SOLUTIONS

Your Name: ____________________

TA name and section time: ____________

T&AM 203 Final Exam

Friday May 19, 2006, 2-4:30 PM

Draft May 15, 2006

5 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A} + \underline{\mathbf{v}}_{rel}$$

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}) + \dot{\underline{\boldsymbol{\omega}}} \times \underline{\mathbf{r}}_{B/A} + 2\underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}_{rel} + \underline{\mathbf{a}}_{rel}$$

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}$$

Problem	0:	/-125
Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	4:	/25
Problem	5:	/25
TOTAL:		

0) -125 pt In order to *not* get -125 points you need to sign your name below. If you do not sign your name you get *negative* 125 points. Whether or not you sign, any violations of the pledge below will be fully prosecuted under the Cornell policies concerning academic integrity. I (Andy) have prosecuted many such cases and no student I have accused has ever been found innocent or had a decision reversed on appeal.

Pledge: I realize that the regularly scheduled final might be identical to this test. No student taking the late final should have any more foreknowledge of the test than have students taking this early final now. Between now and 3 PM Friday May 19 I promise not to discuss any aspect of this test with anyone, or within earshot of anyone, with the exception of TAM 203 staff and other TAM 203 students who also took this early test (assuming I know and recognize them and saw them taking this test). That is, there should be *no possible means* by which any student in TAM 203 who is not taking the test with me now could learn by any direct or indirect way from me (for example though a third person overhearing me or reading my email or through my parents talking to their friends etc) anything about this test. For example, and these are only examples, no-one will get in any direct or indirect way from me the answers to any of these questions:

- Did I think the test was easy or hard, fair or unfair?
- Was there a Matlab question on the test?
- Did the test have a statics problem, a problem from the lab, a problem involving pulleys, etc?
- How many questions were on the test?
- Were any formulas given on the test?
- Did the test include material from the final homework?
- How well did I think I did on the test?

If anyone asks me any such questions or tries to get such information from me I will say that I am not allowed to even hint at the answers. If pressed further I will tell the person asking that such pressure is a violation of the rules of academic integrity. If pressed further I will tell 203 staff who was asking. If I know of any violations of this pledge I will promptly inform TAM 203 staff. By signing below I indicate that I understand and agree to the text above on this page.

Signed ____

(sign clearly and legibly)

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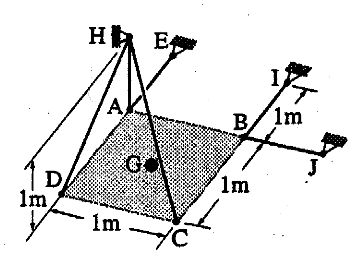
1) (25 pt) A uniform square horizontal rigid plate ABCD has weight mg and is held in place by 6 negligible-mass rods. You need not write long vector formulas if you can confidently justify your answers without them. Find the tension in bar HD.



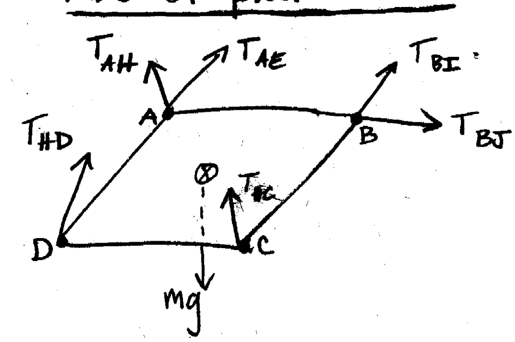
Taking moments about AC, the only force that doesn't pass through it is the force due to tension in bar HD.

Since for static equilibrium

$$\sum \underline{M}_{AC} = 0$$

$\Rightarrow$ The tension HD must be 0.

