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BICYCLE AND MOTORCYCLE BALANCE AND STEER DYNAMICS

I. Linearized Equations of Lateral Motion for a Simple Model

Jim M. Papadopoulos
Assistant Professor
Mechanical Engineering
Northern Illinois University
DeKalb, IL 60115
(815) 753-8053

R. Scott Hand
Company
City, State
Phone

Andy Ruina
Theoretical and Applied Mechanics
Thurston Hall, Cornell University
Ithaca, NY 14853
(607) 255-7108, 5035

Abstract

The linearized equations of motion for lateral motion for a basic bicycle model are presented. The dynamical equations are compared to variously succesful attempts to write such equations in the past ninety years. A basic bicycle consists of four linked rigid bodies which have a symmetry plane. Two of the bodies, the wheels, are also axisymmetric about an axis perpendicular to this plane. With the assumption that the wheels do not leave the ground the linearized dynamical equations reduce to four coupled second order differential equations. When the further assumption that sideways motion is constrained by rolling contact the equations reduce to two coupled second order ordinary differential equations. These equations may be generalized to describe special systems such as a bicycle with compliant tires, and a bicycle augmented with various gyro-stats, dashpots, springs or automatic controllers. The equations also may be reduced to various primitive bicycles, to a skate-board, or to a tricycle. It is shown that bicycles of current design are balanced primarily by means of the horizontal acceleration of the support points.

Introduction

Motivation

How does a bicycle balance, or how does a person balance on a bicycle? This has been a question of popular and technical interest since the late nineteenth century. One can find many papers and theses on the topic, which we review below, and further proprietary research has also been done. But one cannot find any standard models or sets of governing equations that are consensually accepted as correct or useful for addressing steer and balance issues. So although modern bicycles steer and balance remarkably well, how they do this is still something of a mystery. One hopes that good understanding of the mechanics relevant to answer this question might also further evolution of bicycle design. For example, how might radical changes in bicycle frame geometry be expected to improve bicycle handling, or, alternately, how do steer and balance issues restrict radical changes in bicycle geometry that one may want to make for other purposes.

A first step in addressing such questions is to find governing equations for a reasonably complete, though not intractably elaborate, model. Because we could not find a documentably correct set of such equations we thought it useful to present here a carefully checked set. The equations apply to a model which we term the basic bicycle model. We present the linear equations in a form that is more clear than any we have seen. We have derived the equations two different ways, only one of which is presented here in a somewhat abrieviated form.

We have used our equations to check the literature. Equations for this model or closely related models have been presented many times. However, no set of equations of motion has been agreed upon and most of the literature lacks comparisons to previous results. As previously presented, the equations are often in error and/or are impenetrably complex in form. There are few cross references between the papers so that a reader cannot, without substantial work, determine those few references which present correct equations. As a result, no general guidelines or design criteria exist for bicycle designers, builders or consumers.

Determination of correct equations is not profoundly difficult, but the frequent errors in the literature follow from complex features of the model. First, the basic bicycle is composed of four interconnected rigid bodies moving in three dimensions so many parameters are required as well as three dimensional geometry. Second, the constraints in this problem are nonholonomic (the number of degrees of freedom is fewer than the number of generalized coordinates required to define the configuration of system). Third, the geometry is complex so that the exact non-linear governing equations can only be written in an implicit form.

The equations presented below can be used directly to see dynamical effects of frame geometry (e.g.,"trail" and "head angle"), and weight distribution. The basic bicycle model equations may also be applied to other single-track vehicles including motorcycles, scooters. With simplifying constraints the equations may be made to apply to models of skate boards, tricycles, wheel chairs and shopping carts.

The equations of motion for the Basic Bicycle can be directly augmented to be applicable for motion with a given control strategy, given tire models, or given models of external loads (such as wind). The linearized basic bicycle equations may also serve as a special consistancy-check case for more elaborate models, such as those taking account of frame or rider distortion, or of large lean and stear angles.

It is intended that this paper may serve as a reliable basis for future work on bicycle or motorcycle dynamics. A second paper will investigate the dynamics of uncontrolled motion for the basic bicycle model and for some of the related special cases.

Brief History of Bicycle Dynamics Equations

The study of bicycle and motorcycle stability has attracted attention from mechanicians such as Rankine [1869], Sommerfeld and Klein [1903], Timoshenko [1948], Den Hartog [1948], Niemark and Fufaev (1967), Kane [1975] and many others. Investigations have ranged from purely empirical analyses to nonlinear computer simulation studies.

As it turns out, our equations are entirely consistent with those of two previous presentations: Döhring (1955) and Weir's PhD thesis (1972). They also agree with Whipple (1899), after typographical corrections, Neimark & Fufaev (1967), after correcting errors based on an incorrect potential energy, with R. Sharp (1971), with a minor algebraic correction, and with Weir and Zellner (1978), after correcting some minor errors. This paper also agrees with the thesis of Hand (1988) and the unpublished note of Papadopoulos (1987), the two primary sources for this presentation. Seemingly correct equations for somewhat simplified models are presented by Carvallo (1901), Sommerfeld & Klein (1904), Timoshenko and Young (19–), and Kane (19–). Other authors' equations disagree with these and each others', or are so complex in presentation that we have not been able to make detailed comparison.

Table 1 summarizes the history of the governing equations for the basic bicycle model to the extent that we have checked it. A more complete chronological history is presented with more detail in the Appendix.

The Basic Bicycle Model

In building the basic bicycle model we make the following simplifying assumptions about the bicycle-rider system and its behavior. For reference, parts of the bicycle are defined in figure 1 which also shows an outline used for dimensions in later figures.

1. The bicycle consists of four rigid bodies: the rear frame with fixed rigid rider; the front fork/handlebar assembly; and the front and rear wheels with knife-edge outer edges.
2. The bicycle-rider system is symmetric about the vertical plane passing lengthwise through the middle of the rear frame. That is, when the bike is in its vertical equilibrium position the wheel axles are perpendicular to the plane, the steering axis is in the plane, the wheel contacts are on the plane and all mass is symmetrically distributed with respect to the plane.

3 The wheels are rotationally symmetric about their axles.

4. Within the bicycle-rider system there is no friction or propulsion acting on the wheels (e.g. no friction or pedaling torques between the wheels and axle or frame).

5. No fore and aft loads are applied. External forces applied to the bicycle are: a) vertical body forces due to gravity, b) vertical constraint forces from the ground on the wheels (keeping the bicycle from penetrating the ground), c) lateral loads at the base of the wheels (from road contact), d) arbitrary lateral loads on the front and rear assemblies e) arbitrary roll, yaw and steer moments on the front and rear assemblies (such as from side winds, training wheels or controllers).

