

Basant Sharma, Spring 2009

① [5.147]

Two particles connected by elastic spring sliding on a frictionless horizontal table (no external forces)

a) Consider the two masses and the spring as a free body,

$$\dot{P} = \sum \dot{F}_i = \frac{d}{dt} \underline{P} \quad (\text{no external forces})$$

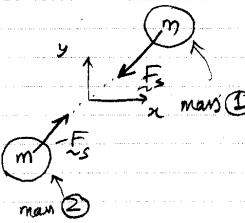
where  $\underline{P}$  is the total linear momentum.

So  $\underline{L} = \text{const.}$  i.e.  $\underline{L}$  is conserved.

b)  $\underline{P} = 2m\underline{v}_c$  where  $\underline{v}_c$  is the velocity of the center of mass

So  $\underline{v}_c = \text{const.}$  i.e. the center of mass doesn't accelerate.

c) FBD of each mass



d) Equations of motion

Let mass ① be located at  $\underline{r}_1(t)$  and ② be at  $\underline{r}_2(t)$ .

By LMB

$$\begin{aligned} m\ddot{r}_1 &= F_s \\ m\ddot{r}_2 &= -F_s \end{aligned} \quad \left. \begin{array}{l} \text{---(1)} \\ \text{---(2)} \end{array} \right.$$

Since ① and ② are connected by elastic spring of constant  $K$  and unextended length  $2R$ ,

$$F_s = -K((r_1 - r_2) - 2R) \frac{(r_1 - r_2)}{|r_1 - r_2|} \quad \text{---(2)}$$

by b) the reference frame attached to the center of mass is inertial so we place the origin at  $(\frac{r_1 + r_2}{2})$ . In this coordinate system

$$m\ddot{r}_1 = -K(|r_1| - R) \frac{r_1}{|r_1|}$$

$$m\ddot{r}_2 = -K(|r_2| - R) \frac{r_2}{|r_2|} \quad (r_2 = -r_1)$$

In cartesian coordinates

$$m\ddot{x}_1 = -K(1 - \frac{R}{\sqrt{x_1^2 + y_1^2}}) x_1$$

$$m\ddot{y}_1 = -K(1 - \frac{R}{\sqrt{x_1^2 + y_1^2}}) y_1$$

$$\text{and } m\ddot{x}_2 = -K(1 - \frac{R}{\sqrt{x_2^2 + y_2^2}}) x_2$$

$$m\ddot{y}_2 = -K(1 - \frac{R}{\sqrt{x_2^2 + y_2^2}}) y_2$$

(actually  $x_2 = -x_1$ ,  $y_2 = -y_1$ ).

e) The total K.E. of the system is

$$\begin{aligned} \text{K.E.} &= \frac{1}{2}m(x_1)^2 + \frac{1}{2}m(y_1)^2 + \frac{1}{2}(2m)(k_c)^2 \\ &= \frac{1}{2}m(x_1^2 + y_1^2) + \frac{1}{2}m(x_2^2 + y_2^2) + m(x_c)^2 \\ &= m(x_1^2 + y_1^2) + m(x_c)^2 \end{aligned}$$

( $x_c$  is determined by the initial conditions of the system)

The total P.E. of the system is

$$\begin{aligned} \text{P.E.} &= \frac{1}{2}K((|r_1| - R)^2 + \frac{1}{2}k((|r_2| - R)^2) \\ &= K(|r_1| - R)^2 \\ &= K(x_1^2 + y_1^2 + R^2 - 2R\sqrt{x_1^2 + y_1^2}) \end{aligned}$$

(we have used that  $x_2 = -x_1$ ,  $y_2 = -y_1$  in the center of mass frame of reference)

f) In a fixed coordinate system the equations of motion are given by (1) and (2). In cartesian coordinates this becomes

$$\ddot{x}_1 = -\frac{k}{m} (x_1 - x_2) \left(1 - \frac{2R}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

$$\ddot{y}_1 = -\frac{k}{m} (y_1 - y_2) \left(1 - \frac{2R}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

$$\ddot{x}_2 = \frac{k}{m} (x_1 - x_2) \left(1 - \frac{2R}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

$$\ddot{y}_2 = \frac{k}{m} (y_1 - y_2) \left(1 - \frac{2R}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

