"Solutions" *

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Your TA: BURNS

T&AM 203 Prelim 1

Tuesday February 29, 2000 7:30 — 9:00$^+$ PM

Draft February 26, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Two pages of formulas, from the front of the text, and a blank page for tentative scrap work are provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;

↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
± all signs and directions are well defined with sketches and/or words;

→| reasonable justification, enough to distinguish an informed answer from a guess, is given;
* you clearly state any reasonable assumptions if a problem seems poorly defined;

- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);

□ your answers are $\boxed{\text{boxed}}$ in; and

≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

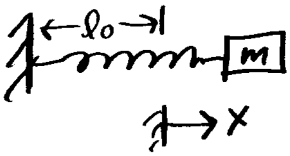
* The quotations are because often my "Solutions" have some remaining errors. (error in 1c corrected here)

I hope!

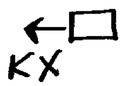
Problem	1:	30/30
Problem	2:	35/35
Problem	3:	35/35
TOTAL:		100 /100

1)(30 pts) The problems below are independent. MATLAB commands are not allowed except in part (c).

1a) (10 pts) A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? (Answer in terms of some or all of m, k, and d.) [Neglect gravity and friction.]



FBD:



LMB: $F = ma$

$\Rightarrow -kx = m\ddot{x}$

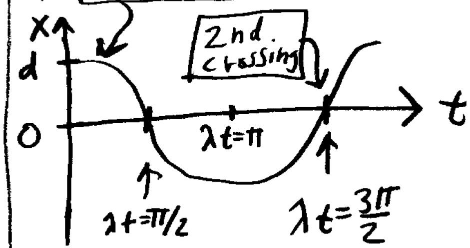
$\Rightarrow \ddot{x} + (k/m)x = 0$

$\Rightarrow x = A\cos(\lambda t) + B\sin(\lambda t)$

$\lambda = \sqrt{k/m}$

$x(0) = d \Rightarrow A = d$, $\dot{x}(0) = 0 \Rightarrow B = 0$

$\Rightarrow x = d\cos(\lambda t)$,



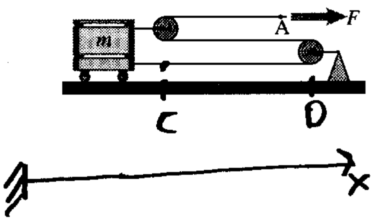
At second crossing

$\lambda t = 3\pi/2$

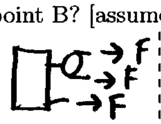
$\Rightarrow t = 3\pi/2\lambda$

$\Rightarrow \boxed{t = \frac{3\pi}{2\sqrt{k/m}}}$

1b) (10 pts) Assuming equal masses and equal forces in the two cases, what is the ratio of the acceleration of point A to that of point B? [assume massless ideal pulleys etc]



FBD



LMB

$3F = m\ddot{x}_C$

$\ddot{x}_C = 3F/m$ (1)

Kinematics

$const = \ell_{CA} + 2\ell_{CD}$

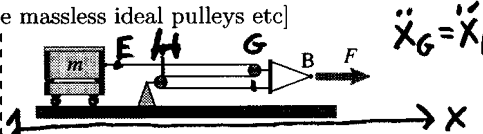
$= (x_A - x_C) + 2(x_D - x_C)$

$\ddot{(const)} = (\ddot{x}_A - \ddot{x}_C) + 2(\cancel{\ddot{x}_D} - \ddot{x}_C)$, $\ddot{x}_D \to 0$

$0 = \ddot{x}_A - 3\ddot{x}_C$ (2)

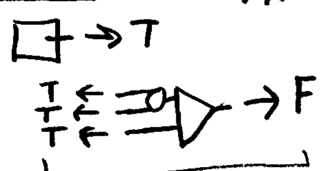
$\Rightarrow \ddot{x}_A = 3\ddot{x}_C$

(1)&(2) $\Rightarrow \ddot{x}_A = 9F/m$ (3)



$\ddot{x}_G = \ddot{x}_B$

FBDs



$\Rightarrow T = F/3$

LMB $T = m\ddot{x}_E$

$F/3 = m\ddot{x}_E$ (4)