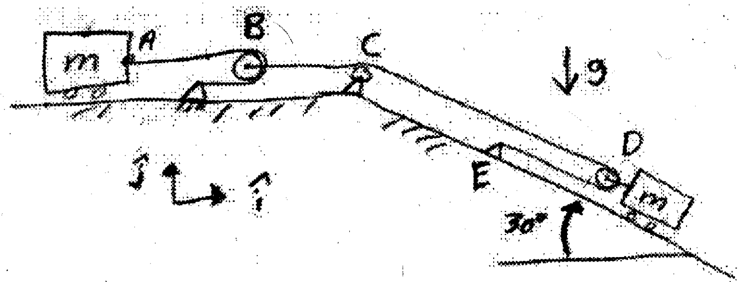
FBD of plate ABCD



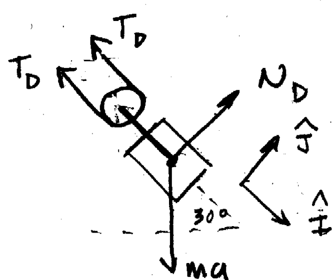
2) **(25 pt)** Make the usual assumptions about pulleys and the like.

a) (20 pt) In terms of some or all of m and g find $\mathbf{a}_D \cdot \hat{\mathbf{j}}$. That is, find the y component of the acceleration of point D.

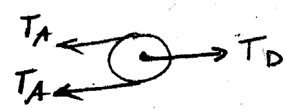
b) (5 pt) Roughly speaking can you explain the answer to part (a). Hint: the answer to part (a) is a number multiplied by a symbol or symbols. That number is close to $2^{\pm n}$ where n is an integer. For example, if the answer to part (a) was $9m/g$ (it isn't) then we could say that answer was close to $2^3 m/g$ and we would have $n = 3$. Use words and/or diagrams to rationalize the appropriate value of n from part (a). That is, somehow the mechanics has in it, approximately, n factors of two. Can you identify each one of these factors. [A very good answer to this part can make up for lost points in part (a)].



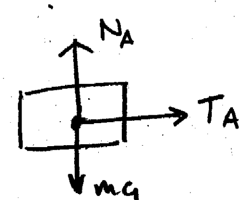
FBD for mass D



FBD for "massless" drum B



FBD for mass A



$(\Sigma F)\cdot\hat{I} = \boxed{mg\cos 60^\circ - 2T_D = m\ddot{x}_D}$

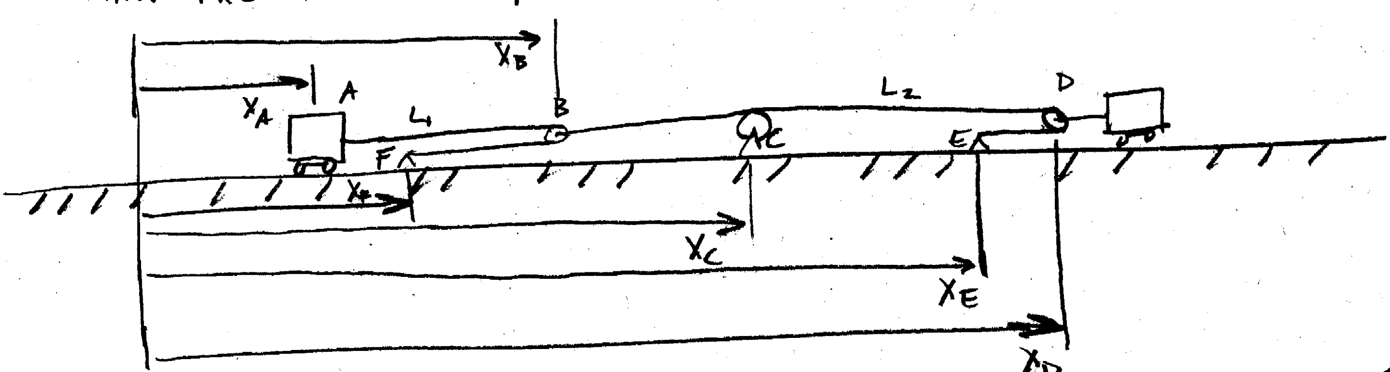
$\Sigma F = -2T_A + T_D = 0$

$\Rightarrow \boxed{T_D = 2T_A}$

$(\Sigma F)\cdot\hat{i} = \boxed{T_A = m\ddot{x}_A}$

1 $\boxed{mg\cos 60^\circ - 4m\ddot{x}_A = m\ddot{x}_D}$

To find the relationship between $\ddot{x}_A$ & $\ddot{x}_D$



$L_2 = x_D - x_E + x_D - x_C + x_C - x_B \Rightarrow \boxed{2\ddot{x}_D = \ddot{x}_B}$ 2