We choose the initial conditions

$$x_1(0) = 1, y_1(0) = 1; \dot{x}_1(0) = 1, \dot{y}_1(0) = 1$$

$$x_2(0) = -1, y_2(0) = -1; \dot{x}_2(0) = 1, \dot{y}_2(0) = -1.$$

\* Two equal masses connected by spring on a smooth surface  
\* Numerical Solution of the ODEs involved

[t,z]=ODE23('two mass spring', [0 50], [1 0.2 1 -0.1 -1 0]);

cmx=0.5\*(z(:,1)+z(:,5));

cmy=0.5\*(z(:,3)+z(:,7));

hold on;

plot(z(:,1),z(:,3),'-');

plot(z(:,5),z(:,7),'-');

plot(cmx,cmy);

axis('square');

xlabel('x');

ylabel('y');

title('Basant Sharma, HW6, Problem 1 (5.147)').

\* Equations of Motion two\_mass\_spring.m

function zdot = twomass(t,z)

x1 = z(1);

vx1 = z(2);

y1 = z(3);

vy1 = z(4);

x2 = z(5);

vx2 = z(6);

y2 = z(7);

vy2 = z(8);

\* mass and spring constant

m = 1;

k = 1;

\* unstretched length of spring

R = 1;

\* the ratio k/m

f = k/m;

\* repetition of the same function

expn = 1 - 2\*R/sqrt((x1-x2)^2 + (y1-y2)^2);

x1dot = vx1;

vx1dot = -f\*(x1-x2)\*expn;

y1dot = vy1;

vy1dot = -f\*(y1-y2)\*expn;

x2dot = vx2;

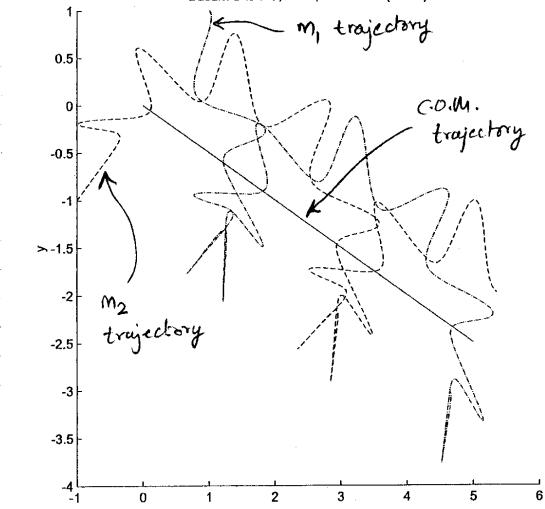
vx2dot = f\*(x1-x2)\*expn;

y2dot = vy2;

vy2dot = f\*(y1-y2)\*expn;

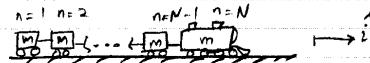
zdot = [x1dot, vx1dot, y1dot, vy1dot, x2dot, vx2dot, vy2dot]';

Basant Sharma, HW6, Problem 1 (5.147)



(2) G.2

Given: N cars of mass m connected by rigid links and no resistance to rolling of the cars. Power of engine is  $P_t$  and speed  $v_t$ . Find: the tension  $T_n$  between car n and car  $n+1$ .

FBD of the train

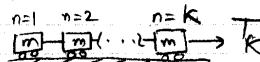
$$\text{By LMB, } F_t = M_t a_t \quad (1)$$

$$\text{By the definition of Power, } P_t = F_t v_t$$

$$\Rightarrow F_t = \frac{P_t}{v_t}$$

$$\text{Also } M_t = Nm, \text{ so (1) } \Rightarrow a_t = \frac{F_t}{M_t} = \frac{P_t}{Nm} = \frac{P_t}{v_t Nm}$$

Since the links between all the cars are rigid the acceleration of all cars is same.

