1. Problem 2.3.2.

Statement. Car $B$ is driving straight toward the point $O$ at a constant speed $v$. An observer, located at $A$, tracks the car with a radar gun (see Figure 1). What is the speed $|\dot{r}_{B/A}|$ that the observer at $A$ records? Note that $r_{B/A}$ means the magnitude (same as the “length”) of vector $\vec{r}_{B/A}$ while $\dot{r}_{B/A}$ means the time derivative of this magnitude.

![Figure 1](image-url)

Solution. Take point $A$ to be the origin of our polar coordinate system (as shown in Figure 1). Then $\|\vec{r}_{B/A}\| = r$ and $\dot{r}_{B/A} = \dot{r}$. In terms $\dot{e}_r$ and $\dot{e}_\theta$, $\vec{v}_{B/A} = \frac{d}{dt} \vec{r}_{B/A}$ can be given as

$$\vec{v}_{B/A} = \dot{r} \dot{e}_r + r \dot{\theta} \dot{e}_\theta \quad (1.1)$$

while in terms of $\hat{i}$ and $\hat{j}$, we have

$$\vec{v}_B = v \cos 45^\circ (\hat{i}) + v \sin 45^\circ (\hat{j}). \quad (1.2)$$

Since $\vec{v}_A = 0$ (A is fixed), we have that $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_{B/A}$. It follows from (1.1) and (1.2) that

$$\dot{r} \dot{e}_r + r \dot{\theta} \dot{e}_\theta = v \cos 45^\circ (\hat{i}) + v \sin 45^\circ (\hat{j}). \quad (1.3)$$

We dot product the both sides of (1.3) with $\dot{e}_r$

$$\left( \dot{r} \dot{e}_r + r \dot{\theta} \dot{e}_\theta \right) \cdot \dot{e}_r = \left( v \cos 45^\circ (\hat{i}) + v \sin 45^\circ (\hat{j}) \right) \cdot \dot{e}_r \quad (1.4)$$

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\[ \dot{r} \hat{e}_r \cdot \hat{e}_r + r \dot{\theta} \hat{e}_\theta \cdot \hat{e}_r = -v \cos 45° \hat{i} \cdot \hat{e}_r - v \sin 45° \hat{j} \cdot \hat{e}_r \]
\[ \Rightarrow \dot{r} = -v \cos 45° \cos \theta - v \sin 45° \sin \theta \]
\[ \Rightarrow \dot{r} = -v(\cos 45° \cos \theta + \sin 45° \sin \theta). \tag{1.4} \]

From the given geometry, we know that
\[ \cos \theta = \frac{0.1 \text{ km}}{r} = \frac{0.1 \text{ km}}{\sqrt{(0.1 \text{ km})^2 + (0.2 \text{ km})^2}} \approx 0.4472; \tag{1.5} \]
\[ \sin \theta = \frac{0.2 \text{ km}}{r} = \frac{0.2 \text{ km}}{\sqrt{(0.1 \text{ km})^2 + (0.2 \text{ km})^2}} \approx 0.8944. \tag{1.6} \]

The substitution of (1.5) and (1.6) into (1.4) gives
\[ \dot{r} = -v(\cos 45° \cdot 0.4472 + \sin 45° \cdot 0.8944) \]
\[ \approx -0.9487v. \]

Thus, \[|\dot{r}_{B/A}| = |\dot{r}| = 0.9487v. \]
2. Problem 2.3.3.

**Statement.** Explain why $\|\frac{d}{dt} \vec{r}(t)\|$ is not in general equal to $\frac{d}{dt} \|\vec{r}(t)\|$.

**Solution.** $\vec{r}(t)$ is a vector. $\frac{d}{dt} \vec{r}(t)$, which is also a vector, is the time derivative of $\vec{r}(t)$. $\|\frac{d}{dt} \vec{r}(t)\|$, which is always positive, is the magnitude of vector $\frac{d}{dt} \vec{r}(t)$.