$L_1 = x_B - x_F + x_B - x_A \Rightarrow \boxed{2\ddot{x}_B = \ddot{x}_A} \longrightarrow \boxed{\ddot{x}_D = \frac{1}{4}\ddot{x}_A}$

Combining ① & ②

$$mg\cos 60^\circ - 4m(4\ddot{x}_D) = m\ddot{x}_D \implies \boxed{\ddot{x}_D = \frac{1}{17} g\cos 60^\circ}$$

The $\hat{j}$-component of $\underline{a}_D$ is then

$$(\underline{a}_D)\cdot\hat{j} = -\frac{1}{17} g\cos 60^\circ \sin 30^\circ = -\frac{g}{17}\sin^2 30^\circ$$

$$= -\frac{g}{17}\left(\frac{1}{2}\right)^2 = \boxed{-\frac{1}{68}g \approx -2^{-6}g}$$

Thus we find that $\boxed{n=-6}$. This number can be explained as follows:

From the kinematics we found that mass A has 4 times the acceleration of mass D $\implies$ mass A has 16 times the energy & thus dominates the inertia of the 1-d.o.f. system. So basically we have the weight at D (w/ negligible mass) pulling on mass A.

1. Only half the weight at D is carried by the rope because of the slope, $\sin 30^\circ = 1/2$.
2. Only half of this weight is carried by rope BD because of the pulley at D.
3. Only half the tension of rope BD is carried by rope AB because of pulley B.

So far we have $\ddot{x}_A = T_A/m = \left(\frac{mg}{8}\right)/m = \frac{g}{8}$ from the force analysis. From kinematics we find:

4. Point B has half the acceleration of mass A because of pulley B.

5. Point D has only half the acceleration of B because of the pulley at D.

6. The $\hat{j}$-component of the acceleration of mass D is half the along-slope acceleration because of the slope, $\sin 30° = 1/2$.

Thus the acceleration of mass D is $\approx 2^6$ times smaller than it would be in free flight. It's actually smaller than that since we neglected the inertia of mass D completely, and that would slow the system down more.

3) (25 pt) A person with mass m stands still at the back of a stationary boat with mass M. Then at $t = 0$ she walks the length L of the boat over time T according to the equation

$$x_{p/b} = \frac{L(1 - \cos(\pi t/T))}{2}$$

where $x_{p/b}$ is how far she has moved relative to the boat. Then for $t > T$ she stands still in the front of the boat.

a) (5 pt) Make a plot of $x_{p/b}$ vs t (put t on the "x" axis). Label key points on the "x" and "y" axes in terms of m, M, T and L.

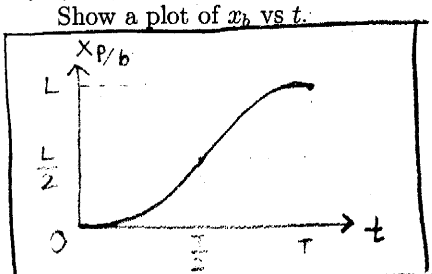
b) (10 pt) Make a plot of x_b vs t, labeling key points on the axis as for part (a). x_b is the absolute position of the boat relative to a fixed reference frame. Assume the boat moves frictionlessly on the water.

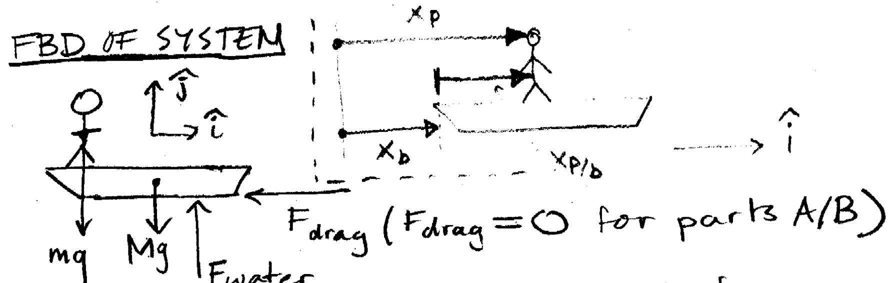
c) (5 pt) For parts (c & d) assume that the boat *has* friction with the water. The drag force is proportional to the boat speed:

