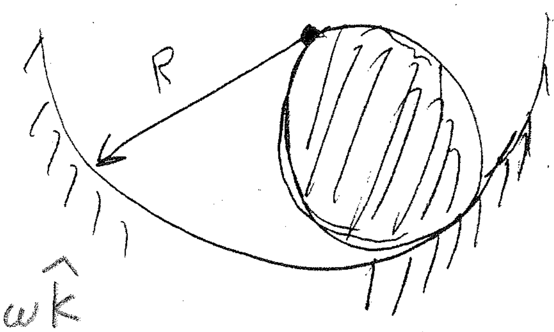
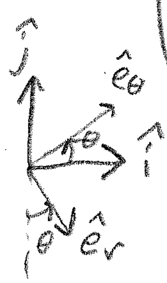
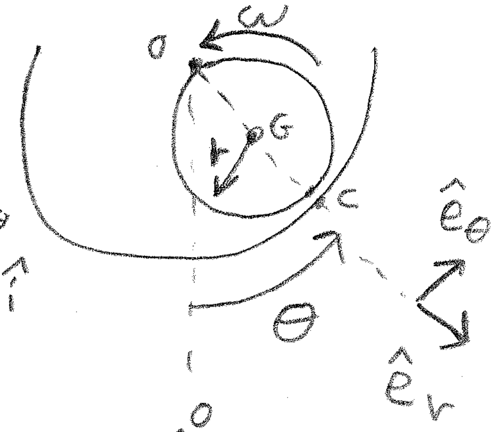
1) A uniform disk with mass m and diameter R rolls back and forth in a trough with radius R. Assuming small oscillations what is the period of oscillation? Answer in terms of some or all of R, g and m.



$\downarrow g$





$\vec{\omega} = \omega \hat{k}$

$\omega = \omega_{disk}$

$2r = R$

<u>Kinematics</u>

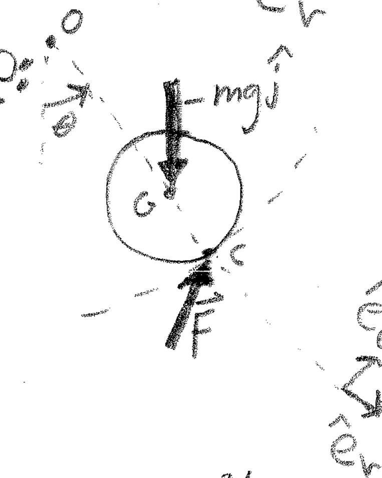
$$\vec{V}_G = \vec{V}_G$$

$$\left\{ \dot{\theta} r \hat{e}_\theta = -\omega r \hat{e}_\theta \right\}$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow \boxed{\dot{\theta} = -\omega} \ (1)$$

$$\hat{e}_r \times \hat{j} = \sin\theta \hat{k}$$

FBD:



<u>AMB/c</u>

$$I^G = \int \rho^2 dm = mr^2/2$$

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$\vec{r}_{G/c} \times -mg\hat{j} = \vec{r}_{G/c} \times (m\vec{a}_G) + I^G \dot{\omega} \hat{k}$$

$$\dot{\omega} = -\ddot{\theta}$$

$$\vec{r}_{G/c} = -r\hat{e}_r \qquad \vec{a}_G = r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

$$\Rightarrow (-r\hat{e}_r) \times (-mg\hat{j}) = (-r\hat{e}_r) \times [m(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r)] + \tfrac{1}{2}mr^2(-\ddot{\theta}\hat{k})$$

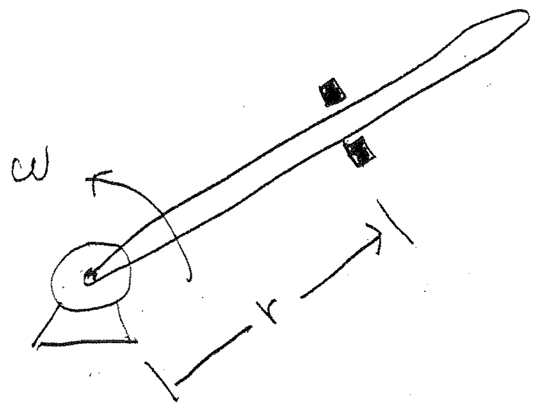
$$\left\{ rmg\sin\theta \hat{k} = -mr^2\ddot{\theta}\hat{k} - \tfrac{1}{2}mr^2\ddot{\theta}\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad rmg\sin\theta = -\ddot{\theta}\,\tfrac{3}{2}mr^2$$

