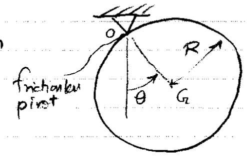
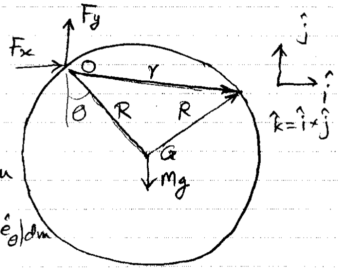
Solutions by Basant Sharma April 12 '02.

① $\boxed{7.134}$

Thin hoop of radius R and mass M hung from a point O on its edge.



a) FBD of hoop



AMB$_{/O}$: $\dot{\underset{\sim}{H}}_{/O} = \underset{\sim}{M}_{/O}$

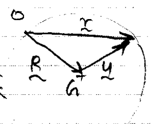
$\dot{\underset{\sim}{H}}_{/O} = \int_{\text{hoop; origin O}} \underset{\sim}{r}\times\underset{\sim}{a}\,dm$

$= \int \underset{\sim}{r}\times(-r\dot\theta^2\hat e_r + r\ddot\theta\hat e_\theta)\,dm$

$= \int r\hat e_r\times(-r\dot\theta^2\hat e_r + r\ddot\theta\hat e_\theta)\,dm$

$= \int r^2\ddot\theta\,\hat k\,dm$

$= \ddot\theta\int(\underset{\sim}{r}\cdot\underset{\sim}{r})\,dm\,\hat k = \ddot\theta\int(\underset{\sim}{R}+\underset{\sim}{u})\cdot(\underset{\sim}{R}+\underset{\sim}{u})\,dm\,\hat k$



$= \ddot\theta\left(\int R^2 dm + \cancel{2\int \underset{\sim}{u}\cdot\underset{\sim}{R}\,dm}^{\,0} + \int \underset{\sim}{u}\cdot\underset{\sim}{u}\,dm\right)\hat k$

$= \ddot\theta\left(MR^2 + \int u^2 dm\right)\hat k$

$= \ddot\theta\,(MR^2 + MR^2)\,\hat k = 2MR^2\ddot\theta\,\hat k$ $\quad I_O = I_G + MR^2$

$\underset{\sim}{M}_{/O} = -MgR\sin\theta\,\hat k$

So by AMB·$\hat k$ $\quad\boxed{2MR^2\ddot\theta = -MgR\sin\theta}$ —— (1)

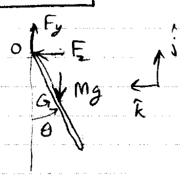
Assume θ is small for oscillations-in-plane considered, so $\sin\theta\approx\theta$

(1) ⇒ $\boxed{\ddot\theta + \frac{g}{2R}\theta = 0}$ —— (2)

Compare (2) with simple harmonic oscillator, so

$\boxed{\text{Time period} = \frac{2\pi}{\sqrt{g/2R}} = 2\pi\sqrt{\frac{2R}{g}}}$ —— (T1)

b) FBD of hoop



AMB$_{/O}$: $\dot{\underset{\sim}{H}}_{/O} = \underset{\sim}{M}_{/O}$

$\dot{\underset{\sim}{H}}_{/O} = I_{xx/O}\,\ddot\theta\,\hat k$,

with $I_{xx/O} = \frac12 I_{zz/G} + MR^2$ (by Parallel axis and Perpendicular axis theorems)

$= \frac32 MR^2$

$\underset{\sim}{M}_{/O} = -MgR\sin\theta\,\hat k$

AMB·$\hat k$ ⇒ $\boxed{\frac32 MR^2\ddot\theta = -MgR\sin\theta}$ —— (3)

Assume θ small, so $\sin\theta\approx\theta$

(3) ⇒ $\boxed{\ddot\theta + \frac{2g}{3R}\theta = 0}$ —— (4)

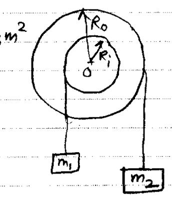
So $\boxed{\text{Time period} = \frac{2\pi}{\sqrt{2g/3R}} = 2\pi\sqrt{\frac{3R}{2g}}}$ —— (T2)

Comparing T1 and T2 we can see that the period of oscillations is less in the case of in and out of plane swinging (second case) as against in-plane swinging (first case)

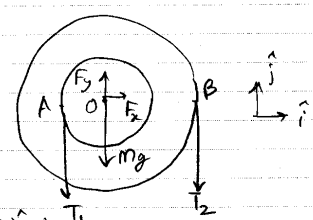
② $\boxed{7.141}$

$R_i = 0.2\,m$, $R_o = 0.4\,m$, $I_{/O} = 2.7\,kg\,m^2$

$m_1 = 40\,kg$, $m_2 = 100\,kg$.