FBD of k cars from the end

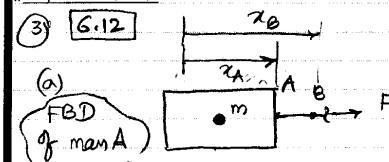
$$\text{By LMB, } T_k = (km) a_t = km \cdot \frac{P_t}{v_t Nm}$$

$$\therefore T_k = \frac{k P_t}{N v_t}$$

This is the tension between car k and car  $k+1$ .

(3) G.12

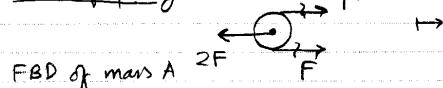
(3) G.12



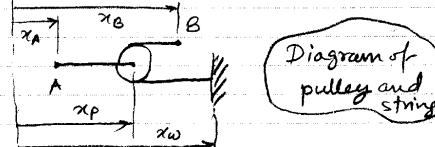
$$\text{By LMB (in x-dir), } m a_A = F \text{ so } a_A = \frac{F}{m}$$

$$\text{Since } x_B - x_A = \text{constant we get } \ddot{x}_B = \ddot{x}_A$$

$$\text{In particular } \ddot{x}_B = \ddot{x}_A \quad \text{or} \quad a_B = a_A = \frac{F}{m}$$

(b) FBD of pulley

$$\text{By LMB, } m a_A = 2F \quad \text{or} \quad a_A = \frac{2F}{m}$$



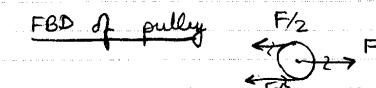
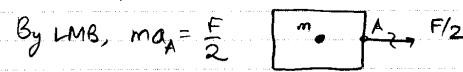
$$\text{Here } x_P - x_A = \text{const}$$

$$x_W - x_P + x_B - x_P = \text{const}$$

$$x_W = \text{const.}$$

$$\text{So } \ddot{x}_P = \ddot{x}_A \quad \text{and} \quad \ddot{x}_B = 2\ddot{x}_P \quad \text{or} \quad \ddot{x}_A = \frac{1}{2}\ddot{x}_B$$

$$\text{or} \quad a_A = \frac{1}{2} a_B = \frac{2F}{m}$$

(c) FBD of pulleyFBD of mass A

$$\text{By LMB, } m a_A = \frac{F}{2}$$

$$\text{or} \quad a_A = \frac{F}{2m}$$

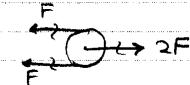
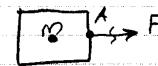
$$\text{Here } x_B - x_P = \text{const.}$$

$$x_P - x_W + x_B - x_A = \text{const.}$$

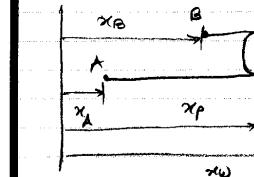
$$x_W = \text{const.}$$

$$\text{So } \ddot{x}_P = \ddot{x}_B \quad \text{and} \quad \ddot{x}_A = 2\ddot{x}_B$$

$$\text{or} \quad a_A = 2a_B = \frac{F}{2m}$$

(d) FBD of pulley:FBD of mass A:

$$\text{By LMB, } F = m a_A \quad \text{or} \quad a_A = \frac{F}{m}$$

Diagram of pulley and string

$$\text{Here } x_P - x_A + x_P - x_B = \text{const.}$$

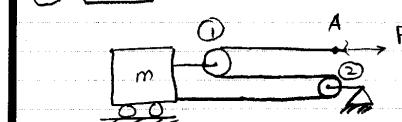
$$x_W - x_P = \text{const.}$$

$$x_W = \text{const.}$$

$$\text{So } \ddot{x}_W = \ddot{x}_P = 0 \quad \text{and} \quad \ddot{x}_A + \ddot{x}_B = 0 \quad \text{or} \quad \ddot{x}_A = -\ddot{x}_B$$