$$F_{drag} = cv_b.$$

Eventually, as $t \to \infty$, the boat speed tends to zero and the system comes to rest. What is the net impulse of the force of the water on the boat? That is, evaluate $\int_0^\infty F_{drag}\, dt$ (using basic mechanics principles this is a short calculation).

d) (5 pt) What is $x_b(\infty)$? That is, after all has come to rest how far will the boat have moved? Show a plot of x_b vs t.

a) 



b) Taking man & boat as system, there is no external force on the system ⇒ Linear momentum of system is constant

$\Rightarrow \quad \vec{P}(t<0) = \vec{P}(0<t<T) = \vec{P}(t>T)$

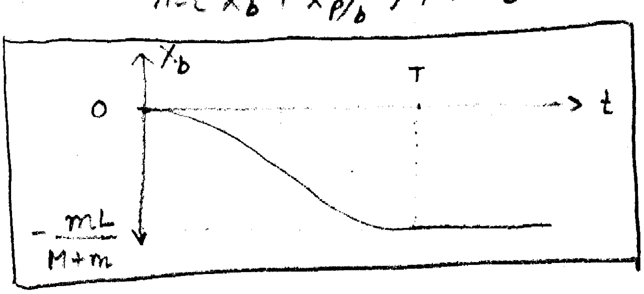
$\Rightarrow \quad 0 = m\vec{v}_p + M\vec{v}_b = 0$

note $\vec{v}_p = \vec{v}_b + \vec{v}_{P/b}$

$\Rightarrow \vec{x}_p = \vec{x}_b + \vec{x}_{P/b}$

integrating $\quad m\vec{x}_p + M\vec{x}_b = 0$

$m(\vec{x}_b + \vec{x}_{P/b}) + M\vec{x}_b = 0 \Rightarrow \boxed{x_b = -\frac{m}{M+m} x_{P/b}} \quad t \le T$



for $t > T$ the system comes to rest.

c) $\int_0^\infty F_{drag}\, dt = \Delta \vec{P}_{system} = (m+M)(V_{after} - V_{before}) = \boxed{0}$