Kinematics

$const = \ell_{GE} + 2\ell_{GH}$

$= (x_G - x_E) + 2(x_G - x_H)$

$\ddot{(const)} = (\ddot{x}_G - \ddot{x}_E) + 2(\ddot{x}_G - \cancel{\ddot{x}_H})$, $\ddot{x}_H \to 0$

$0 = 3\ddot{x}_G - \ddot{x}_E$

$\Rightarrow \ddot{x}_G = \ddot{x}_B = \ddot{x}_E/3$ (5)

(4)&(5) $\Rightarrow \ddot{x}_B = F/9m$

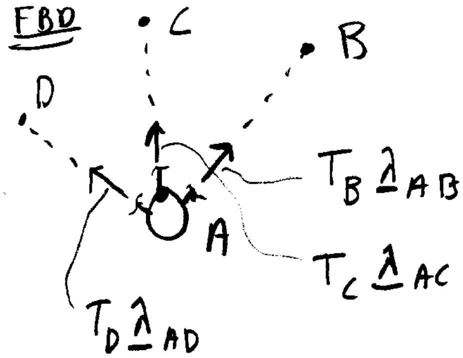
Comparison

$\frac{\ddot{x}_A}{\ddot{x}_B} = \frac{9F/m}{F/9m}$

$= \boxed{81}$ (b)

(Point B "feels" 81 times less massive than pt. A)

1c) (10 pts) In three-dimensional space with no gravity a particle with $m = 3\,\text{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\underline{\mathbf{a}} = (1\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}})\,\text{m/s}^2$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]



LMB $\quad T_B \underline{\lambda}_{AB} + T_C \underline{\lambda}_{AC} + T_D \underline{\lambda}_{AD} = m\underline{a}$

$$\Rightarrow \overbrace{\underbrace{\left[\begin{bmatrix} \underline{\lambda}_{AB} \end{bmatrix} \Bigg| \begin{bmatrix} \underline{\lambda}_{AC} \end{bmatrix} \Bigg| \begin{bmatrix} \underline{\lambda}_{AD} \end{bmatrix} \right]}_{\text{comps of unit vectors in columns}}}^{A} \begin{bmatrix} T_A \\ T_B \\ T_C \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underbrace{[A][T] = m[a]}_{\text{need to solve for } [T]}$$

```
% a MATLAB script file to find 3 tensions
m   = 3;
a   = [ 1 2 3]';

rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';

uAB = rAB/norm(rAB); % norm gives vector magnitude
% You write the code below (4 to 5 lines).
% Don't copy any of the numbers above.
% Don't do any arithmetic on the side.
```

This code does the job.

```
uAC = rAC/norm(rAC);   % The other two
uAD = rAD/norm(rAD);   % unit vectors

A = [uAB  uAC  uAD];   % assemble A

T = A\(m*a)            % Solve with backslash
```

2)(35 pts) A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\underline{\mathbf{v}}|$, where $\underline{\mathbf{v}}$ is the velocity. Do not neglect gravity.

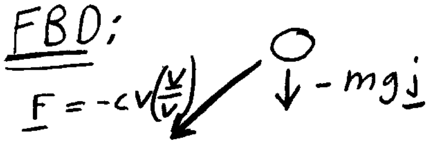
a) (10 ptss) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) (15 pts) Starting with a free body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (10 pts) (challenge, do last, long calculation) Assume the particle is thrown from $\underline{\mathbf{r}} = \underline{\mathbf{0}}$ with $\underline{\mathbf{v}} = v_{x0}\,\hat{\mathbf{i}} + v_{y0}\,\hat{\mathbf{j}}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.)

[Hint: i) find $x(t)$ and $y(t)$ like in the homework, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.). If you use Matlab, start your code by assigning any non-trivial values to all constants.]