6. Only small disturbances from the vertical straight-ahead (at finite speed) steady-state motion position are considered. The equations we consider are linear equations. Terms not affecting these equations are often neglected (rather than being presented and dropped).

7. We assume constant forward velocity, though in fact constant (to first order) velocity is a consequence of the linearization.

Further assumptions are made for various special cases of interest:

8. The lateral load at the wheel contacts is just that required to cause no side slip between the wheel and the ground.

9. The wheels rotate at a rate such that their outer edges contact the ground with no relative velocity (ie, forward motion is directly coupled to spin angular momentum).

10. Various external loads are combined or neglected entirely so that steering torque, training wheels, wind loads, and uncontrolled riding can be described.

With the first assumption we have neglected any deformation of the rear frame, wheels, and front fork/handlebar assembly. All motion of the rider relative to the bicycle rear frame (desired or undesired) is also neglected. We also neglect the motion of the chain and crank assembly. The linearization obviously forbids investigation of various non-linear phenomena or even a determination of the amplitudes of motion for which the linearized description is usefully accurate.

The basic bicycle is symmetric in that the front and rear assemblies are endowed with an identical set of attributes. A basic bicycle moving backwards is still a basic bicycle (though quantitatively different).

Definition of Terms

Terms and variables used in the analysis are defined in this section. Figures 2a-d illustrate many of the variables.

Parts of the bicycle

The letters F, R and T are used as superscripts and subscripts to identify which part of the bicycle a mass, length or inertia term applies.

- F denotes the Front assembly including the handlebars, fork and the front wheel, basket (e.g. of groceries).
- R denotes the Rear assembly including the rear part of the frame, the rigidly attached rigid rider, and the rear wheel.
- T denotes the Total bicycle including all the parts of the frame, wheels and the rider with the front wheel in the straight ahead configuration.

Points on the bicycle

Points are all labelled P with a subscript. They are defined in the upright, straight ahead configuration. The same subscripts are also used for the description of length, mass, and inertial quantities (e.g. the height of the point $P_{\bar r}$ is $h_{\bar r}$).

- P_r is the point where the rear wheel touches the ground.
- $P_{\bar r}$ is the center of mass of the rear (R) part of the bicycle.
- P_f is the point where the front wheel touches the ground.
- $P_{\bar f}$ is the center of mass of the front (F) part of the bicycle.
- $P_{\bar t}$ is the center of mass of the whole bicycle.
- P_g is the point where the steering axis hits the ground.
- P_a is the point on the steering axis which is also vertically above (or below) the rear contact.

Lengths and angles on the bicycle frame.

All lengths are described with the letters l, c and h.

- $l_{\bar r}, h_{\bar r}$ are the forward distance and height of the rear assembly center of mass ($P_{\bar r}$) relative to the rear wheel contact (P_r).
- $l_{\bar f}, h_{\bar f}$ are the forward and vertical position of the front center of mass ($P_{\bar f}$) relative to the front contact P_f.
- $l_{\bar t}, h_{\bar t}$ are the forward distance and height of the total bicycle center of mass relative to P_r.
- λ is tilt of steering axis back from vertical ($\pi/2$ minus the "head angle").
- c_f is "mechanical trail", the perpendicular distance of P_f (backward) from steering axis. It is $\cos\lambda$ times the ground length "trail".
- d is perpendicular distance of $P_{\bar f}$ (forward) from the steering axis. d is also given by $d = h_{\bar f}\sin\lambda + l_{\bar f}\cos\lambda - c_f$
- c_w is the "wheelbase", the distance between the rear and front wheel contact points (P_r and P_f).
- a_F, a_R are the front and wheel radius, respectively.

Reference Directions on the Bicycle Frame

x, y, z are righthanded orthogonal coordinate directions relative to an origin on the ground at the instantaneous position of the rear contact P^r. When the bicycle is in its straight ahead-upright reference direction, y is the forward direction, z is the vertical upwards direction and x the lateral (transverse or sideways) direction.

$\mathbf{j}, \mathbf{k}$ are unit vectors along y, z directions.

$\mathbf{n}_v$ $= -\sin\lambda\ \mathbf{j} + \cos\lambda\ \mathbf{k}$ or in component form $\mathbf{n}_v = (0, -\sin\lambda, \cos\lambda)$ is a more upwards than downwards unit vector parallel to the steering axis (in the y, z plane in the reference configuration).

u is distance in the generally forward direction perpendicular to the steering axis (perpendicular to the $\mathbf{n}_v$ direction).

v is distance along the steering axis (in $+\mathbf{n}_v$ direction).

Mass and Inertia Terms

Moment of Inertia matrices have components which are proper components of an inertia tensor (e.g. $I_{yz} = -\int yz\, dm$), the negative of an old fashioned notation for the off diagonal terms. All moments of inertia components are denoted with a capital I and three labels:

1) A pre-superscript denoting which object is being described (e.g. ${}^T I$ is the moment of inertia of the Total bicycle)
2) A superscript denoting the reference point (e.g. I^r is the moment of inertia about the point P_r).

Two subscripts denoting the components. Skew components are used so that, for example, I_{vz} means the scalar determined by the matrix $[I]$ dotted once with $\mathbf{k}$ and again with $\mathbf{n}_v$. So, for example,

$$
\begin{aligned}
{}^F I^{\bar{f}}_{vz} &= \mathbf{n_v} \cdot [{}^F I^{\bar{f}}] \cdot \mathbf{k} \\
&= -\int uy\,dm \qquad\qquad \text{(if the bicycle is planar)}
\end{aligned}
$$

where u and y are distance in the u and y directions relative to the front center of mass $P_{\bar{f}}$ and the integral is over all mass in the front (F) assembly, including the wheels. The brackets [] indicate the full inertia tensor (or matrix). Note that $I_{xy} = I_{xz} = I_{xv} = I_{xu} = 0$ for all bodies and all reference points since the yz plane is a symmetry plane for the bicycle in its reference configuration. Also, $I_{xx} = I_{yy} + I_{zz}$ if the bicycle is planar (ie, all mass is on the center plane of the rear frame). In cartesian form all inertia matrices are of the form:

$$
[I] = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & I_{yz} \\ 0 & I_{yz} & I_{zz} \end{bmatrix}.
$$

m_R is the mass of the rear assembly including the rear part of frame, the rigidly attached rider, and the rear wheel.

m_F is mass of the front assembly including the fork, handlebar, front wheel, and groceries.

m_T is the total mass of the bicycle and all attached componants $= m_R + m_F$

J_R, J_F are the polar mass moment of inertia of the rear and front wheels.

