoided issues likely to make the resulting math too esoteric to apply easily.

The inclusion of factors such as rider behavior, frame flexibility, and sophisticated tire phenomena rapidly takes the mathematics into computer simulation and out of the realm of practical guidelines to designers. CBRP elected to work with a *basic bicycle model* that had rigid knife-edge wheels, a rigid rear frame including a rigidly mounted and immobile rider, and a rigid steerable front fork, including the front wheel, stem, and handlebar (Figure 1).

For the purposes of calculation, this basic model was represented by the equivalent but highly abstract skeleton model shown in Figure 2. Note the important dimension called "mechanical trail," the perpendicular distance from the front contact point to the steering axis.

Complications were also minimized by analyzing only the limited steer and lean angles of approximately straight-line riding. Thus, CBRP could keep the motion equations basic enough to be useful and could write simple and effective computer programs to answer other questions. By researching past mathematical models of bicycles, re-deriving the basic equations with new techniques, and comparing the results from other researchers, CBRP has developed a solid, usable mathematical base.

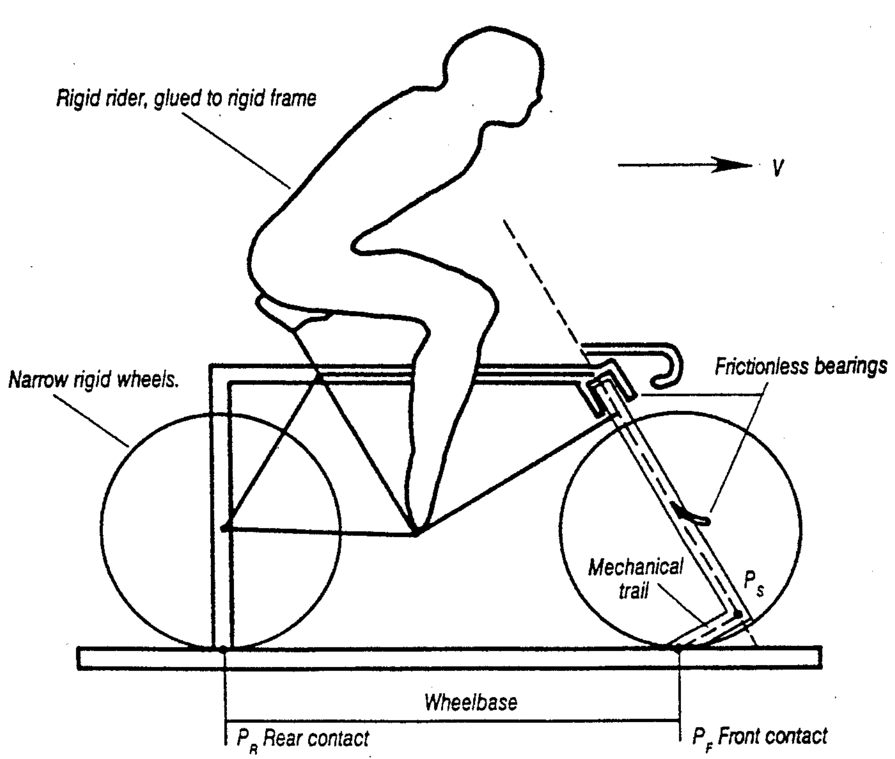
Handling Qualities

The subjectivity of bicycle handling poses a problem for mathematics and physics that requires precise variable definitions. For instance, how do you define good handling? No handling standard specifies factors such as the acceptable minimum and maximum path curvature for a given steering torque input.

In its initial efforts, CBRP has focused on the **self-stability** of the bicycle in no-hands riding; i.e., the tendency of a bicycle ridden by a "statue" no-hands-style to recover automatically from side winds or other tipping disturbances. This approach using common stability criteria from the science of dynamics is relatively easy and convenient but can't answer all handling questions.