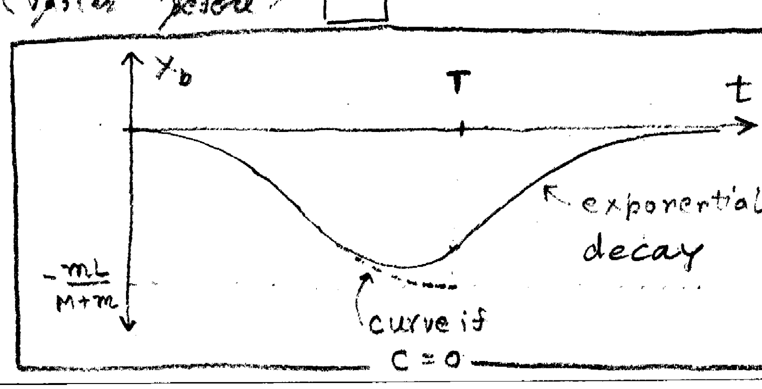
d) $X_b(\infty) = X_b(\infty) - X_b(0)$

$= \int_0^\infty V_b\, dt$

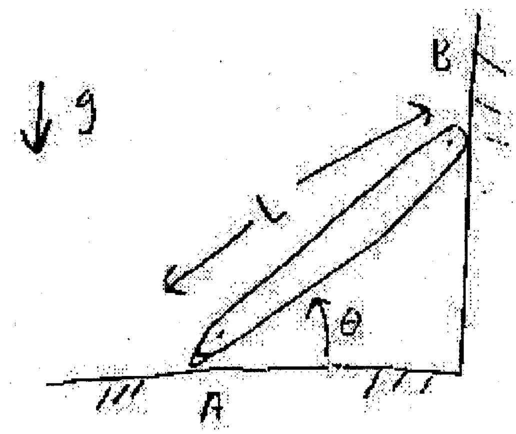
$= \frac{1}{c}\int_0^\infty c V_b\, dt$

$= \frac{1}{c}\int_0^\infty F_{drag}\, dt = \boxed{0}$

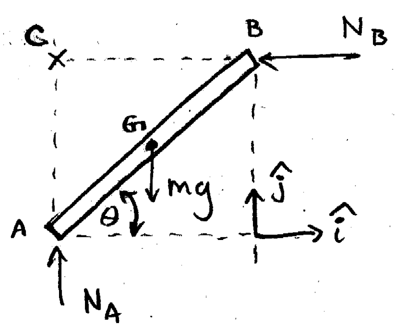
See part (c) above



4) (25 pt) A uniform ladder with mass m and length L slides on a slippery floor and against a slippery wall. It is released from rest at angle θ. Immediately after release find the angular acceleration of the rod. Answer in terms of some or all of θ, g, L, m, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. If you think you need I_G, I_A or I_B you can recall them or derive them or, for less credit, leave them in your final answer.



FBD of ladder AB



LMB

$$\sum \underline{F} = \boxed{-N_B \hat{i} + (N_A - mg)\hat{j} = m\underline{a}_G}$$

AMB about point C

$$\sum \underline{M} = \boxed{\underline{r}_{G/C} \times -mg\hat{j}} \quad \boxed{= I_G \ddot{\theta}\hat{k} + \underline{r}_{G/C} \times m\underline{a}_G}$$

To find $\underline{a}_G$ we use the rigidity of AB to write

$$\boxed{\underline{a}_G = \underline{a}_A + \underline{a}_{G/A} = \underline{a}_A + \underline{\alpha} \times \underline{r}_{G/A} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{G/A}}$$

To find $\underline{a}_A$ we use rigidity & the fact that ends A & B are constrained to move along their respective walls.

$$\underline{a}_B = a_B\hat{j} = \underline{a}_A + \underline{a}_{B/A} = a_A\hat{i} + \underline{\alpha} \times \underline{r}_{B/A} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{B/A}$$

$$\underline{\alpha} \times \underline{r}_{B/A} = \ddot{\theta}\hat{k} \times (L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}) = -L\ddot{\theta}\sin\theta\,\hat{i} + L\ddot{\theta}\cos\theta\,\hat{j}$$

$$\underline{\omega} \times \underline{\omega} \times \underline{r}_{B/A} = \dot{\theta}\hat{k} \times \dot{\theta}\hat{k} \times (L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})$$

$$= -L\dot{\theta}^2\cos\theta\,\hat{i} - L\dot{\theta}^2\sin\theta\,\hat{j}$$

$$\{\ \}\cdot\hat{i} \Rightarrow 0 = a_A - L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta$$

$$\Rightarrow \boxed{a_A = L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta}$$

Thus we have

$$\boxed{\underline{a}_G = (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)\hat{\imath} + \left(-\frac{L\ddot{\theta}}{2}\sin\theta\,\hat{\imath} + \frac{L\ddot{\theta}}{2}\cos\theta\,\hat{\jmath}\right) + \left(-\frac{L\dot{\theta}^2}{2}\cos\theta\,\hat{\imath} - \frac{L\dot{\theta}^2}{2}\sin\theta\,\hat{\jmath}\right)}$$

$$\boxed{= \left(\frac{L\ddot{\theta}}{2}\sin\theta + \frac{L\dot{\theta}^2}{2}\cos\theta\right)\hat{\imath} + \left(\frac{L\ddot{\theta}}{2}\cos\theta - \frac{L\dot{\theta}^2}{2}\sin\theta\right)\hat{\jmath}}$$

Plugging this back into AMB we get

$$-\frac{mgL}{2}\cos\theta\,\hat{k} = \frac{1}{12}mL^2\hat{k} + \left(\frac{L}{2}\cos\theta\,\hat{\imath} - \frac{L}{2}\sin\theta\,\hat{\jmath}\right) \times m\underline{a}_G$$