On the other hand, $\|\vec{r}(t)\|$ is the magnitude of vector $\vec{r}(t)$. $\frac{d}{dt} \|\vec{r}(t)\|$ is the time derivative of this magnitude. Thus, $\frac{d}{dt} \|\vec{r}(t)\|$ can be positive, negative, or zero as the magnitude of $\vec{r}(t)$ can increase, decrease, or be constant, with respect to time.

In terms of polar coordinates, we have

\[
\frac{d}{dt} \vec{r}(t) = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta \implies \left\| \frac{d}{dt} \vec{r}(t) \right\| = \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2}
\]

while

\[
\vec{r}(t) = r \hat{e}_r \implies \|\vec{r}(t)\| = r \implies \frac{d}{dt} \|\vec{r}(t)\| = \dot{r}.
\]

It is obvious that $\dot{r} \neq \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2}$ in general since $\dot{\theta} \neq 0$ in general.

Consider a particle going in circles at speed $v$. Let the radius of the circle be $r$. Then $\|\frac{d}{dt} \vec{r}(t)\| = v \neq 0$ while $\frac{d}{dt} \|\vec{r}(t)\| = \dot{r} = 0$ since $r$ is constant for the circle.
3. Problem 2.3.19.

**Statement.** Assume that the distance between the hill and the cloud layer is 150 ft. Neglect the dimensions of the hill. Initially, the light from the headlights makes a $20^\circ$ angle with the horizontal, and the light beam rotates around at the constant rate of $20^\circ/s$ (clockwise) as the car moves up the hill. What are the initial speed and acceleration of the light spot on the cloud’s underside? What are they when the light beam makes a $5^\circ$ angle with the horizontal?

![Figure 3](image_url)

**Solution.** As depicted in Figure 3, using polar coordinates, we can describe the motion of the light spot by

$$\vec{r}_{L/O} = r \hat{e}_r;$$

$$\vec{v}_L = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta;$$

$$\vec{a}_L = (\ddot{r} - r \ddot{\theta}^2) \hat{e}_r + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{e}_\theta.$$ (3.3)

Now we express $\hat{e}_r$ and $\hat{e}_\theta$ in terms of $\hat{i}$ and $\hat{j}$ (see Figure 4)

$$\hat{e}_r = (\hat{e}_r \cdot \hat{i}) \hat{i} + (\hat{e}_r \cdot \hat{j}) \hat{j}$$

$$= \cos \theta \hat{i} + \cos(90^\circ - \theta) \hat{j}$$

$$= \cos \theta \hat{i} + \sin \theta \hat{j};$$ (3.4)

$$\hat{e}_\theta = (\hat{e}_\theta \cdot \hat{i}) \hat{i} + (\hat{e}_\theta \cdot \hat{j}) \hat{j}$$

$$= \cos(90^\circ + \theta) \hat{i} + \cos \theta \hat{j}$$

$$= -\sin \theta \hat{i} + \cos \theta \hat{j}.$$ (3.5)

![Figure 4](image_url)
With (3.4) and (3.5), (3.2) and (3.3) become

$$\vec{v}_L = \dot{r} \left( \cos \theta \hat{i} + \sin \theta \hat{j} \right) + r \dot{\theta} \left( -\sin \theta \hat{i} + \cos \theta \hat{j} \right)$$
$$= (\dot{r} \cos \theta - r \dot{\theta} \sin \theta) \hat{i} + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta) \hat{j}; \quad (3.6)$$

$$\vec{a}_L = (\ddot{r} - r \ddot{\theta}^2) \left( \cos \theta \hat{i} + \sin \theta \hat{j} \right) + (2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \left( -\sin \theta \hat{i} + \cos \theta \hat{j} \right)$$
$$= \left( (\ddot{r} - r \ddot{\theta}^2) \cos \theta - (2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \sin \theta \right) \hat{i} + \left( (\ddot{r} - r \ddot{\theta}^2) \sin \theta + (2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \cos \theta \right) \hat{j}. \quad (3.7)$$