Equations describing the motions (i.e., velocity and acceleration) of systems are called differential equations. For example, one simple type describes the motion of a weight

FIGURE 1



Exaggerated representation of the basic bicycle model. Note the definition of the highly important mechanical trail: the lever arm from the front contact P_F to the steering axis at P_S. The equivalent skeleton model is shown for clarity without its masses

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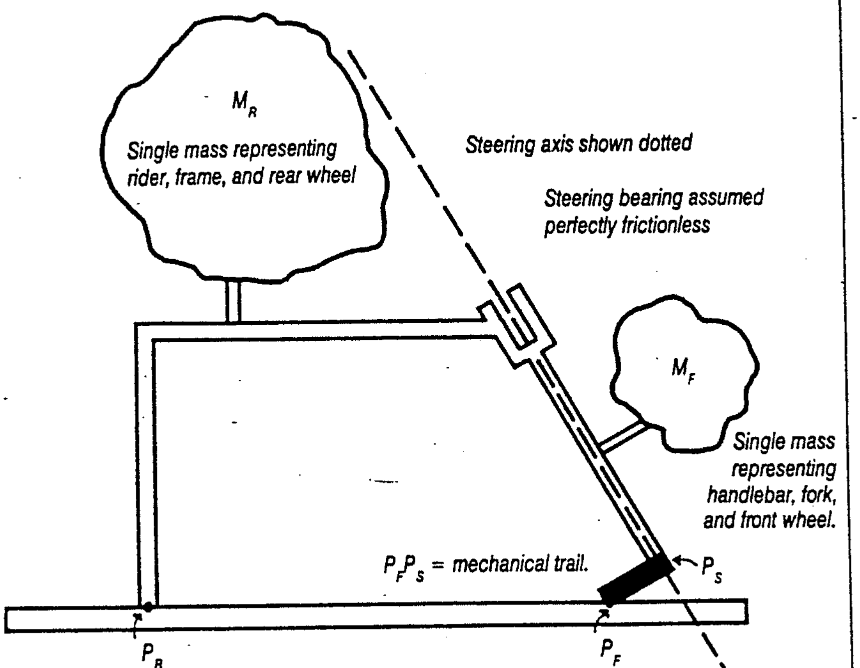
FIGURE 2 *An equivalent skeleton representation of the basic bicycle model, shown with rigid masses representing rider, handlebar, etc.*



P_R and P_F = front and rear contact. (It is legitimate to assume infinitesimal wheels if lean and steer angles are slight and gyro effects are accounted for.)

bouncing at the end of a spring. The equations describing a basic bicycle are not simple but are understandable. Two differential equations are needed to describe the lateral motion of the bicycle: one governing lean angle, the other steer angle. The terms of these two equations are as straightforward as mass or spring stiffness, but rather are involved functions of bicycle velocity, frame geometry, and various characteristics of the bicycle and rider's distribution of mass. The equations are also complicated by being "coupled"; e.g., leaning affects steering and vice versa.

Deriving Equations of Motion

The equations can be developed by considering the following variables and their derivatives, which define the lateral position and motion of the bicycle (Figure 3):

X—The rightward lateral position of the rear wheel contact point.

L—The rightward lean angle of the rear frame.

H—The leftward heading (or yaw) angle of the rear frame.

S—The leftward steer angle of the front fork relative to the rear frame. (It's initially assumed that the tires have no lateral grip, so the variables X, H, and S can each be varied independently.)

Using these variables, we first specify the total force required in the X direction as a sum of the forces needed for arbitrary "accelerations" of X,L,H and S, each considered individually. Using S = MA, we then set this total equal to the external forces acting on the bicycle in the X direction (in this case, the horizontal forces exerted by the ground on the tires).

For a second equation, we specify the total moment or torque around the ground contact line required to cause the accelerations of X, L, H and S, plus the gyroscopic effects. We set this moment equal to the external tipping moment acting on the bike because of gravity. Then we repeat this process and create a total yaw or heading moment (third equation) for torques acting around a vertical axis through the rear tire/ground contact point. The external moment here is attributable to the horizontal ground force at the front tire contact point.

Finally, we repeat this for the front fork and wheel to create a fourth equation describing torques around the steering axis. The external turning moments here come from the rider's steering effort, the side force of the road on the front tire times the lever arm of the trail, and gravity effects trying to turn the handlebar.

Once these equations are complete and correct, we use algebra to combine the four equations into two by eliminating the two unknown horizontal tire forces. Next, the grip is restored to both tires, which interrelates X, H, and S (but not lean angle, L, which can always vary without changes in the other three variables).

Now changes in X and H can be expressed in terms of S, so the two equations involve only the unknowns L, rightward lean angle and S, leftward steer angle, both measured in radians.