$$\text{or} \quad a_A = -a_B = \frac{F}{m}$$

(4) G.14

FBD of pulley ②FBD of pulley ①FBD of massFBD of mass

$$\text{By LMB, } m a_m = 3F$$

$$\text{or} \quad a_m = \frac{3F}{m}$$

Diagram of the string

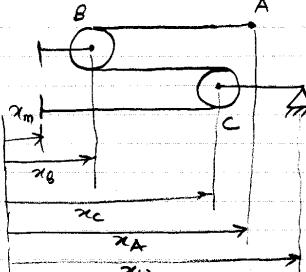
$$\text{Here}$$

$$x_C - x_m + x_C - x_B$$

$$+ x_A - x_B = \text{const.}$$

$$x_W - x_C = \text{const.}$$

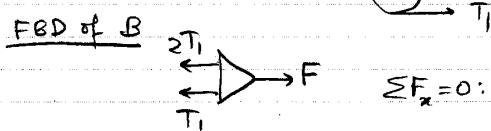
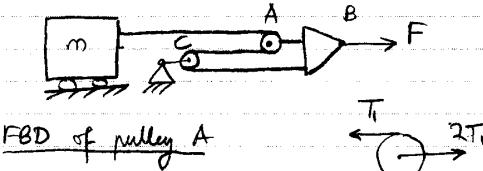
$$x_B - x_m = \text{const.}$$



$$\text{So } \ddot{x}_B = \ddot{x}_C = 0, \ddot{x}_m = \ddot{x}_B \quad \text{and } -\ddot{x}_m + \ddot{x}_B + \ddot{x}_A = 0 \quad \text{or} \quad \ddot{x}_A = 3\ddot{x}_m$$

$$\text{or } a_A = 3a_m = 3 \times \frac{3F}{m}$$

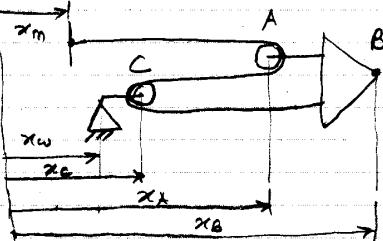
$$\text{So } a_A = \frac{9F}{m}$$



$$\text{By LMB, } ma_m = T_1 \quad \text{or } a_m = \frac{F}{3m}$$

Diagram of the string

$$\begin{aligned} \text{Here } & x_A - x_m + x_A - x_C \\ & + x_B - x_C = \text{const.} \\ & x_B - x_A = \text{const.} \\ & x_C - x_W = \text{const.} \end{aligned}$$



$$\text{So } \dot{x}_c = \dot{x}_w = 0$$

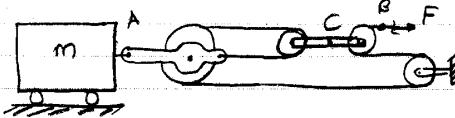
$$\dot{x}_B = \dot{x}_A \quad \text{and } 2\dot{x}_A - \dot{x}_m + \dot{x}_B = 0$$

$$\text{or } 3\dot{x}_B = \dot{x}_m \quad \text{or } a_B = \frac{1}{3}a_m = \frac{F}{9m}$$

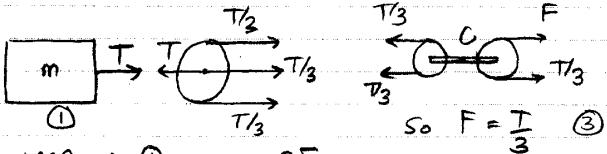
$$\text{Therefore } \frac{a_A}{a_B} = \frac{9F/m}{F/9m}$$

$$\text{or } \frac{a_A}{a_B} = 81$$

(5) [G.22]