Since the light spot is constrained to move along the cloud’s underside, which is parallel to the \( \hat{i} \) direction, it follows that \( \vec{v}_L \cdot \hat{j} = 0 \text{ m/s} \) and \( \vec{a}_L \cdot \hat{j} = 0 \text{ m/s}^2 \). Then we have

$$\vec{v}_L \cdot \hat{j} = 0 \text{ m/s} \implies (\dot{r} \cos \theta - r \dot{\theta} \sin \theta) \hat{i} \cdot \hat{j} + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta) \hat{j} \cdot \hat{j} = 0 \text{ m/s}$$
$$\implies \dot{r} \sin \theta + r \dot{\theta} \cos \theta = 0 \text{ m/s}$$
$$\implies \dot{r} = -\frac{r \dot{\theta} \cos \theta}{\sin \theta}; \quad (3.8)$$

$$\vec{a}_L \cdot \hat{j} = 0 \text{ m/s}^2 \implies \left( (\ddot{r} - r \ddot{\theta}^2) \cos \theta - (2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \sin \theta \right) \hat{i} \cdot \hat{j}$$
$$+ \left( (\ddot{r} - r \ddot{\theta}^2) \sin \theta + (2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \cos \theta \right) \hat{j} \cdot \hat{j} = 0 \text{ m/s}^2$$
$$\implies (\ddot{r} - r \ddot{\theta}^2) \sin \theta + (2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \cos \theta = 0 \text{ m/s}^2$$
$$\implies \ddot{r} = -\frac{(2 \ddot{r} \dot{\theta} + r \ddot{\theta}) \cos \theta}{\sin \theta} + r \ddot{\theta}^2. \quad (3.9)$$

Since \( \dot{\theta} = -20^\circ/\text{s} \), we have \( \ddot{\theta} = 0^\circ/\text{s}^2 \). By this fact and (3.8), (3.9) becomes

$$\ddot{r} = -\frac{2 \ddot{r} \dot{\theta} \cos \theta}{\sin \theta} + r \ddot{\theta}^2$$
$$= -\frac{2 \left( \frac{\ddot{r} \dot{\theta} \cos \theta}{\sin \theta} \right) \dot{\theta} \cos \theta}{\sin \theta} + r \ddot{\theta}^2$$
$$= \frac{2 \ddot{r} \dot{\theta}^2 \cos \theta}{\sin^2 \theta} + r \ddot{\theta}^2. \quad (3.10)$$

The substitution of (3.8) into (3.6) yields

$$\vec{v}_L = \left[ \left( -\frac{\ddot{r} \dot{\theta} \cos \theta}{\sin \theta} \right) \cos \theta - \ddot{r} \sin \theta \right] \hat{i} + 0 \text{ m/s} \hat{j}$$
$$= \left[ \frac{r \dot{\theta} \cos^2 \theta}{\sin \theta} - \ddot{r} \sin \theta \right] \hat{i}$$
$$= -\frac{r \dot{\theta}}{\sin \theta} \hat{i}$$
$$= -\frac{(r \sin \theta) \dot{\theta}}{\sin^2 \theta} \hat{i}$$
$$= -\frac{h \dot{\theta}}{\sin^2 \theta} \hat{i}, \quad (3.11)$$
where the \( \hat{j} \) component is automatically zero since that is how we have derived (3.10). Similarly, the substitution of (3.8) and (3.10) into (3.7) yields (note that \( \dot{\theta} = 0^\circ/s^2 \))