a) FBD of pulley



AMB$_{/O}$:

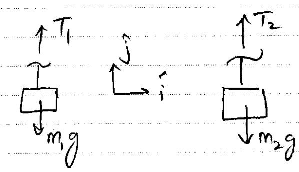
$\dot{\underset{\sim}{H}}_{/O} = \underset{\sim}{M}_{/O}$

So

$\{I_{/O}\alpha\hat k = (T_1R_i - T_2R_o)\hat k\}$

$\{\ \}\cdot\hat k$: $\quad I_{/O}\alpha = T_1R_i - T_2R_o$ —— (1)

FBD of masses m_1 and m_2



LMB: $\quad m_1\underset{\sim}{a}_1 = (T_1 - m_1g)\hat j$, $\quad m_2\underset{\sim}{a}_2 = (T_2 - m_2g)\hat j$

By Kinematics: $\quad \underset{\sim}{a}_1 = \underset{\sim}{a}_A = +\alpha\hat k\times R_i(-\hat i) = -R_i\alpha\hat j$

$\underset{\sim}{a}_2 = \underset{\sim}{a}_B = \alpha\hat k\times(R_o\hat i) = R_o\alpha\hat j$

So $\begin{Bmatrix} -m_1R_i\alpha\hat j = (T_1 - m_1g)\hat j \\ m_2R_o\alpha\hat j = (T_2 - m_2g)\hat j \end{Bmatrix}$

$\{\ \}\cdot\hat j$ ⇒ $\begin{rcases} T_1 = m_1(g - R_i\alpha) \\ T_2 = m_2(g + R_o\alpha)\end{rcases}$ —— (2)

(1) and (2) give

$\boxed{\alpha = \frac{(m_1R_i - m_2R_o)\,g}{I_{/O} + (m_1R_i^2 + m_2R_o^2)}}$

Using data $\quad\boxed{\alpha \approx -15.45\ rad/s^2}$ —— (3)

b) by (2), (3) and data

$\boxed{T_1 \approx 515.6\,N}$

$\boxed{T_2 \approx 362\,N}$

③ $\boxed{7.143}$

Given $R_A = 400\,mm$

$R_B = 200\,mm$

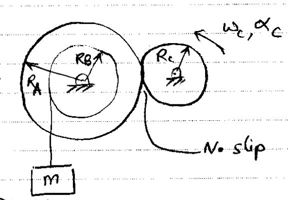
Combined $I_{/O}$ of A, B $= 0.5\,kg\,m^2$

$R_C = 300\,mm$

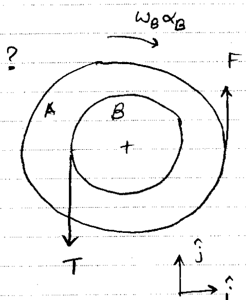
$m = 5\,kg$

$\omega = 60\,rpm = 2\pi\ rad/s$

$\alpha = 12\,rpm/s = \frac{2\pi}{5}\ rad/s^2$



acceleration of mass m $\quad\underset{\sim}{a} = ?$



By kinematics,

$\underset{\sim}{a} = \alpha_B\hat k\times(-R_B\hat i)$

$= -R_B\alpha_B\hat j$

and $R_C\omega_C = R_A\omega_B$

$R_C\alpha_C = R_A\alpha_B$

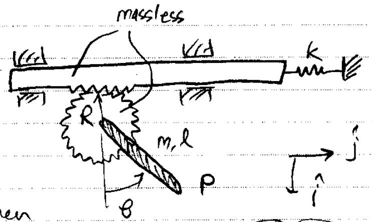
So $\quad\underset{\sim}{a} = -R_B\left(\frac{R_C}{R_A}\right)\alpha_C\,\hat j$

Using data $\quad\underset{\sim}{a} = -\frac{3}{20}\times\frac{2\pi}{5}\,\hat j\ \ m/s^2$

$\boxed{\underset{\sim}{a} \approx -0.188\ m/s^2}$

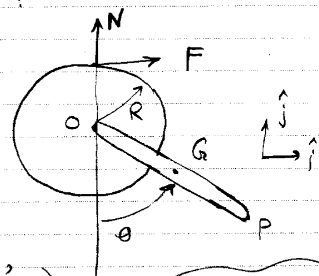
④ 7.149

(rack and gear)
Spring relaxed when $\theta = 0$.
at $t=0$, $\theta = \theta_0$.
acceleration of P (when $\theta = 0$) = ?