H_R, H_F are the spin angular momentum of the front and rear assemblies, respectively. If, for example, the front wheel has polar moment of inertia J_F and radius a_F and there are no further attached spinning objects, then $H_F = VJ_F/a_F$.

H_T is the total spin angular momentum of the bicycle due to spinning of wheels or attached gyros, that is $H_T = H_R + H_F$.

The inertial components which appear in the final equations of motion are written below in terms of center-of-mass inertias and also in terms of other previously defined masses, lengths and angles.

$$^TI^r_{yy} = {}^RI^r_{yy} + {}^FI^r_{yy} = \left[{}^RI^{\bar{r}}_{yy} + m_R h^2_{\bar{r}}\right] + \left[{}^FI^{\bar{f}}_{yy} + m_F h^2_{\bar{f}}\right]$$

$$^TI^r_{zz} = {}^RI^r_{zz} + {}^FI^r_{zz} = \left[{}^RI^{\bar{r}}_{zz} + m_R l^2_{\bar{r}}\right] + \left[{}^FI^{\bar{f}}_{zz} + m_F(c_w + l_{\bar{f}})^2\right]$$

$$^TI^r_{yz} = {}^RI^r_{yz} + {}^FI^r_{yz} = \left[{}^RI^{\bar{r}}_{yz} - m_R h_{\bar{r}} l_{\bar{r}}\right] + \left[{}^RI^{\bar{r}}_{yz} - m_F h_{\bar{f}}\ (c_w + l_{\bar{f}})\right]$$

$$^FI^g_{vv} = m_F d^2 + \mathbf{n}_v \cdot [{}^FI^{\bar{f}}] \cdot \mathbf{n}_v$$
$$= m_F d^2 + \left[{}^FI^{\bar{f}}_{yy}\ \sin^2\lambda - 2\,{}^FI^{\bar{f}}_{yz}\ sin\lambda\ cos\lambda + {}^FI^{\bar{f}}_{zz}\ cos^2\lambda\right]$$

$$^FI^g_{vy} = -m_F h_{\bar{f}} d + \mathbf{n}_v \cdot [{}^FI^{\bar{f}}] \cdot \mathbf{j} = -m_F h_{\bar{f}} d + \left[-{}^F\,I^{\bar{f}}_{yy}\ \sin\lambda + {}^FI^{\bar{f}}_{yz}\ \cos\lambda\right]$$

$$^FI^a_{vz} = -m_F d\ (c_w + l_{\bar{f}}) + \mathbf{n}_v \cdot [{}^FI^{\bar{f}}] \cdot \mathbf{k}$$
$$= m_F d\ (c_w + l_{\bar{f}}) + \left[-{}^FI^{\bar{f}}_{yz}\ \sin\lambda + {}^FI^{\bar{f}}_{zz}\ \cos\lambda\right]$$

Generalized Coordinates

At any time there is a coordinate system fixed to the stationary ground which is instantaneously aligned with the bicycle so that Y is on the ground along the line from rear wheel contact towards front wheel contact. The X asis is on the plane and perpendicular to Y. The wheels are assumed to be on the ground. We do not pay heed to the rotation angle of the wheels since they are assumed to be rotationally symmetric and, to first order in the perturbations we consider, the wheel rotation speed is constant. The forward speed of the bike is similarly constant), and equal at the moment in question to dY/dt.

We describe the configuration of the bicycle in terms of 4 generalized coordinates.

X is lateral motion of the rear wheel contact P_r from the Y-axis.

χ is lean (or roll) to the right of the rear assembly ("right hand rule" about the Y axis).

θ is heading (or yaw) of rear wheel relative to y-axis (positive is a right handed rotation about the z-axis)

ψ is steer to the left of the front assembly relative to rear assembly (right handed rotation of the front assembly about the v-axis)

Forces and Generalized Forces

In addition to the forces from gravity and ground reactions we also include the possibility of additional external forces which are conjugate to the generalized coordinates. The coordinates we use are simple enough so that these conjugate forces have simple interpretations.

Q_X is the additional external lateral force on the whole bicycle assembly.

Q_χ is the net additional external roll torque (moment about the Y axis) acting on the whole bicycle.

Q_θ is the net additional external yaw torque (moment about the Z axis) acting on the whole bicycle.

Q_ψ is the net additional external steering torque. It is the net torque on the front assembly about the steering axis.

The gravity forces and ground reactions are given below.

$m_R g$ is the downwards gravitational force acting on the entire rear assembly at $P_{\bar{r}}$.

$m_F g$ is the downwards gravitational force on the front assembly acting at $P_{\bar{f}}$.

$G_f = m_T g l_{\bar{t}}/c_w$ is the vertically up ground reaction at P_f.

$G_r = m_T g(1 - l_{\bar{t}}/c_w)$ is the vertically up ground reaction at P_r

$\mathcal{F}_{xf}$ is the transverse constraint force (in the positive X-direction) at P_f.

$\mathcal{F}_{xr}$ is the transverse constraint force (in the positive X-direction) at P_r.

Other Quantities

V is forward velocity, which is constant for the purposes of linearized analysis.

g is the gravitational acceleration.

$\nu = m_F d + m_T l_{\bar{t}} c_f/c_w$ is defined for convenience since it appears often in the equations.

Summary of parameters and variables

Seventeen parameters can fully define the bicycle mechanically. The front and rear assemblies each need 7 quantities: their masses, two center-of-mass co-ordinates relative to the wheel contact point, three center-of-mass inertia tensor components, and the ratio of spin angular momentum to forward speed. In addition to these 14, we need the head angle (the angle of the steering axis), and the distances from the steering axis to the two ground-wheel contacts. These quantities can be described various ways.

The configuration of the bicycle is described by 4 generalized coordinates.

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Equations of Motion

A basic bicycle with stiff non-slipping tires has three instantaneous degrees of freedom: roll, steer, and forward motion. It can be shown that (e.g. Hand 1978), assuming small perturbations from vertical, the perturbations in forward speed are second order in the roll and steer variables. To simplify this presentation, we assume that forward speed V is a constant. In order to derive roll and steer equations we first find four coupled equations for roll, steer, yaw and side-slipping motion. To get these equations we make use of the holonomic constraint that the wheels always just touch the ground. These ~~equations~~ four equations have sideways wheel contact forces which can be used to apply various tire models. Our central interest is with the simplest tire model: pure rolling of a disc. Using this model, the sideways wheel forces can be eliminated from the governing equations by enforcing the non-holonomic rolling constraint.