FBD:



LMB: $\underline{F} = m\underline{a}$

$$\Rightarrow \left\{-c\underline{V} - mg\underline{j} = m(\dot{V}_x\underline{i} + \dot{V}_y\underline{j})\right\}$$

$$\{\}\cdot\underline{i} \Rightarrow \quad -cV_x = m\dot{V}_x \Rightarrow \boxed{m\ddot{x} + c\dot{x} = 0}$$

$$\{\}\cdot\underline{j} \Rightarrow \quad -cV_y - mg = m\dot{V}_y \Rightarrow \boxed{m\ddot{y} + c\dot{y} = -mg}$$

(b)

Steady State $\Rightarrow \dot{V}_x = 0,\ \dot{V}_y = 0 \Rightarrow V_x = 0,\ V_y = \frac{-mg}{c}$

$\Rightarrow$ steady state falling speed $= \boxed{mg/c}$ (a)

Trajectory eqs

(1) $\dot{V}_x + \frac{c}{m}V_x = 0$

(3) $\dot{x} = V_x$

(5) $\dot{V}_y + \frac{c}{m}V_y = -g$

(7) $\dot{y} = V_y$

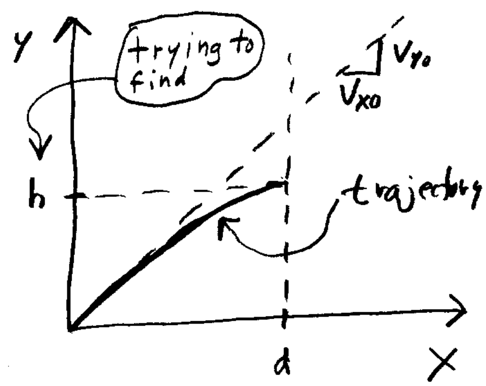
IC's

(2) $V_x(0) = V_{xo}$

(4) $x(0) = 0$

(6) $V_y(0) = V_{yo}$

(8) $y(0) = 0$



Find $x(t)$

(1) $\Rightarrow \quad V_x = C_1 e^{-(c/m)t}$, (2) $\Rightarrow C_1 = V_{xo} \Rightarrow V_x = V_{xo}e^{-(c/m)t}$

(3) $\Rightarrow \quad x(t) = -\frac{mV_{xo}}{c}e^{-(c/m)t} + C_2$, (4) $\Rightarrow C_2 = \frac{mV_{xo}}{c} \Rightarrow \boxed{x = \frac{mV_{xo}}{c}\left(1 - e^{-(c/m)t}\right)}$

(like homework)

Find t from x

$$\frac{cx}{mV_{xo}} = 1 - e^{-(c/m)t} \Rightarrow e^{-(c/m)t} = 1 - \frac{cx}{mV_{xo}} \Rightarrow -(c/m)t = \ln\left(1 - \frac{cx}{mV_{xo}}\right)$$

$$\Rightarrow \boxed{t = \frac{-m}{c}\ln\left(1 - \frac{cx}{mV_{xo}}\right)} \ (1)$$

{note, this inversion is possible because particle always moves to right.}

Solve for $y(t)$

$$(5) \Rightarrow \quad V_y = C_3 e^{-(c/m)t} - \frac{mg}{c}, \quad (6) \Rightarrow C_3 = V_{y0} + \frac{mg}{c}$$

$$\Rightarrow \quad V_y(t) = \left(V_{y0} + \frac{mg}{c}\right) e^{-(c/m)t} - \frac{mg}{c}$$

$$(7) \Rightarrow \quad y(t) = -\frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right) e^{-(c/m)t} - \frac{mg}{c} t + C_4$$

$$(8) \Rightarrow \quad 0 = -\frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right) + C_4 \Rightarrow C_4 = \frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right)$$

$$\Rightarrow \quad \boxed{y(t) = \frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{mg}{c} t} \quad (2)$$

(like homework)

[Define: char. time, $t_c = m/c$ | V steady state, $V_s = \frac{mg}{c}$
{to simplify algebra} max. X, $X_m = mV_{x0}/c$]

$$(1) \Rightarrow \quad t = -t_c \ln(1 - X/X_m) \quad (3)$$

$$(2) \Rightarrow \quad y(t) = t_c(V_{y0} + V_s)\left(1 - e^{-t/t_c}\right) - V_s t \quad (4)$$