The lean equation resembles the following:

$$M_{LL}\ddot{L} + C_{LL}\dot{L} + K_{LL}L + M_{LS}\ddot{S} + C_{LS}\dot{S} + K_{LS}S = 0$$

The dots represent derivatives—one for first derivative or "rate" and two for second derivative or "acceleration."

The right side of the equation actually represents external forces affecting the lean of the frame and wouldn't be equal to zero if, for instance, training wheels were supporting the bicycle or a sidewind were tipping it.

The steer equation resembles the following:

$$M_{SL}\ddot{L} + C_{SL}\dot{L} + K_{SL}L + M_{SS}\ddot{S} + C_{SS}\dot{S} + K_{SS}S = \text{steer torque}$$

(Steer torque is equal to zero for no-hands riding.)

Note how the 12 M, C, and K coefficients (fixed numbers at any given speed) are labeled: The first subscript refers to the equation (lean equation or steer equation), while the second refers to the variable or its derivative that the coefficient multiplies. Both equations represent rotational effort; i.e., torque or moment. Thus, each product of a coefficient and variable is one part of the total lean moment or steer torque.

The M,C,K coefficients look like simple, singular quantities, but that's an intentional illusion. Most are fairly complicated combinations of basic measurable parameters. The C terms are crucial to a bicycle's stability. The K terms are responsible for changes in bicycle behavior as speed increases; e.g., at low speeds gravity is more important, while at high speeds centrifugal and gyroscopic aspects predominate.

Explaining the Math

When equations such as these are derived rigorously, they include about every physical phenomenon uncovered by technically inclined cyclists, including the following:

- With a more shallow head angle, turning the front wheel involves less "steering" and more "flop."
- When the frame is held vertical and the handlebar is turned to the left, mechanical trail causes the frame to pivot leftward slightly about a vertical line through the rear contact.
- Any sideways force of the ground acting on the lever arm of the mechanical trail tends to turn the steering.
- The gyroscopic effects of the rotating wheels include the need for a tipping effort to sustain a turning motion, and the need for a turning effort to sustain a leaning motion.
- Turning the handlebar of an upright bicy-

cle *lowers* the bicycle's center of mass (the reason the front wheel turns to one side when the bicycle is at rest).

Wheel Forces and Stability

What's the payoff? An engineer could use these equations for many purposes, including designing a bicycle-riding robot, a tricycle with neutral handling, or a self-balancing skateboard. He could also calculate potential wheel-damaging side forces in a rapid swerve or the destabilizing aerodynamic effects of disk wheels and fairings. Finally, the equations could lead to an understanding of more complex problems, such as shimmy in which neglected factors like frame flex play a part.

But what about bicycle handling, or at least no-hands self-stability? One approach is to solve the equation and note whether the bicycle falls over. However, there's an easier method for predicting the stability of systems known to dynamicists as the Routh-Hurwitz tests. If certain combinations of the M, C, K coefficients are all greater than zero, the bicycle will be self-stable; i.e., even if it's knocked sideways it will straighten without rider intervention. (Balance disturbances automatically tend to reduce to zero instead of growing.) If the forward traveling bicycle fails any of these requirements, it must be unstable and requires some rider input.

In fact, a typical riderless bicycle is unstable at low speeds, stable in some speed range, and slightly unstable at higher speeds. But note that a bicycle that is not self-stabilizing is still often ridable; it's just that one which assists in balancing should be easier to ride and will require less concentration than one with a tendency to fall over.