The generalized coordinates we have chosen X, χ, θ, ψ all correspond to rigid rotations or translations of a rigid body or bodies. If X alone varies, the whole bike translates sideways. If χ alone varies, the whole bike is rigidly tipped (or rolled) around the line between P_r and P_g. If θ alone varies the whole bike's heading (or yaw) is rigidly changed. Lastly, if ψ alone is varied, the front assembly is rotated rigidly about the steer axis. Note that for pure steer, P_g remains fixed and P_f slides laterally.

It is possible to define coordinates so that, even with the no-slip rolling constraint is enforced, the coordinates may still be independently varied on a stationary bicycle. The price one would pay for using such variables is a simplicity in the variable definitions. Also, the resulting set of 4 governing equations would not have such simple interpretations.

D'Alambert's Principle

Virtual work of all forces, including inertial forces, is zero.

Momentum Balance

Momentum and Angular momentum balance.

The Four Governing Equations

(1) <u>FOR THE WHOLE BIKE:</u> the total X-force required to effect the lateral acceleration of all mass points in a general motion = the sum of applied X-forces.

$$m_t\ddot{X} + m_t\bar{h}_t\ddot{\chi} - m_t\bar{l}_t\ddot{\theta} - m_f d\ddot{\psi} = \mathcal{F}_{Xr} + \mathcal{F}_{Xf}$$

LHS:

— the lateral acceleration of m_t due to lateral acceleration of the rear contact, no lean or yaw

— the X acceleration of m_t due to yawing (only) of the whole bike

— the X acceleration of m_t due to lean (only) of the whole bike

— the X acceleration of m_f due to steering only

RHS: sum of forces in X-direction

(2a) <u>FOR THE WHOLE BIKE:</u> the total χ-moment (about the heading line of the rear assembly), required when accelerating mass-points laterally in a general motion, = sum of χ moments of external forces about the same line.

$$m_t\bar{h}_t\ddot{X} + T_{yy}\ddot{\chi} + T_{yz}\ddot{\theta} + F'_{\lambda y}\ddot{\psi} - H_t\dot{\theta} - H_f\cos\lambda\dot{\psi} = gm_t\bar{h}_t\chi - gm_fd\psi - c_f(m_tg\frac{\bar{l}_t}{c_w})\psi$$

$$= gm_t\bar{h}_t\chi - g\nu\psi \ ,$$

where for convenience we have defined $\nu = (m_fd + m_t\frac{\bar{l}_t}{c_w}c_f)$.

LHS:

- the y (or χ) moment required to accelerate the center of mass of the whole bike laterally
- the y-moment required for angular acceleration ($\ddot{\chi}$, $\ddot{\theta}$) of the whole bicycle about the rear contact point
- the y-moment required for angular acceleration of the front assembly about the steering axis $\vec{\lambda}$ (remember that the contacts can slip sideways, and that X, χ, θ are being held fixed
- the gyroscopic moment (from wheel spin angular momenta H_r, H_f) required for yawing of the whole bike about a vertical axis
- the gyroscopic y-moment required for the precession about the vertical ($\dot{\psi}\cos\lambda$) of the front wheel

RHS:

- the moment of gravity force acting at the c.m. of the total bicycle which is leaned only (Fig.A8a)
- the moment of gravity force acting at the center of mass of the front assembly, for a non-leaning bicycle (Fig.A8b)
- the moment of vertical front contact force ($m_tg\bar{l}_t/c_w$) which is offset due to steer only (Fig.A8b)
- the steer moment $\mathcal{M}_\psi$ does not contribute because $-\mathcal{M}_\psi$ also acts on the bicycle. The forces $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$ do not contribute because they are in the ground plane and so have no moments about the rear-assembly heading line.

(2b) <u>FOR THE WHOLE BIKE:</u> the total θ-moment (about the z-axis through the rear contact P_r, which for small angles is equivalent to a *vertical* axis) required when accelerating mass-points laterally in a general motion = sum of moments of external forces about the same line.

$$-m_t\bar{l}_t\ddot{X} + T_{zy}\ddot{\chi} + T_{zz}\ddot{\theta} + F''_{\lambda z}\ddot{\psi} + H_t\dot{\chi} - H_f\sin\lambda\dot{\psi} = -c_w\mathcal{F}_{Xf}$$

LHS:

— the z (or θ) moment required to accelerate the center of mass of the whole bike
— the z-moment required for angular acceleration ($\ddot{\chi}$, $\ddot{\theta}$) of the whole bicycle about the rear contact point
— the z-moment required for angular acceleration of the front assembly about the steering axis $\vec{\lambda}$
— the gyroscopic moment required for tipping of the whole bike about the heading line of the rear assembly
— the gyroscopic z-moment required for the precession about the horizontal ($-\dot{\psi}\sin\lambda$) of the steered front wheel

RHS: moment of $\mathcal{F}_{Xf}$ about the z axis.

(3) <u>FOR THE FRONT ASSEMBLY ONLY:</u> the total ψ-moment about the steering axis $\vec{\lambda}$ required for a general bicycle motion = sum of external moments about the same axis.

$$-m_f d\ddot{X} + F'_{\lambda y}\ddot{\chi} + F''_{\lambda z}\ddot{\theta} + F'_{\lambda\lambda}\ddot{\psi} + H_f(\dot{\chi}\cos\lambda + \dot{\theta}\sin\lambda)$$

$$= \mathcal{M}_\psi + c_f\mathcal{F}_{Xf} - g(m_f d + m_t \tfrac{l_t}{c_w} c_f)\chi + g\sin\lambda(m_f d + m_t\tfrac{l_t}{c_w}c_f)\psi$$

$$= \mathcal{M}_\psi + c_f\mathcal{F}_{Xf} - g\nu\chi + g(\sin\lambda)\nu\psi\ ,$$

(where ν is defined in (2a) above).

LHS:

— ψ-moment required to support lateral acceleration of the front assembly
— Moments about $\vec{\lambda}$ axis required when the front assembly is given angular acceleration about the χ (y) axis and θ (z) axis. Because inertia tensors about any point are symmetric matrices, the moment about one axis required for angular acceleration about another is the same as the moment about the second required for angular accleration about the first.
— the moment about the steering axis required for angular acceleration of the front assembly about that axis (the coefficient is the polar moment of inertia)
— the moment about the steering axis required for precession ($\dot{\chi}\cos\lambda + \dot{\theta}\sin\lambda$) about an axis in the plane of the bicycle which is perpendicular to the steering axis.