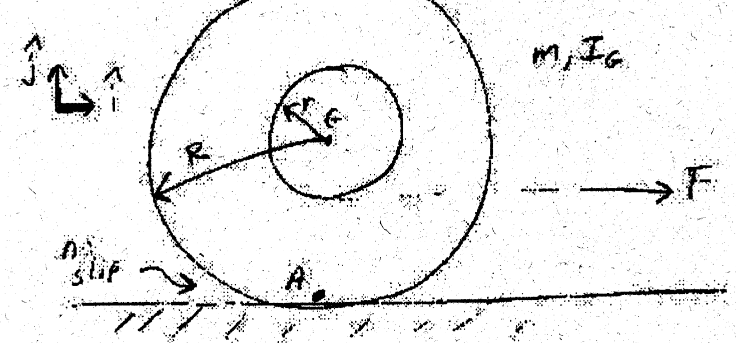
$$\{\ \}\cdot\hat{k} \Rightarrow -\frac{mgL}{2}\cos\theta = \underbrace{\frac{1}{12}mL^2\ddot{\theta} + \frac{1}{4}mL^2\ddot{\theta}}_{\frac{1}{3}mL^2\ddot{\theta}}$$

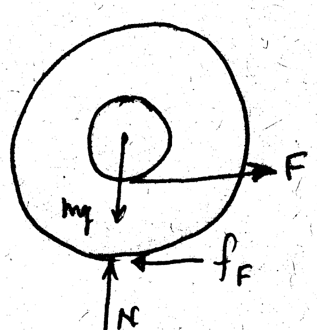
NOTE: The terms with $\dot{\theta}^2$ cancel out after the cross-product.

$$\Rightarrow \ddot{\theta} = -\frac{3g}{2L}\cos\theta \quad \Rightarrow \quad \boxed{\underline{\alpha} = -\frac{3g}{2L}\cos\theta\,\hat{k}}$$

<u>NOTE #1</u> - An alternative method would be to use conservation of energy. Set $T + V = E$ (constant) and differentiate w.r.t. time.

5) (25 pt) A spool (like the movie *Heat Treatment of Aluminum* shown in lecture), with outer radius R rolls without slip on a flat horizontal surface. The film is at a radius r and is being pulled with a horizontal force F. At the moment in question the velocity of the middle of the spool is $v\,\hat{\mathbf{i}}$. The mass of the spool is m and its moment of inertia about its center of mass is I_G. What is the acceleration of point A on the spool which is, at the instant in question, touching the ground. Answer in terms of some or all of m, I_G, r, R, g, v and F.





$\underline{V}_A = \underline{V}_A$ no slip

$\Rightarrow \cancel{\underline{V}_{Ground}}^{\,0} + \cancel{\underline{V}_{A/Ground}}^{\,0} = \underline{V}_G + \underline{V}_{A/G}$

$\Rightarrow \underline{0} = \underline{V}_G + \underline{\omega} \times \underline{r}_{A/G}$

$\Rightarrow \underline{0} = v\hat{i} + \omega\hat{k} \times (-R\hat{j})$

$\Rightarrow \underline{0} = v\hat{i} + \omega R\hat{i}$

$\{\ \}\cdot\hat{i} \Rightarrow \omega = -\dfrac{v}{R} = \dot{\theta}$

$\underline{a}_G = -R\ddot{\theta}\,\hat{i}$

α

$\underline{r}_{A/G} = -R\hat{j}$

$\underline{a}_A = \underline{a}_G + (\ddot{\theta}\hat{k}) \times \underline{r}_{A/G} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{r}_{A/G})}_{-\omega^2 \underline{r}_{A/G}}$

$\Rightarrow \underline{a}_A = \cancel{-R\ddot{\theta}\hat{i}} + \cancel{R\ddot{\theta}\hat{i}} + \omega^2 R\hat{j}$

$\Rightarrow \boxed{\underline{a}_A = \dfrac{v^2}{R}\hat{j}}$

Hence the answer doesn't depend on using momentum balance.