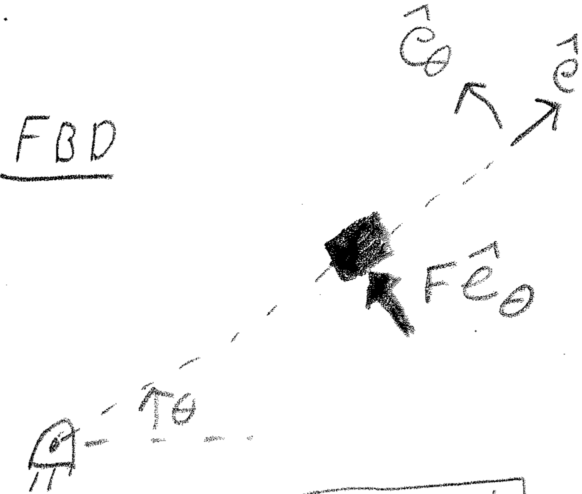
$$\theta << 1 \Rightarrow \sin\theta \doteq \theta \Rightarrow \ddot{\theta} + \frac{2g}{3r}\theta = 0 \Rightarrow \text{say, } \theta = \theta_0 \cos\sqrt{\frac{2g}{3r}}\, t$$

$$r = R/2, \quad t^* = 1 \text{ period} \Rightarrow \sqrt{\frac{4g}{3r}}\, t^* = 2\pi \Rightarrow \boxed{t^* = 2\pi\sqrt{\frac{3r}{4g}} = \pi\sqrt{\frac{3r}{g}}}$$

2) A small bead m slides without friction on a straight rod which is rotating at constant ω about a point on the rod. Neglect gravity. 2D. Find $\ddot{r}$ in terms of some or all of r, $\dot{r}$ and ω.



FBD



$\omega = \text{const}$
$\Rightarrow \dot{\omega} = 0$

LMB:

$$\sum \vec{F} = m\vec{a}$$

$$\left\{ F\hat{e}_\theta = m\left[(\ddot{r} - \omega^2 r)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta\right.\right.$$

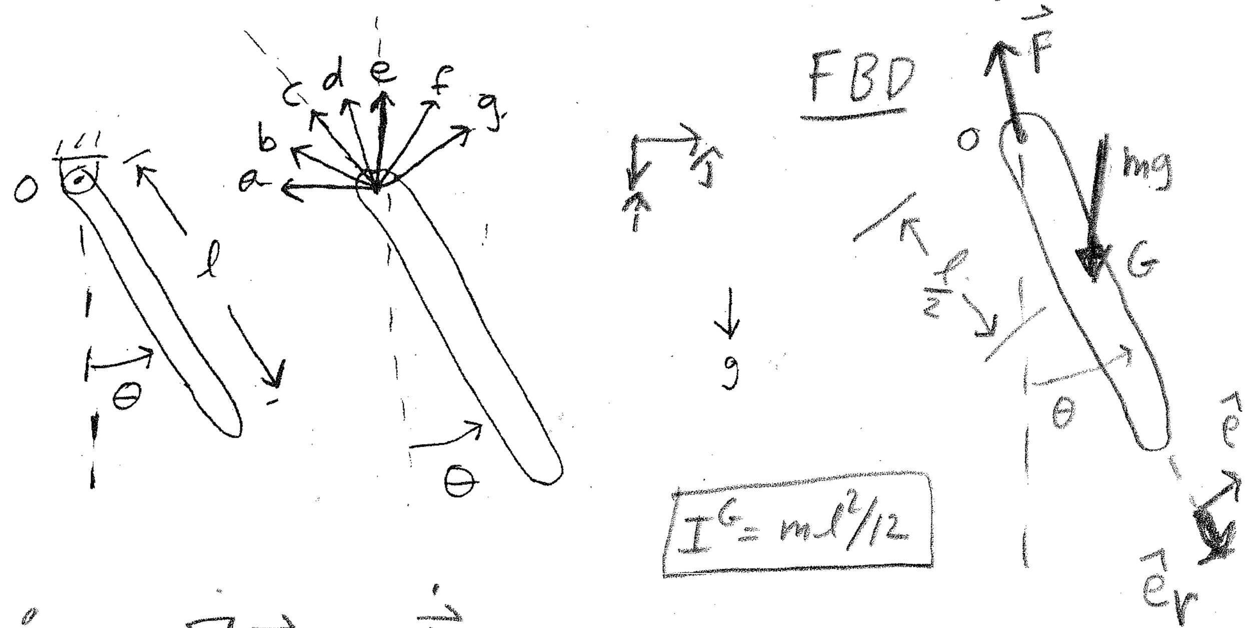
$$\{\}\cdot\hat{e}_r \Rightarrow \quad 0 = \ddot{r} - \omega^2 r$$