RHS:

— the steering moment $\mathcal{M}_\psi$, and the moment about the steering axis of the horizontal force, are easy to see. (Fig.A9a)
— when the bicycle is leaned only, the vertical reaction force at the front contact and the vertical gravitational force on m_f both have components proportional to χ

which are perpendicular to the plane of the bike. These forces act on lever arms c_f and d. (Fig.A9b)

— when the bicycle is steered only, these two forces are displaced from the plane of the bicycle, and no longer pass through the steering axis. Resolve them initially into components perpendicular and parallel to the steering axis; then when they are displaced, it is easy to see that only the components initially *perpendicular* to the steering axis (which are multiplied by $\sin\lambda$) exert moments, with lever arms equal to their lateral displacements ψd and ψc_f. (Fig.A9c)

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Constraints

REDUCED EQUATIONS OF MOTION

Equations $1, 2a, 2b, 3$ are true whatever horizontal forces $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$ act at the wheel contacts, and in particular they are true if the forces are just right to prevent the wheels from side-slipping relative to their instantaneous headings.

[Need a discussion of side slip here......]

For the rear wheel, zero side slip is expressed by the equation $\dot{X} = -V\theta$. (See Fig.A10a.) For the front contact, an analogous relation is needed in terms of the X-co-ordinate and heading of the front wheel: $\dot{X}_f = -V\theta_f$. To write this in terms of the variables describing the motion, we need $X_f = X - c_w\theta + c_f\psi$ for the X-co-ordinate of the front contact P_f (Fig.A10b), and $\theta_f = \theta + \psi\cos\lambda$ for the heading of the front wheel (Fig.A10c). ($\cos\lambda$ comes in because a small rotation of the front wheel about the steering axis can be considered a sum of small rotations about horizontal and vertical axes — and only the latter changes the front wheel's heading.)

With these relations, we can express such quantities as $\dot{\theta}$, $\ddot{\theta}$ and $\ddot{X}$ in terms of and ψ and $\dot{\psi}$. By subtracting the rear-wheel constraint from the front-wheel constraint, we obtain

$$-c_w\dot{\theta} + c_f\dot{\psi} = V\psi\cos\lambda \ ,$$

which may be solved for $\dot{\theta}$ to give

$$\dot{\theta} = \frac{c_f}{c_w}\dot{\psi} + V\frac{\cos\lambda}{c_w}\psi \ .$$

This may be differentiated once to give $\ddot{\theta}$:

$$\ddot{\theta} = \frac{c_f}{c_w}\ddot{\psi} + V\frac{\cos\lambda}{c_w}\dot{\psi} \ .$$

Finally, the rear constraint relation may be differentiated once, and $\dot{\theta}$ may be replaced:

$$\ddot{X} = -V\dot{\theta} = -V\frac{c_f}{c_w}\dot{\psi} - V^2\frac{\cos\lambda}{c_w}\psi$$

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After substituting these relations into Eqs. 1,2,3, to eliminate $\dot{\theta}, \ddot{\theta}$ and X, we have four equations but apparently only two variables. However, demanding that the horizontal contact forces should prevent the wheel sideslip means that these forces are no longer freely selectable, but must be exactly the right magnitude at every instant. In essence, they are the two remaining variables.

Since we are concerned at present only with the *motion,* the most convenient thing is to rearrange the equations so that the unkown forces do not appear in two of them; these two will allow us to solve for the unknowns χ, ψ.

Equation (2a) (with X and θ eliminated) is already in that form; because it dealt with moments about a line in the ground it will be called the lean equation. For the other equation we simply eliminate $\mathcal{F}_{Xf}$ from (2b) and (3), and leave $\mathcal{M}_\psi$ on the right hand side; this is called the steer equation. (Evidently equation (1) is not needed, unless we wish to find $\mathcal{F}_{Xr}$.)

We write these two equations in the form:

$$M_{\chi\chi}\ddot{\chi} + M_{\chi\psi}\ddot{\psi} + C_{\chi\psi}\dot{\psi} + K_{\chi\chi}\chi + K_{\chi\psi}\psi = 0 \text{ , the lean equation*}$$

(note that there is no $C_{\chi\chi}\dot{\chi}$ term); and

$$M_{\psi\chi}\ddot{\chi} + M_{\psi\psi}\ddot{\psi} + C_{\psi\chi}\dot{\chi} + C_{\psi\psi}\dot{\psi} + K_{\psi\chi}\chi + K_{\psi\psi}\psi = \mathcal{M}_\psi \text{ , the steer equation.}$$

The coefficients to the lean equation are

$$\begin{aligned}
M_{\chi\chi} &= T_{yy} \\
M_{\chi\psi} &= F'_{\lambda y} + \frac{c_f}{c_w}T_{yz} \\
C_{\chi\chi} &= 0 \\
C_{\chi\psi} &= -\left(H_f\cos\lambda + \frac{c_f}{c_w}H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w}m_t\bar{h}_t\right) \\
K_{\chi\chi} &= -\,g m_t\bar{h}_t \\
K_{\chi\psi} &= g\nu - H_t V\frac{\cos\lambda}{c_w} - V^2\frac{\cos\lambda}{c_w}m_t\bar{h}_t
\end{aligned}$$

and the coefficients to the steer equation are:

$$M_{\psi\chi} = F'_{\lambda y} + \frac{c_f}{c_w}T_{yz}$$

* If we had allowed flexible training wheels (say) to help support the rear assembly against leaning, the lean equation would have to have the supporting moment $\mathcal{M}_\chi$ on the right hand side.

$$M_{\psi\psi} = F'_{\lambda\lambda} + 2\frac{c_f}{c_w}F''_{\lambda z} + \frac{c_f^2}{c_w^2}T_{zz}$$

$$C_{\psi\chi} = H_f \cos\lambda + \frac{c_f}{c_w}H_t$$

$$C_{\psi\psi} = V\left(\frac{\cos\lambda}{c_w}F''_{\lambda z} + \frac{c_f}{c_w}\left(\frac{\cos\lambda}{c_w}T_{zz} + \nu\right)\right)$$

$$K_{\psi\chi} = g\nu$$

$$K_{\psi\psi} = -g\nu\sin\lambda + VH_f\frac{\sin\lambda\cos\lambda}{c_w} + V^2\frac{\cos\lambda}{c_w}\nu$$

Note that most coefficients are functions of velocity. In fact the angular momentum H_f for the front wheel typically could be written as $H_f = V(J_f/a_f)$, where a_f is front wheel radius and J_f is front wheel polar moment of inertia; and similarly for the rear. However there is also the possibility of adding independent high-speed gyros to the bicycle, in which case H_f and/or H_r might be constant, or a negative multiple of speed, etc.

When developed in this form, the equations display a degree of symmetry.