Substitute (4) into (3)

$$Y = t_c(V_{y0} + V)\left(1 - (1 - X/X_m)\right) + V_s t_c \ln(1 - X/X_m)$$

$$Y = t_c(V_{y0} + V_s)\left(\frac{X}{X_m}\right) + V_s t_c \ln(1 - X/X_m)$$

$$\Rightarrow h = t_c(V_{y0} + V_s)\left(\frac{d}{X_m}\right) + V_s t_c \ln(1 - d/X_m)$$

$$\boxed{h = V_s t_c \left[\left(\frac{V_{y0}}{V_s} + 1\right)\left(\frac{d}{X_m}\right) + \ln(1 - d/X_m)\right]} \quad (c)$$

$\boxed{X_m = mV_{x0}/c}$ $\boxed{V_s = mg/c}$ $\boxed{t_c = m/c}$

Checks:

$d \to 0 \Rightarrow$
$\ln(1 - d/X_m) \approx -d/X_m$

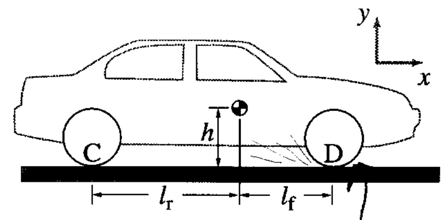
$$\Rightarrow h \sim \cancel{V_s} t_c \left(\frac{V_{y0}}{\cancel{V_s}}\right)\frac{d}{X_m} = \frac{\cancel{m}}{\cancel{c}} V_{y0} \frac{d}{\cancel{m}V_{x0}/\cancel{c}} = \frac{V_{y0}}{V_{x0}} d \quad \text{(which makes sense, see sketch)}$$

$$\underline{c \to 0} \Rightarrow \ln(1 - d/X_m) \approx -d/X_m - \tfrac{1}{2}(d/X_m)^2 \Rightarrow h \sim \frac{V_{y0}}{V_{x0}} d - \frac{1}{2}\left(\frac{m}{c}\right)\left(\frac{mg}{c}\right)\left(\frac{d}{mV_{x0}/c}\right)^2$$

$\boxed{\text{Taylor series}}$

$$\sim \frac{V_{y0}}{V_{x0}} - \frac{1}{2} g\left(\frac{d}{V_{x0}}\right)^2 \quad \text{(parabolic trajectory)}$$

3) 35 pts) Car accelerating. A car (mass $=m$) with a big motor, front-wheel drive, and a stiff suspension accelerates to the right with the front wheels over-powered and skidding (friction coefficient $=\mu$) and back wheels turning freely.



skids w/ μ

a) (5 pts) Assuming the car starts from rest and has constant acceleration a, how far has it travelled in time t? (Answer in terms of a and t.) [Not a trick, just easy.].

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1$$

$$v(0)=0 \Rightarrow \quad v(t) = at$$

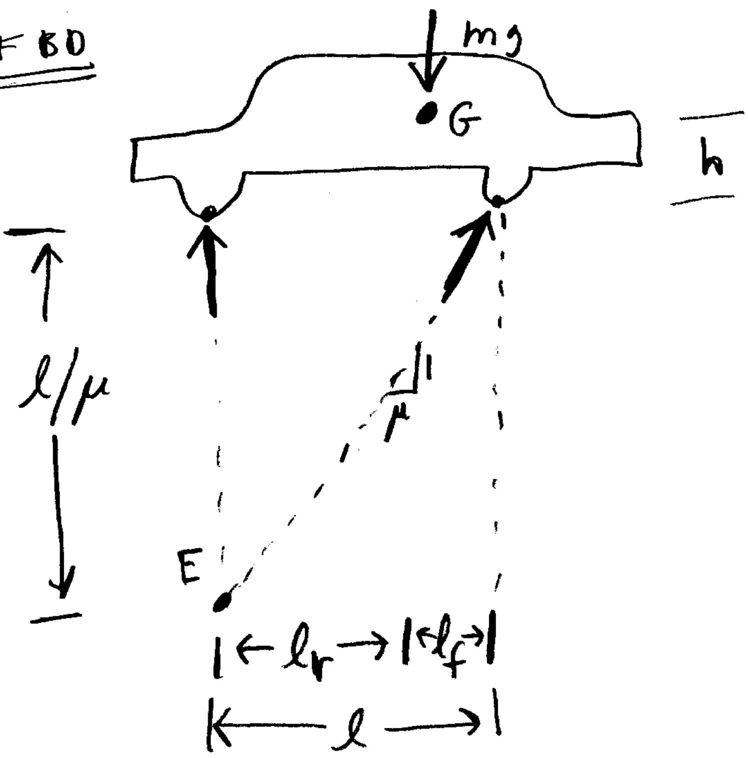
$$x(t) = \int v(t)\,dt = \int at\,dt = \frac{at^2}{2} + C_2$$