\[
\vec{a}_L = \left\{ \left[ \left( \frac{2r\ddot{\theta}^2 \cos^2 \theta}{\sin^2 \theta} + r\dot{\theta}^2 \right) \cos \theta - 2 \left( -\frac{r\dot{\theta} \cos \theta}{\sin \theta} \right) \dot{\theta} \sin \theta \right] + 0 \text{m/s}^2 \hat{j} \right\} \hat{i} + 0 \text{m/s}^2 \hat{j} \\
= \left\{ \frac{2r\ddot{\theta}^2 \cos^2 \theta}{\sin^2 \theta} \cos \theta + 2r\dot{\theta}^2 \cos \theta \right\} \hat{i} \\
= \frac{2r\ddot{\theta}^2 \cos \theta}{\sin^2 \theta} \hat{i} \\
= \frac{2(r \sin \theta)\dot{\theta}^2 \cos \theta}{\sin^3 \theta} \hat{i} \\
= \frac{2h\ddot{\theta}^2 \cos \theta}{\sin^3 \theta} \hat{i},
\]

(3.12)

where the \( \hat{j} \) component is again automatically zero since that is how we have derived (3.10).

Finally, evaluating (3.11) and (3.12) at \( \theta = 20^\circ \), \( \dot{\theta} = -20^\circ/s = -\frac{\pi}{9} \text{ rad/s} \), and \( h = 150 \text{ ft} \) gives

\[
\vec{v}_L = -\frac{h\dot{\theta}}{\sin^2 \theta} \hat{i} \\
= \frac{150 \text{ ft} \cdot (-\frac{\pi}{9} \text{ rad/s})}{\sin^2 20^\circ} \hat{i} \\
\approx 447.6 \text{ ft/s} \hat{i};
\]

\[
\vec{a}_L = \frac{2h\ddot{\theta}^2 \cos \theta}{\sin^3 \theta} \hat{i} \\
= \frac{2 \cdot 150 \text{ ft} \cdot (-\frac{\pi}{9} \text{ rad/s})^2 \cos 20^\circ}{\sin^3 20^\circ} \hat{i} \\
\approx 858.6 \text{ ft/s}^2 \hat{i}.
\]

Evaluating (3.11) and (3.12) at \( \theta = 5^\circ \), \( \dot{\theta} = -20^\circ/s = -\frac{\pi}{9} \text{ rad/s} \), and \( h = 150 \text{ ft} \) gives

\[
\vec{v}_L = -\frac{h\dot{\theta}}{\sin^2 \theta} \hat{i} \\
= \frac{150 \text{ ft} \cdot (-\frac{\pi}{9} \text{ rad/s})}{\sin^2 5^\circ} \hat{i} \\
\approx 6893 \text{ ft/s} \hat{i};
\]

\[
\vec{a}_L = \frac{2h\ddot{\theta}^2 \cos \theta}{\sin^3 \theta} \hat{i} \\
= \frac{2 \cdot 150 \text{ ft} \cdot (-\frac{\pi}{9} \text{ rad/s})^2 \cos 5^\circ}{\sin^3 5^\circ} \hat{i} \\
\approx 5.500 \times 10^4 \text{ ft/s}^2 \hat{i}.
\]
**Alternative Solution.** Using unit vectors \( \hat{i} \) and \( \hat{j} \), we have

\[
\vec{r}_{L/O} = \frac{h}{\tan \theta} \hat{i} + h \hat{j}.
\]

Next, \( \dot{h} = 0 \) since \( h \) is constant. \( \theta \) is a function of \( t \) with \( \dot{\theta} = -20^\circ/s = -\frac{\pi}{9} \text{ rad/s} \) and \( \ddot{\theta} = 0 \). Then it follows that

\[
\vec{v}_L = \frac{d}{dt} \vec{r}_{L/O} = \frac{d}{dt} \left( \frac{h}{\tan \theta} \right) \hat{i} + \frac{d}{dt} h \hat{j} = -\frac{h \dot{\theta}}{\sin^2 \theta} \hat{i},
\]

which is exactly the same as (3.11). In addition,

\[
\vec{a}_L = \frac{d}{dt} \vec{v}_L = \frac{d}{dt} \left( -\frac{h \dot{\theta}}{\sin^2 \theta} \right) \hat{i} = \frac{2h \dot{\theta}^2 \cos \theta}{\sin^3 \theta} \hat{i},
\]

which is exactly the same as (3.12).