Representation of the Equations of Motion

Equations (3.14) and (3.15) can be expressed either as two second order differential equations or one fourth order differential equation. For the case of two second order differential equation it is common to write the equations in matrix form as follows,

$$\left[\overbrace{\begin{pmatrix}\mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi}\\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi}\end{pmatrix}}^{\mathcal{M}}D^2 + \overbrace{\begin{pmatrix}\mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi}\\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi}\end{pmatrix}}^{\mathcal{C}}D + \overbrace{\begin{pmatrix}\mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi}\\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi}\end{pmatrix}}^{\mathcal{K}}\right]\begin{pmatrix}\chi_r\\ \psi\end{pmatrix} = \begin{pmatrix}M_{\chi_r}\\ M_\psi\end{pmatrix}$$

where $\mathcal{M}$, is the mass matrix , $\mathcal{C}$ is the damping matrix, $\mathcal{K}$ is the stiffness matrix, and D is the differential operator. The components of $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{K}$ are defined above.

$$\mathcal{M} = \begin{pmatrix}\mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi}\\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi}\end{pmatrix} = \begin{pmatrix}need & need\\ need & need\end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix}\mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi}\\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi}\end{pmatrix} = \begin{pmatrix}need & need\\ need & need\end{pmatrix}$$

$$\mathcal{K} = \begin{pmatrix}\mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi}\\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi}\end{pmatrix} = \begin{pmatrix}need & need\\ need & need\end{pmatrix}$$

Figure Captions

FIGURE 1. The basic bicycle is shown in shaded gray. A dark line sketch of the essential skeleton is superposed for definition in later figures. The wheels are not included in the basic skeleton since their properties only enter the linearized equations in an indirect manner.
Some parts of the bicycle are labeled for reference.

FIGURE 2. Terms used in the basic bicycle model are defined on the essential skeleton. **a)** Names of points, lengths, angles and directions on the frame, **b)** Names of inertia quantities. The quantities shown are not all independent, some can be found from others **c)** Coordinates used to describe the bicycle motion. **d)** Free body diagram of a bicycle showing applied forces. In addition to the four generalized forces applied to the frame are gravity and vertical and horizontal wheel contact forces.

Normal rider & bike in uniform grey.
Superposed frame in black.

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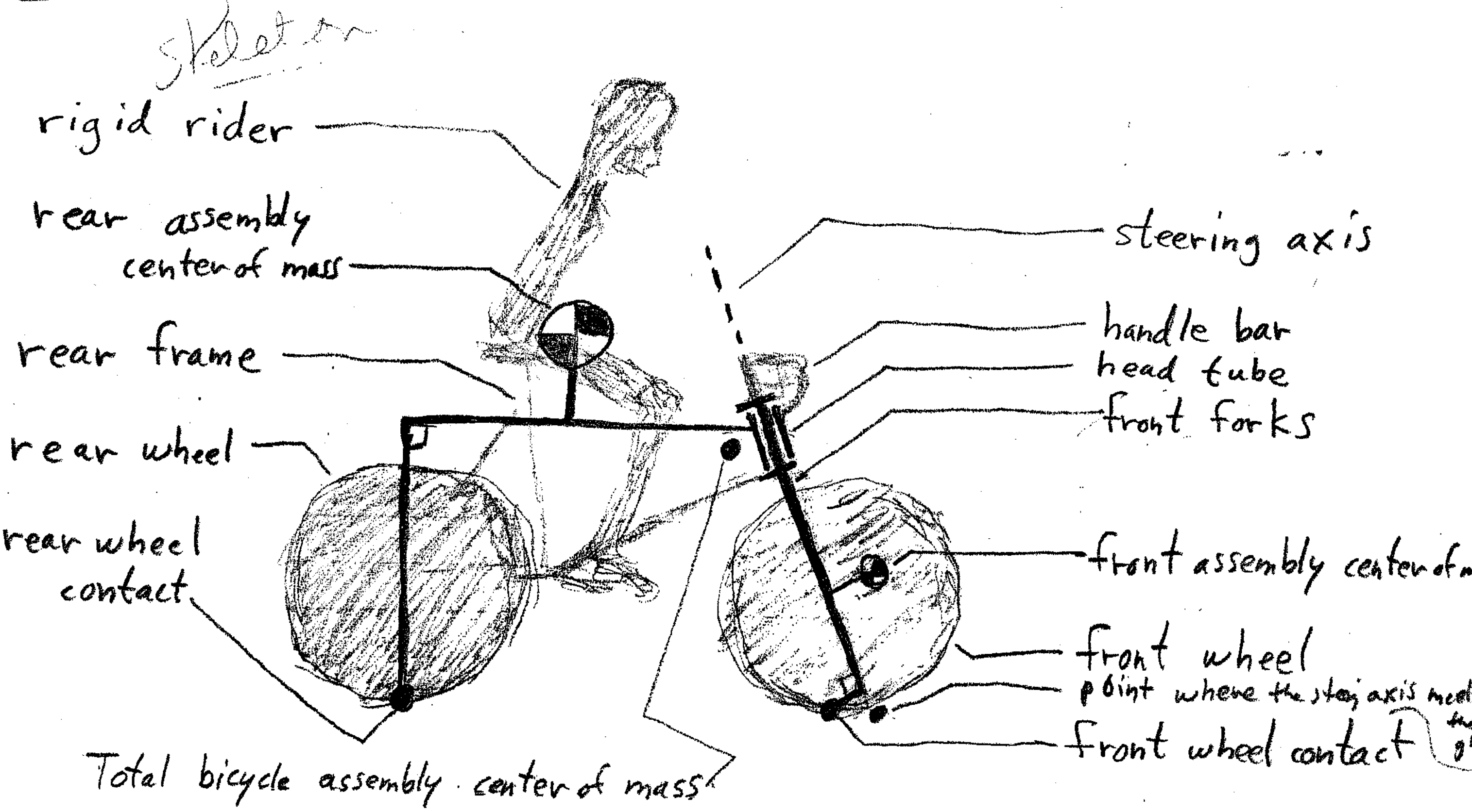

Figure 1

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All unit vectors (bold face **j**, **k**, **n**$_u$, **n**$_v$) have same length.

