Outline

I. Restate book’s solution
II. Show the error in the book’s reasoning
III. Derive the correct solution.

The reasoning is that to minimize the total velocity, we need to minimize its vertical velocity, i.e., get the ball as high as the window. The ball will only go through the window if that max height is 2 m away from the origin.

In short: a trajectory whose maximum point is the position (2, 7). See Fig. 1.

The solution:

\[ y = y_0 + V_{oy}t + \frac{g}{2}t^2 \Rightarrow g = -9.81 \text{ m/s}^2, y_0 = 0 \]
\[ y = V_{oy}t + \frac{g}{2}t^2 \]
\[ y_{max} + g t = 0 \Rightarrow t_{max} = \frac{V_{oy}}{g} \]
\[ y_{max} = V_{oy} \left( \frac{V_{oy}}{g} \right) + \frac{g}{2} \left( \frac{V_{oy}}{g} \right)^2 = -\frac{V_{oy}^2}{2g} \]
\[ V_{oy} = \sqrt{-2(-9.81)(7)} = 11.7192 \text{ m/s} \]
\[ x = V_{ox}t, x_{max} = V_{ox}t_{max} = \frac{-V_{ox}V_{oy}}{g} \Rightarrow V_{ox} = -\frac{x_{max}g}{V_{oy}} \]
\[ = \frac{-2(-9.81)}{11.7192} = 1.67418 \text{ m/s} \]
\[ V = \sqrt{V_{oy}^2 + V_{ox}^2} = \sqrt{1.67418^2 + 11.7192^2} = 11.8382 \text{ m/s} \]
II - On the surface, this strategy seems reasonable, but we must remember that a minimum $V_{oy}$ doesn't lead to a minimum $V$. If the decrease in $V_{ox}$ is enough to compensate for the increases in $V_{oy}$, the overall velocity may be lower. Now the question is: can we find such a velocity vector. If we disprove that the book's solution works for the general case, we have good reason to believe there exists a better solution for this specific problem. We'll use the simplest specific case to show the book's strategy won't yield a minimum $V$ for all cases.

Suppose there is a window $x_1$ meters away from the origin, a shade above the ground (see Fig. 2.

The book's strategy would suggest we reach the maximum altitude (of close to 0), $x_1$, meters from the origin. See Fig. 3. Using our formula for $V_{ox}$, we find $V_{ox} = \frac{x_{1g}}{V_{oy}} = \frac{x_{1g}}{\sqrt{2gy_1}}$.

All the values in that expression are finite and do not approach 0, except for $y_1$, which is very close to 0. Thus the whole expression yields a large number that approaches $\infty$ as $y_1 \to 0$. 

- a2 -
Surely we don't need an infinite initial velocity to cover a finite distance. Consider a trajectory where the ball goes through the window on its way down. See Fig. 4.

The maximum $y$ is not important; what is important is that we can make it as small or as large as we want to, i.e. it doesn't have to approach 0. New $V_{oy} = \sqrt{2gy_{max}}$ and $V_{ox} = \frac{x_1}{\sqrt{2gy_{max}}}$.

All the values in the two expressions are finite and don't approach 0. Thus, both are finite, so $V$ is finite, which seems to be better than the infinite velocity yielded by the book's strategy. Thus, we have shown that the book's strategy won't work for all cases, and maybe it won't work for our specific example.

III - Now we are going to try to find a solution for the problem where the trajectory's max point isn't at the window. See Fig. 5a, b.

We can't conclusively say yet whether the ball will go through the window before or
or after it reaches the peak of its flight, but we'll be able to answer that after we solve the main problem.

\[ V_{ox} = V \cos \theta \]
\[ V_{oy} = V \sin \theta \]

\[ y = y_0 + v \sin \theta t + \frac{1}{2} a t^2 = 7 \]
\[ y_0 = 0, a = -9.81 \text{ m/s}^2 \]

\[ x = V \cos \theta t = 2 \]

\[ t = \frac{2}{v \cos \theta} \]

Now plugging \( t \) into \( y \).

\[ y = v \sin \left( \frac{2}{v \cos \theta} \right) + \frac{a}{2} \left( \frac{2}{v \cos \theta} \right) \]

\[ = 2 \tan \theta + \frac{2a}{2v^2 \cos^2 \theta} = 7 \]

\[ \frac{2a}{2v^2 \cos^2 \theta} = 7 - 2 \tan \theta \]

\[ 2a = v^2(7 \cos^2 \theta - 2 \sin \theta \cos \theta) \]

\[ v(\theta) = \sqrt{\frac{2a}{7 \cos^2 \theta - 2 \sin \theta \cos \theta}} \]

Now we have an explicit equation that relates all angle-velocity pairs that pass through \((2, 7)\). We need to minimize \( v \), so \( 7 \cos^2 \theta - 2 \sin \theta \cos \theta \) need to be maximized. Because \( 2a < 0 \), that means \( 7 \cos^2 \theta - 2 \sin \theta \cos \theta < 0 \), \( 7 \cos^2 \theta - 2 \sin \theta \cos \theta \) needs to be minimized.
\[ f(\theta) = 7\cos^2 \theta - 2 \sin \theta \cos \theta, \] Double angle formulas yield:

\[ f(\theta) = 7 \left( \frac{1 + \cos 2\theta}{2} \right) - \sin 2\theta \]

\[ = \frac{7}{2} + \frac{7}{2} \cos 2\theta - \sin 2\theta, \quad \text{let } 2\theta = \phi \]

\[ f(\phi) = \frac{7}{2} + \frac{7}{2} \cos \phi - \sin \phi \]

\[ f'(\phi) = -\frac{7}{2} \sin \phi - \cos \phi = 0 \]

\[ \tan \phi = -\frac{7}{2} \]

\[ \phi = \tan^{-1} \left(-\frac{7}{2}\right) = 164.055^\circ \text{ (only result in the right range)} \]

\[ \theta = \frac{\phi}{2} = \frac{164.055}{2} = 82.0273^\circ \]

By the second derivative:

\[ f''(\phi) = -\frac{7}{2} \cos \phi + \sin \phi \]

\[ f''(164.055) = -\frac{7}{2} \cos(164.055) + \sin(164.055) = 3.6405 > 0 \]

\[ : \phi = 164.055^\circ \text{ is a minimum of } f(\phi). \]

Now, by substituting \( \theta = 82.0273^\circ \):

\[ v(82.0273) = \sqrt{\frac{2g}{\sqrt{\cos^2(82.0273) - 2 \cdot 164.055 \cdot \cos(82.0273) \cos(82.0273)}}} = \sqrt{-1.862} \]

\[ = \sqrt{140.088} = 11.8359 \text{ m/s} \]

11.8359 < 11.8382 (the book's solution). Now, one final thing to sort out, does the ball go through the window before or after it reaches \( y_{\text{max}} \)?

\[ \text{Correct trajectory} \]

\[ \frac{164^\circ}{2} = 82^\circ \]