$$x(0)=0 \Rightarrow C_2 = 0$$

$$\Rightarrow \boxed{x(t) = \tfrac{1}{2}at^2} \quad (a)$$

b) (30 pts) Find a in terms of any or all of ℓ_r, ℓ_f, h, m, g and μ. [Hint: all the directions on the cover page apply. Your answer should reduce to $a = \ell_r g/h$ in the limit $\mu \to \infty$.]

FBD



$\ell = \ell_f + \ell_r$

Assume: massless tires, rigid body,

$\underline{a}_G = \underline{a} = a\,\underline{i} = a_G\,\underline{i}$

AMB

$$\sum \underline{M}_{/E} = \dot{\underline{H}}_{/E} \quad (1)$$

Evaluate left hand side of (1)

$$\sum \underline{M}_{/E} = \underline{r}_{EG} \times (-mg\,\underline{j})$$

$$= \left(\ell_r\,\underline{i} + \left(\frac{\ell}{\mu} + h\right)\underline{j}\right) \times (-mg\,\underline{j})$$

$$= -\ell_r mg\,\underline{k} \quad (2)$$

Ev. right hand side of (1)

$$\dot{\underline{H}}_{/E} = \underline{r}_{EG} \times m\,\underline{a}_G$$

$$= \left(\ell_r\,\underline{i} + \left(\frac{\ell}{\mu} + h\right)\underline{j}\right) \times (ma\,\underline{i})$$

$$= -ma\left(\frac{\ell}{\mu} + h\right)\underline{k} \quad (3)$$

Plug (2) & (3) back into (1)

$$\Rightarrow \quad \sum \underline{M}_{/E} = \dot{\underline{H}}_{/E}$$

$$\left\{ -\ell_r m g \underline{K} = -ma\left(\frac{\ell}{\mu} + h\right)\underline{K} \right\}$$

$$\{\ \}\cdot\underline{K} \Rightarrow \ell_r \cancel{m} g = \cancel{m} a\left(\frac{\ell}{\mu} + h\right)$$

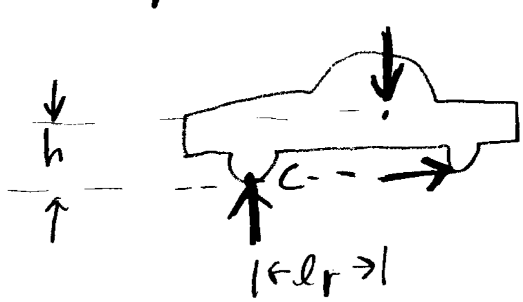
$$a = g\frac{\ell_r}{\left(\frac{\ell}{\mu} + h\right)}$$

$$\boxed{a = g\left[\frac{\ell_r}{\frac{\ell_r + \ell_f}{\mu} + h}\right]} \quad (b)$$

Check: $a \underset{\mu\to\infty}{=} g\frac{\ell_r}{h}$

[No matter how big is μ, a front wheel drive car can't have more accel. than this.]

Why? when $\mu \to 0$ FBD looks like this



If we look at

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{C}$$

$$\Rightarrow mg\ell_r = ahm$$

$$\Rightarrow a = g\ell_r/h$$

(gravity "balances" acceleration.)