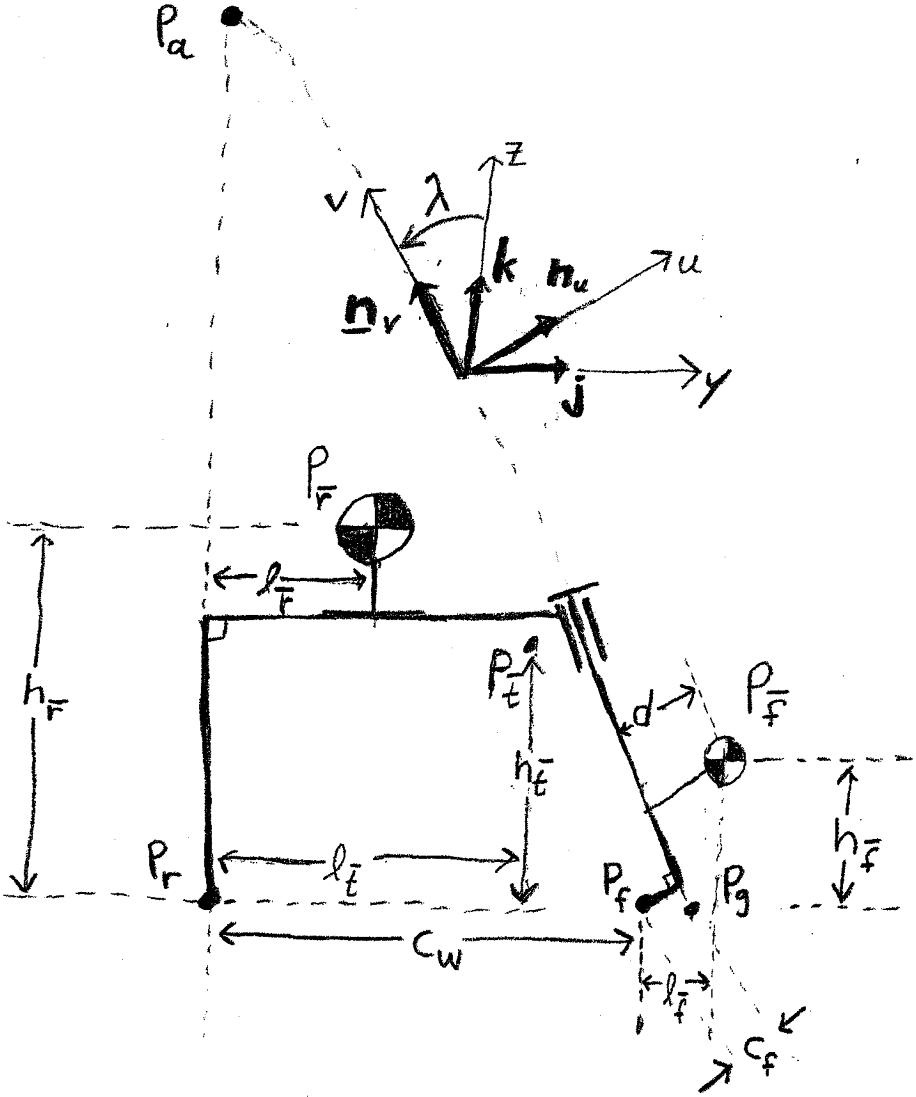

Figure 2a

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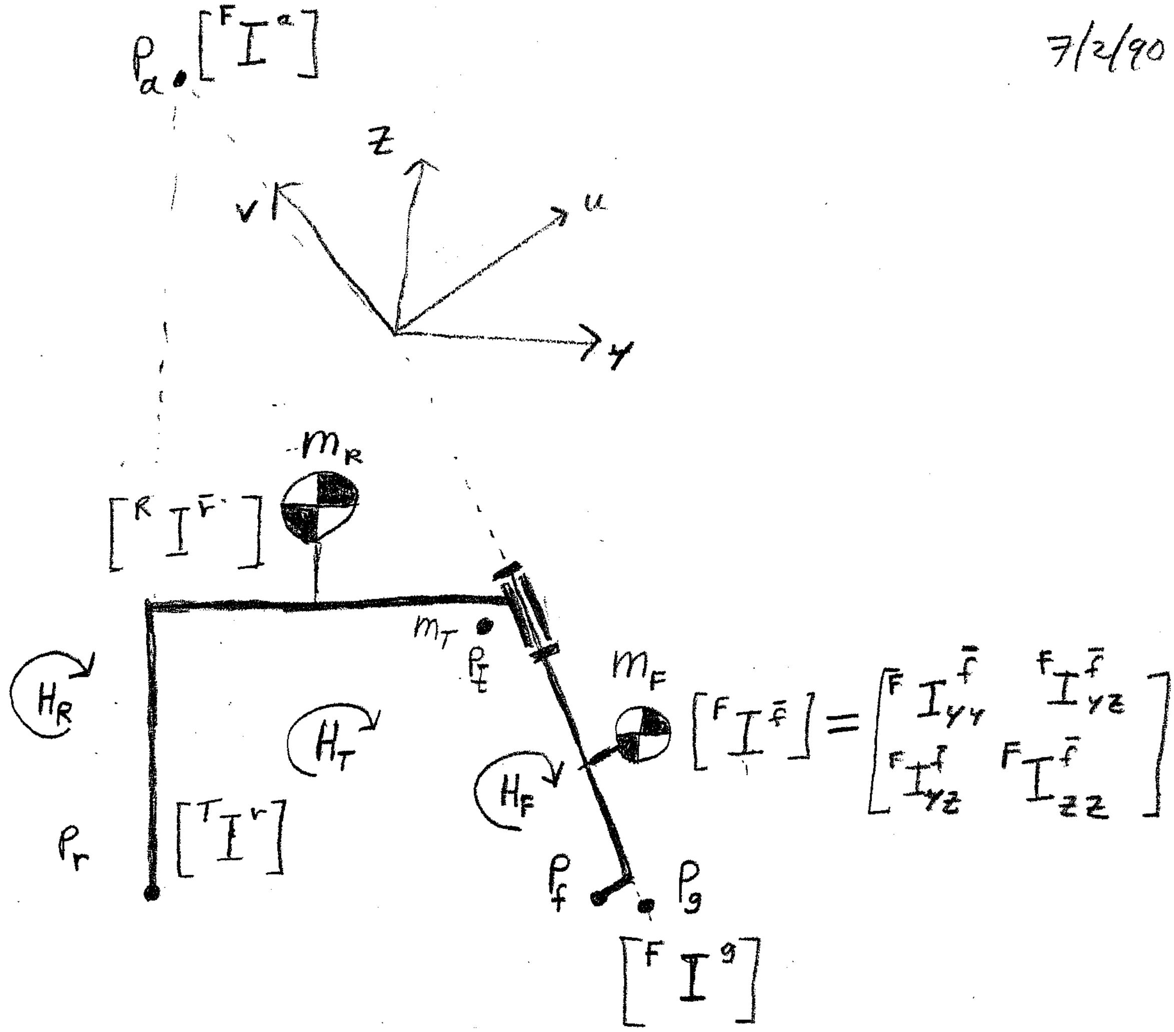