No gravity

<u>FBD of gear with stick</u> :



<u>AMB/o</u> : $\dot{\underset{\sim}{H}}_{/o} = \underset{\sim}{M}_{/o}$

$\dot{\underset{\sim}{H}}_{/o} = I_{zz/o}\, \ddot{\theta}\, \hat{k}$

$\Big(I_{zz/o} = I_{zz/G} + m\left(\frac{\ell}{2}\right)^2$

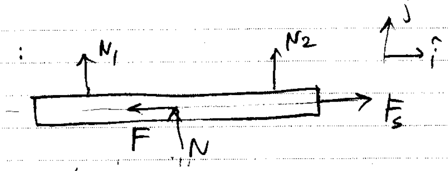
$= \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$,
Using parallel axes theorem $\Big)$

Since gear is massless we need only $I_{zz/o}$ of rod.

$\underset{\sim}{M}_{/o} = -FR\,\hat{k}$

$(AMB)\cdot\hat{k}$: $\boxed{\frac{m\ell^2}{3}\ddot{\theta} = -FR}$ ——— (1)

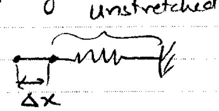
<u>FBD of rack</u> :



$(LMB)\cdot\hat{i}$: $F_s - F = \cancel{m}\, a_x$
0 (∵ rack is massless)

∴ $F = F_s$ ——— (2)

By kinematics the stretch in spring



$\Delta x = R\theta$
So $F_s = k\Delta x$
$= kR\theta$

∴ (2) ⇒ $F = kR\theta$

By (1) $\boxed{\frac{m\ell^2}{3}\ddot{\theta} = -kR\theta}$ ——— (1′)

Since $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$ (rests at $t=0$)

$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{3kR}{m\ell^2}}\right)t$

So the stick first crosses $\theta = 0$ at $t = \frac{\pi/2}{\sqrt{3kR/m\ell^2}}$

In general, $\theta = 0$ when $t = \frac{(2n+1)\pi/2}{\sqrt{3kR/m\ell^2}}$, n any integer.

Now $\dot{\theta}(t) = -\theta_0\sqrt{\frac{3kR^2}{m\ell^2}}\, \sin\left(\sqrt{\frac{3kR^2}{m\ell^2}}\right)t$

So when $\theta = 0$, $\dot{\theta}^2 = \theta_0^2\,\frac{3kR^2}{m\ell^2}$ ——— (3)

Clearly by (1′), when $\theta = 0$, $\ddot{\theta} = 0$ ——— (4)

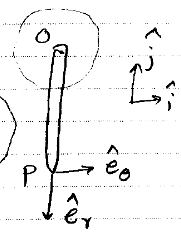
By rigid body kinematics, when $\theta = 0$

$\underset{\sim}{a}_P = -\ell\dot{\theta}^2\hat{e}_r + \ell\ddot{\theta}\,\hat{e}_\theta$
$= -\ell\dot{\theta}^2(-\hat{j}) + \ell\ddot{\theta}\,\hat{i}$

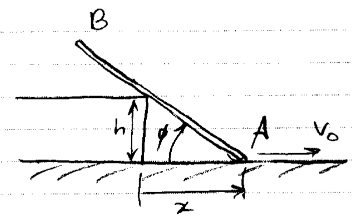
by (3), (4)

$\boxed{\underset{\sim}{a}_P = \frac{3kR^2}{m\ell}\theta_0^2\,\hat{j}}$

Picture of stick at $\theta = 0$



⑤ 8.1



Clearly only two quantities change with time here x, ϕ. ie. $x = x(t)$
$\phi = \phi(t)$.

Since $h = x\tan\phi$ is constant

$x(t) = h\,\mathrm{Cot}\,\phi(t)$

∴ $\frac{d}{dt}x(t) = h\frac{d}{dt}\mathrm{Cot}\,\phi(t)$ (using chain rule)

$= h\left.\frac{d}{d\phi}\mathrm{Cot}\,\phi\right|_{\phi=\phi(t)} \cdot \frac{d}{dt}\phi(t)$

∴ $\dot{x}(t) = -h\,\mathrm{Cosec}^2\phi(t)\,\dot{\phi}(t)$ ——— (1)

Given $\dot{x}(t) = V_0$ = constant

So $\dot{\phi}(t) = \frac{-V_0}{h\,\mathrm{Cosec}^2\phi(t)} = \frac{-V_0}{h(1+\mathrm{Cot}^2\phi(t))}$

∴ $\boxed{\dot{\phi}(t) = \frac{-V_0}{h\left(1+\left(\frac{x}{h}\right)^2\right)} < 0}$ ——— (2)

Differentiating (2) w.r.t t, $\ddot{\phi} = \frac{-(-V_0)}{h\left(1+\left(\frac{x}{h}\right)^2\right)^2}\cdot\frac{2x}{h^2}\dot{x}$

Using $\dot{x} = V_0$, $\boxed{\ddot{\phi} = \frac{2V_0^2 x}{h^3\left(1+\left(\frac{x}{h}\right)^2\right)^2} > 0}$