$$\Rightarrow \quad \boxed{\boxed{\ddot{r} = \omega^2 r}}$$

3) A uniform stick with mass m and length ℓ swings from a frictionless hinge at 0.

a) Find the equation of motion. That is, find $\ddot{\theta}$ in terms of some or all of θ, $\dot{\theta}$, ℓ, m and g.

b) The partial free body diagram shows some conceivable reaction forces at O. Which of these are you confident are in the wrong direction (a,b,c,d,e,f,g)? Justify your answer in a manner that is fully convincing.



$$\boxed{I^G = m\ell^2/12}$$

AMB/O:
$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\underbrace{\vec{r}_{G/O}}_{\frac{\ell}{2}\hat{e}_r} \times mg\hat{\imath} = \vec{r}_{G/O} \times m\underbrace{\vec{a}_G}_{\ddot{\theta}\frac{\ell}{2}\hat{e}_\theta - \frac{\ell}{2}\dot{\theta}^2\hat{e}_r} + I^G\ddot{\theta}\hat{k}$$

$$\Rightarrow \quad \frac{\ell}{2}\hat{e}_r \times mg\hat{\imath} = \frac{m\ell}{2}\hat{e}_r \times \left(\ddot{\theta}\frac{\ell}{2}\hat{e}_\theta - \frac{\ell}{2}\dot{\theta}^2\hat{e}_r\right) + I\ddot{\theta}\hat{k}$$

$$\left\{ -\frac{\ell}{2}\cancel{m}g\sin\theta\,\hat{k} = \cancel{m}\frac{\ell^2}{4}\ddot{\theta}\hat{k} + \frac{\cancel{m}\ell^2}{12}\ddot{\theta}\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad -\frac{\ell g}{2}\sin\theta = \frac{\ell^2}{3}\ddot{\theta}$$

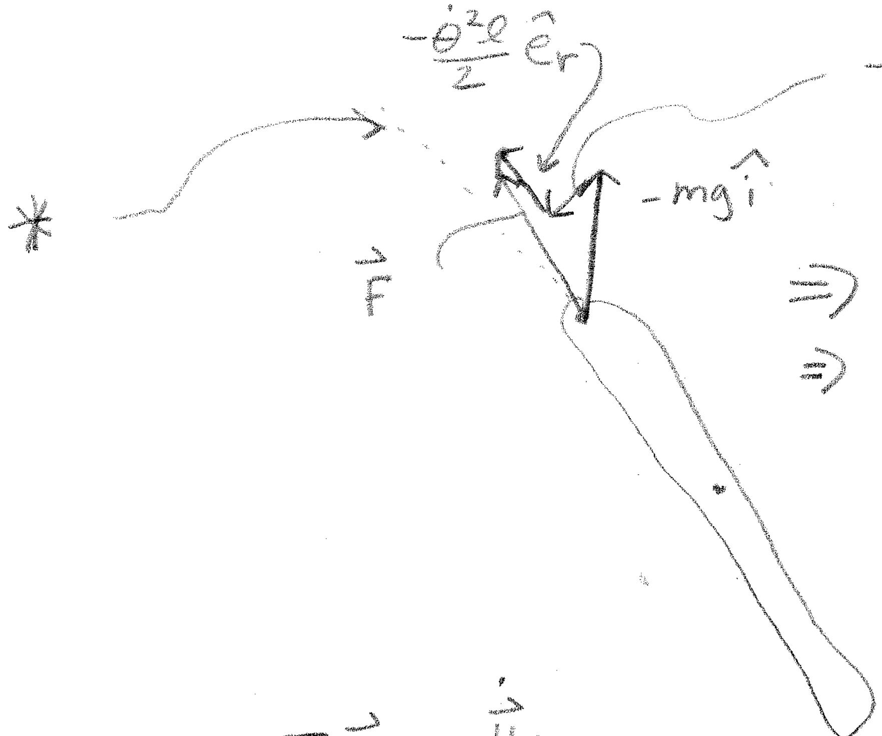
$$\Rightarrow \quad \boxed{\ddot{\theta} = \frac{-3g}{2\ell}\sin\theta}$$