Figure 2b

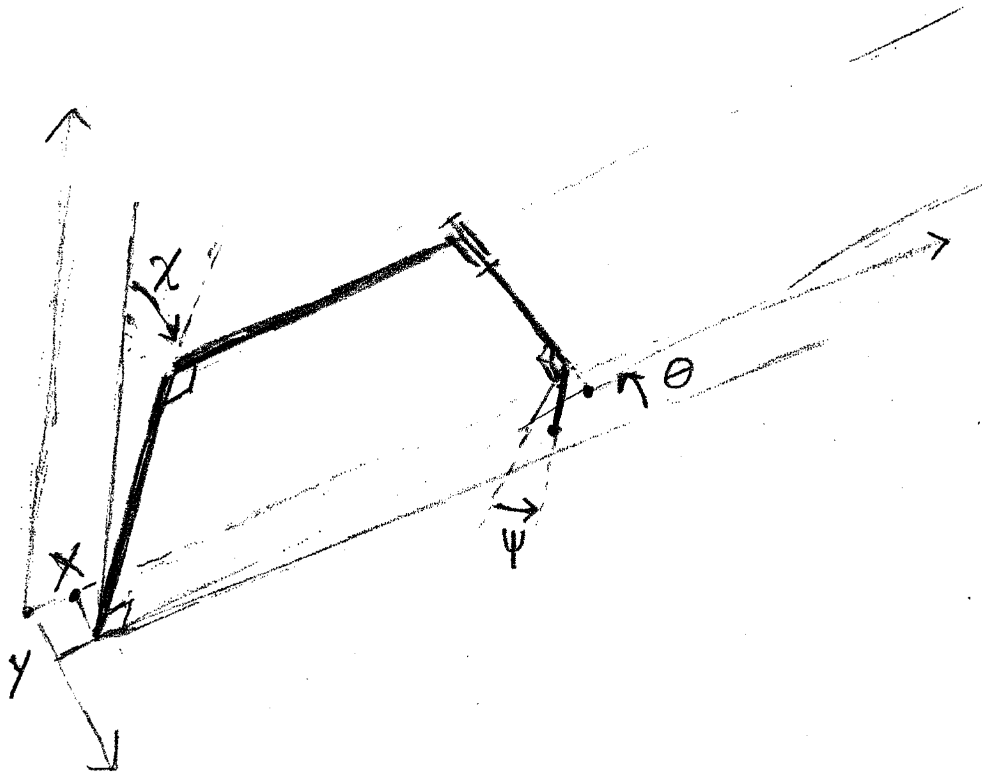

Figure 2C

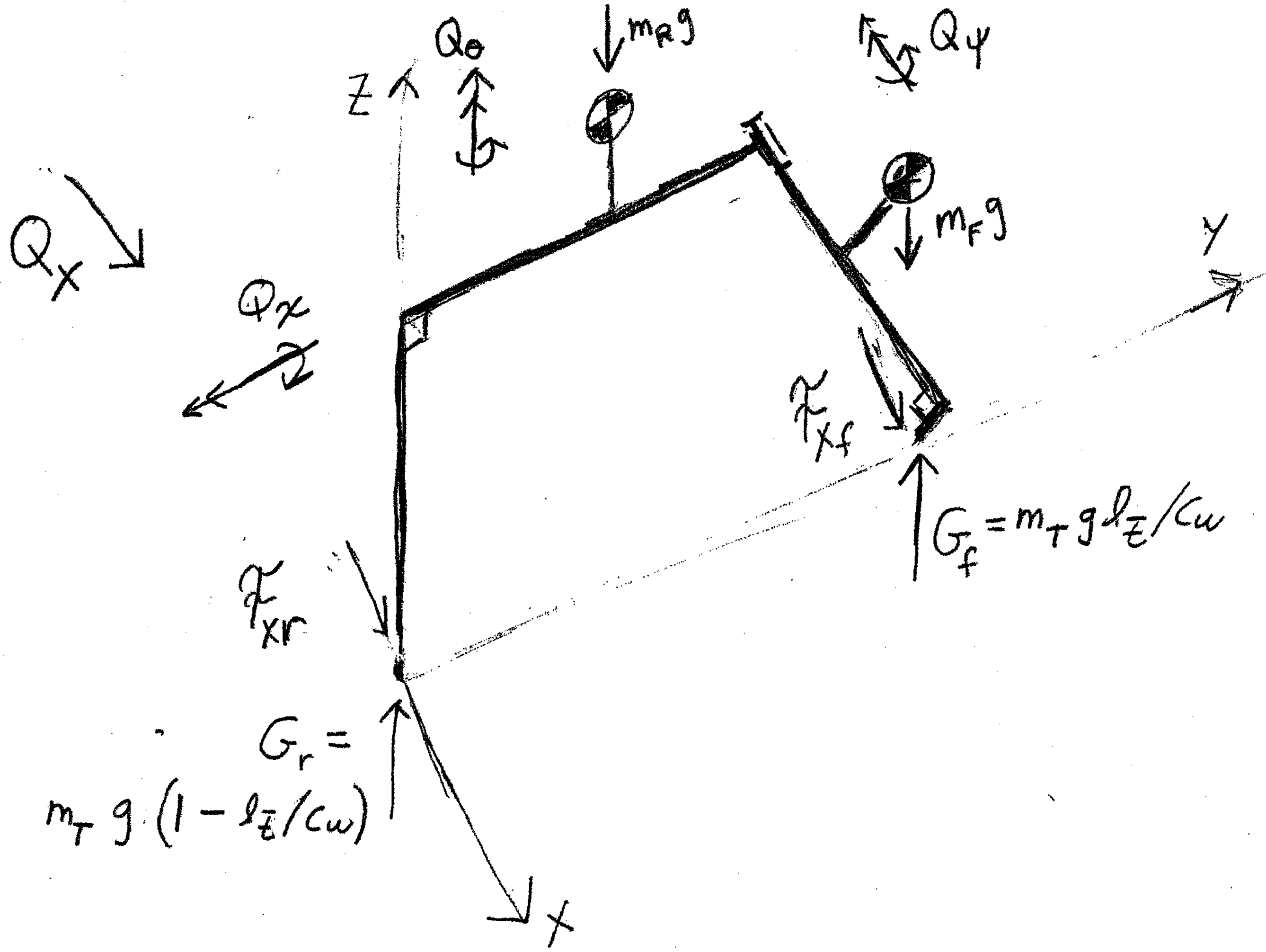

Figure 2d