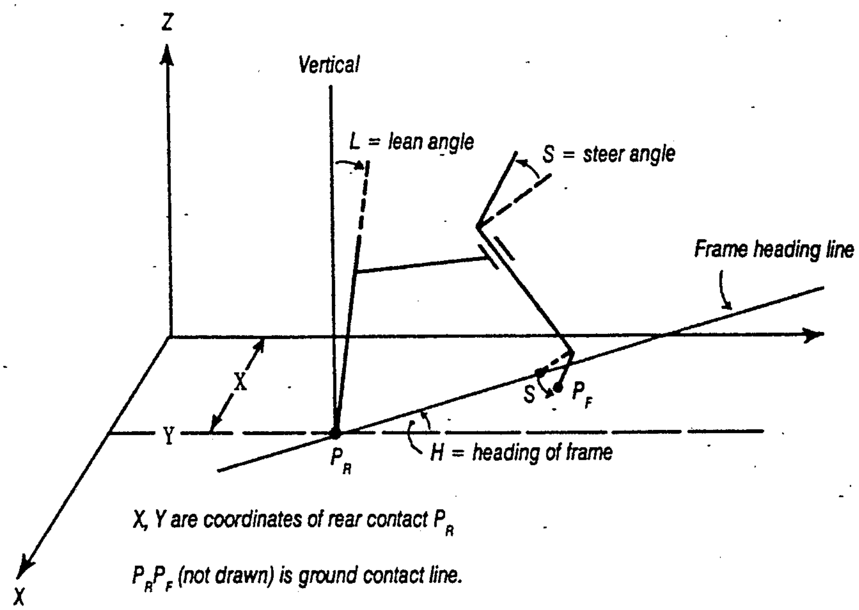
Consider these additional observations resulting from the equations:

- While many cyclists assume that increasing self-stability will make a bicycle sluggish and unresponsive, this isn't necessarily true. With further refinements to the typical bicycle's geometry and mass distribution, we may be able to combine improved self-stability with improved maneuverability and agility.
- A bicycle with *conventional* geometry and mass distribution would never be self-stable if the gyroscopic effects of the front and rear wheels were sufficiently reduced. Neither could a conventional bicycle be self-stable with zero or negative trail; i.e., with the front tire/ground contact point either on or ahead of the steering axis.
- Special unconventional bicycles have been designed that lack either gyroscopic effects or positive trail, yet are still extremely stable.
- A "primitive" bicycle with a vertical steering axis and no trail or handlebar extension is always slightly unstable.
- For a standard bicycle, the stable speed range has upper and lower limits. Below the lower limit the instability is oscillatory, with the bicycle overcorrecting its tilt and weaving increasingly back and forth. Above the upper limit the instability is exponential: The steering automatically tends to correct a lean imbalance, for example, but never quite enough. The bicycle leans more, which results in a further steering correction, then leans still more, eliciting a greater correction, etc., resulting in a slowly spiralling path and a crash.

At the upper stability limit called the *capsize speed*, an unguided bike remains in a steady turn by itself; just below this speed, it slowly straightens up. Below the capsize speed, a small steering torque is needed to prevent the steering from tightening the turn; at the capsize speed, this torque is unneeded—an uncontrolled bicycle remains in a turn by itself. Above the capsize speed, the torque has the opposite sign: The handlebar automatically moves to straighten out the steering even though the bicycle is leaning. This causes a capsizing. Paradoxically, this mild high-speed instability is gyroscopic in origin.

- Reversing the fork on a *standard* bicycle can make it self-stable at much higher speeds. Increasing the inertia of the front wheel increases a bicycle's low-speed stability.

FIGURE 3



Skeleton model (shown without its masses for clarity) illustrating definitions of lateral motion variables X (rightward translation), L (rightward lean angle), H (leftward heading angle), and S (leftward steer angle). All angles are in radians.

Computers to the Rescue

The equations are still being explored mathematically by CBRP, with hopes to develop useful, general design rules. But, to make the practical application of the bicycle handling model easier, CBRP researchers have also developed a computer program that calculates the Routh-Hurwitz quantities from the basic measurable parameters used to describe a real bicycle. This program, written in FORTRAN and intended for the IBM PC and compatibles, determines the numerical values of the individual coefficients for any chosen bicycle configuration and also with the no-hands stable speed range. The program is still being refined and expanded.

The aura of inscrutability surrounding the physics of bicycle handling has begun to crack. In principle, information regarding the handling behavior of bicycles still on the drawing board is now available to anyone with a personal computer. Cyclists have come a long way without much help from the science of dynamics. Who knows where we can go with this powerful discipline working for us.

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The Cornell Bicycle Research Project will share the current version of its Bicycle Dynamics computer program with interested parties. For more information, contact Jim Papadopoulos at M&AE, Upson Hall, Cornell University, Ithaca, NY 14853; (607) 255-3618. Also available are several papers and a master's thesis containing a detailed derivation of the bicycle handling equations in this article and a review and thorough comparison of previous work in the field.