<u>LMB:</u> $\sum \vec F = m\vec a$

$$-\frac{3g}{2\ell}\sin\theta$$

$$\vec F + mg\hat i = m\left(\ddot\theta\,\frac{\ell}{2}\hat e_\theta - \dot\theta^2\frac{\ell}{2}\hat e_r\right)$$

$$\Rightarrow \quad \vec F = m\left[-g\hat i - \frac{3g}{4}\sin\theta\,\hat e_\theta + \left(\dot\theta^2\frac{\ell}{2}\right)\hat e_r\right]$$



$\Rightarrow \vec F$ is up & to left.

$\Rightarrow \vec F\cdot\hat j < 0$

$\vec F\cdot\hat i < 0$

<u>AMB/G:</u> $\sum \vec M_{/G} = \dot{\vec H}_{/G}$

$$\underbrace{\left(-\frac{\ell}{2}\hat e_r\right)}_{\vec r_{O/G}} \times \vec F = \ddot\theta I\hat k$$

$$\ddot\theta = -\frac{3g}{2\ell}\sin\theta$$

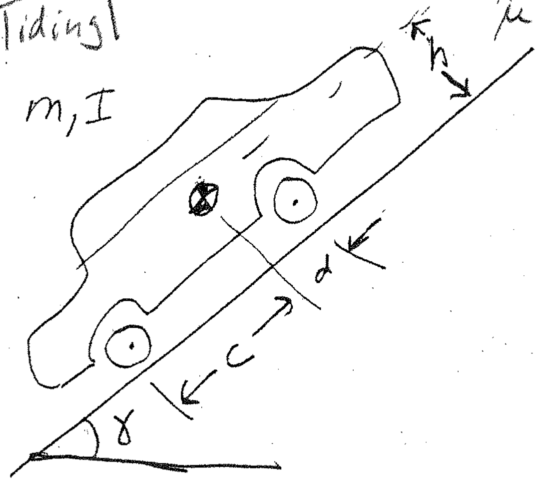
for $\theta > 0 \Rightarrow \ddot\theta < 0 \Rightarrow \vec F$ causes CW rotation

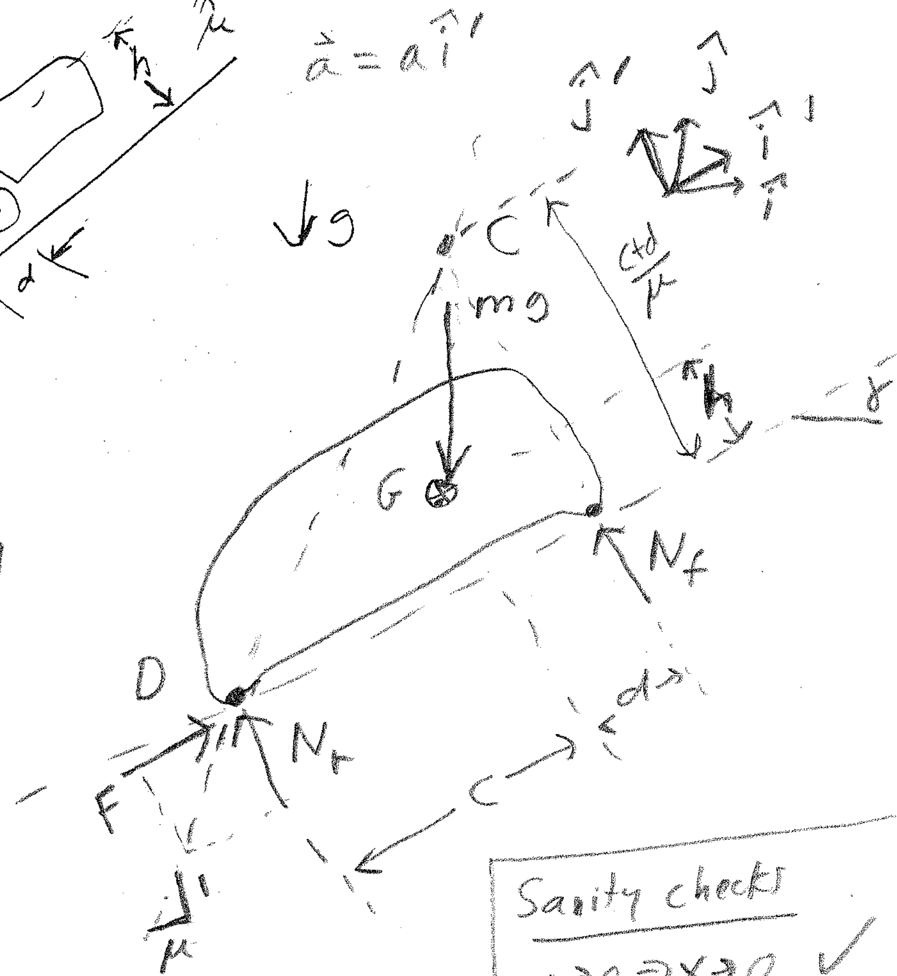
$\Rightarrow \vec F$ points to right of line *