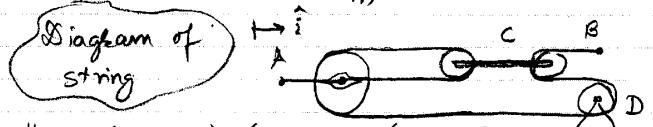


FBDs of mass and pulleys



$$\text{By LMB of } ①, ma_m = 3F$$

$$\text{so } a_m = \frac{3F}{m}$$



$$\text{Here } (x_B - x_D) + (x_D - x_C) + (x_C - x_A) + (x_C - x_A)2 = \text{const.}$$

$$x_D = \text{const.}$$

$$\text{So } \ddot{x}_B - 3\ddot{x}_A = 0 \quad \text{or } \ddot{x}_B = 3\ddot{x}_A = 3a_m$$

$$\text{so } a_A = \frac{3F}{m} \quad \text{and } a_B = \frac{9F}{m}$$

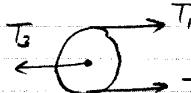
$$\text{By LMB of } ③, ma_c = 0 \Rightarrow a_c = 0 \text{ if } m \neq 0$$

but if C is massless then  $a_c$  is undetermined.

Thus  $a_c$  is determined by things we have neglected; most especially friction in pulleys, also mass of "block" at C.

Note:

For a pulley



we have always (as in FBD ③) had that  $T_2 = T_1$ , irrespective of  $T_3$ .

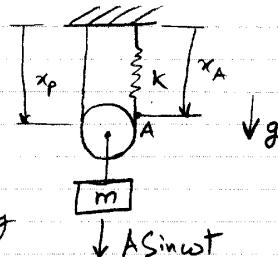
This follows if we take moments of all the forces about the center of the pulley and equate it to zero (as there is no torque applied and pulley is massless).

(6) 6.27

Given: a mass  $m$  hanging on a spring with constant  $k$  and a string as shown, and is forced by an external agent with the force  $F = A \sin \omega t$ .

a) Here  $x_p - x_A + x_p = \text{const}$ .  
so  $\Delta x_A = 2\Delta x_p$ .

So if mass  $m$  moves by a distance  $x$  downward then the spring stretches by  $2x$ .



FBD of pulley

$$\sum M_O \cdot \hat{k} = 0 : T = 2kx$$

$$\sum M_O \cdot \hat{k} = 0 : T = 2kx$$

FBD of mass

By LMB:  $m\ddot{x} = -4kx + mg + A \sin \omega t$

or  $m\ddot{x} + 4kx = mg + A \sin \omega t \quad (1)$

Solution of (1) is homogeneous solution

$$x(t) = a \cos 2\sqrt{\frac{k}{m}}t + b \sin 2\sqrt{\frac{k}{m}}t$$

$$+ \frac{mg}{4k} + \frac{A}{(4k-m\omega^2)} \sin \omega t$$

b) We neglect the homogeneous solution,

$$\text{so, } x(t) = \frac{mg}{4k} + \frac{A}{(4k-m\omega^2)} \sin \omega t$$

$$\Rightarrow T = 2kx \quad (2)$$

$$= \frac{mg}{2} + \frac{2kA}{(4k-m\omega^2)} \sin \omega t$$

The string becomes slack whenever  $T \leq 0$

i.e.  $\frac{mg}{2} + \frac{2kA}{(4k-m\omega^2)} \sin \omega t \leq 0$

has minimum value at  $\omega t = \frac{3\pi}{2}$

i.e.  $\frac{mg}{2} - \frac{2kA}{4k-m\omega^2} \leq 0$

Call  $\omega_0 = \sqrt{\frac{k}{m}}$ , so we need  $\frac{mg}{2} - \frac{2A}{4 - (\omega/\omega_0)^2} \leq 0$

i.e.  $\left(\frac{\omega}{\omega_0}\right)^2 \geq 4 \left(1 - \frac{A}{mg}\right)$

i.e.  $\omega \geq 2\omega_0 \sqrt{1 - \frac{A}{mg}}$  if  $A \leq mg$

If  $A > mg$  then any  $\omega$  works.

Therefore,  $\omega \geq 2\sqrt{\frac{k}{m} - \frac{A}{g}}$  for  $A \leq mg$

and any  $\omega$  for  $A > mg$ , so that the

string becomes slack at some point in the cyclical motion.