$\boxed{\text{Only (d) is possible.}}$

4) A rear-wheel drive car attempts to drive uphill. Assume it does not tip over. What is the steepest slope γ it can go up? Answer in terms of some or all of m, c, d, h, g, μ and the moment of inertia about the center of mass I.

without sliding





Borderline case:

$\vec{a} = \vec{0} \Rightarrow$ statics!

AMB/C:

$$\sum \vec{M}_{/C} = \vec{0}$$

$$\vec{r}_{G/C} \times (-mg\hat{j}) = \vec{0}$$

$$\left[\left(\frac{-(c+d)}{\mu} + h\right)\hat{j}' - d\,\hat{i}'\right] \times (-mg\hat{j}) = \vec{0}$$

$$\Rightarrow \left\{ -\left(\frac{-(c+d)}{\mu} + h\right)\underbrace{\hat{j}' \times \hat{j}}_{-\sin\gamma \hat{k}} + d\,\underbrace{\hat{i}' \times \hat{j}}_{\cos\gamma \hat{k}} = \vec{0} \right\}$$

$$\{\ \} \cdot \hat{k} \Rightarrow \boxed{\tan\gamma = \frac{d}{\frac{c+d}{\mu} - h}}$$

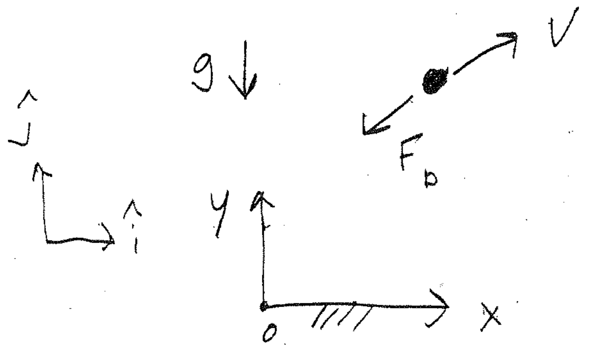
Sanity checks

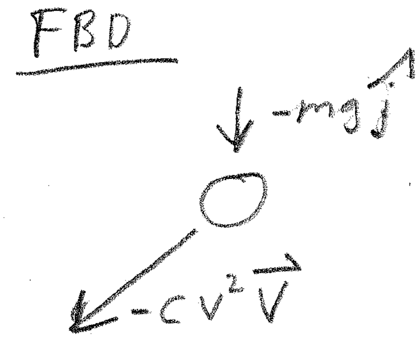
$\mu \to 0 \Rightarrow \gamma \to 0$ ✓

$c = h = 0 \Rightarrow \tan\gamma = \mu$ ✓

$d = 0 \Rightarrow \gamma = 0$ ✓ (interesting)

5) A particle m is acted on by gravity and a cubic drag force $F_D = cv^3$ that opposes its motion. Find $\ddot{x}$ in terms of some or all of $x, y, \dot{x}, \dot{y}, m, g$ and c.





LMB

$$\sum \vec{F} = m\vec{a}$$

$$\left\{ -mg\hat{j} - cv^2\vec{v} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) \right\}$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\dot{x}^2 + \dot{y}^2$$

$$\{\ \}\cdot\hat{i} \Rightarrow \quad \ddot{x} = \frac{-c}{m} v^2 \dot{x}$$

$$\boxed{\ddot{x} = \frac{-c}{m}(\dot{x}^2 + \dot{y}^2)\dot{x}}$$