Your TA, Section # and Section time:

"SOLUTION"

Your name:

ANDY RUINA

Cornell TAM 2030

No calculators, books or notes allowed.

5 Problems, 150 minutes total.

Final Exam

May 8, 2009

Directions. To ease your TA's grading and to maximize your score, please:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- √+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$\dot{\theta}_7$ `= 18`" instead of, say, "`theta7dot = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60 – 100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- □ Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
  Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 1: ____ /25

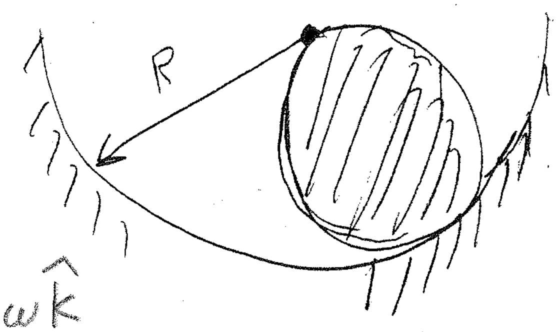
Problem 2: ____ /25

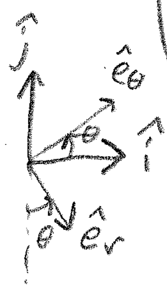
Problem 3: ____ /25

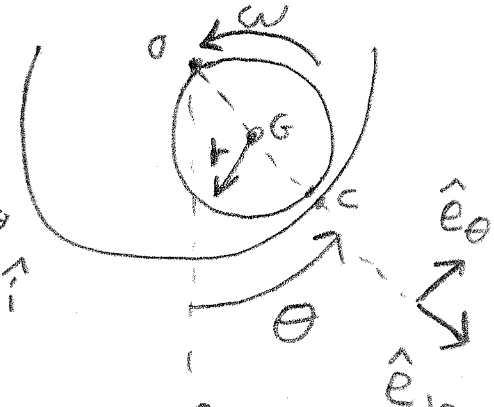
Problem 4: ____ /25

Problem 5: ____ /25

1) A uniform disk with mass m and diameter R rolls back and forth in a trough with radius R. Assuming small oscillations what is the period of oscillation? Answer in terms of some or all of R, g and m.





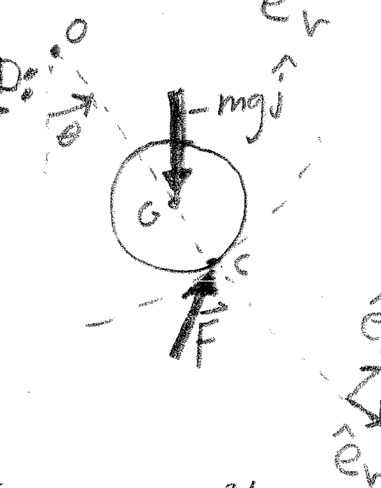


$\vec{\omega} = \omega\hat{k}$

$\omega = \omega_{disk}$

$2r = R$

FBD:



<u>Kinematics</u>

$$\vec{V}_G = \vec{V}_G$$

$$\{\dot{\theta} r \hat{e}_\theta = -\omega r \hat{e}_\theta\}$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow \boxed{\dot{\theta} = -\omega} \quad 1$$

$$\hat{e}_r \times \hat{j} = \sin\theta\,\hat{k}$$

<u>AMB/c</u>

$$I^G = \int \rho^2 dm = mr^2/2$$

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$\vec{r}_{G/c} \times -mg\hat{j} = \vec{r}_{G/c} \times (m\vec{a}_G) + I^G \dot{\omega}\hat{k}$$

$$\dot{\omega} = -\ddot{\theta}$$

$$\vec{r}_{G/c} = -r\hat{e}_r \qquad \vec{a}_G = r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

$$\Rightarrow (-r\hat{e}_r)\times(-mg\hat{j}) = (-r\hat{e}_r)\times[m(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r)] + \tfrac{1}{2}mr^2(-\ddot{\theta}\hat{k})$$

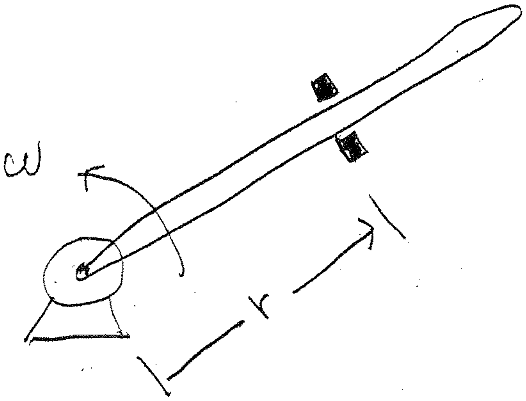
$$\{rmg\sin\theta\,\hat{k} = -mr^2\ddot{\theta}\hat{k} - \tfrac{1}{2}mr^2\ddot{\theta}\hat{k}\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad rmg\sin\theta = -\ddot{\theta}\,\tfrac{3}{2}mr^2$$

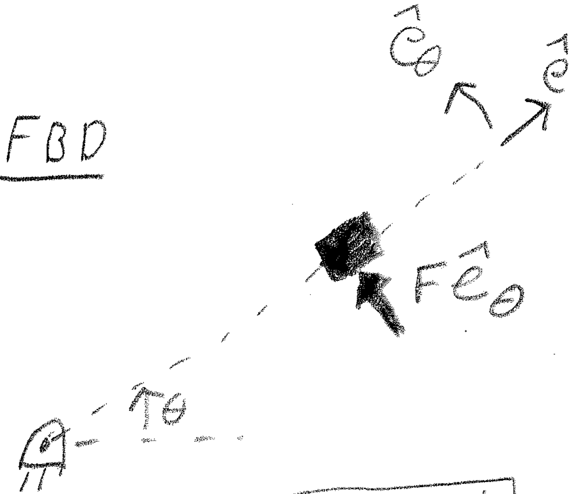
$$\theta << 1 \Rightarrow \sin\theta \doteq \theta \Rightarrow \ddot{\theta} + \frac{2g}{3r}\theta = 0 \Rightarrow \text{say, } \theta = \theta_0 \cos\sqrt{\frac{2g}{3r}}\,t$$

$$r = R/2, \quad t^* = 1 \text{ period} \Rightarrow \sqrt{\frac{4g}{3R}}\,t^* = 2\pi \Rightarrow \boxed{t^* = 2\pi\sqrt{\frac{3R}{4g}} = \pi\sqrt{\frac{3R}{g}}}$$

2) A small bead m slides without friction on a straight rod which is rotating at constant ω about a point on the rod. Neglect gravity. 2D. Find $\ddot{r}$ in terms of some or all of r, $\dot{r}$ and ω.



FBD



$$\boxed{\begin{array}{l}\omega = \text{const} \\ \Rightarrow \dot{\omega} = 0\end{array}}$$

LMB:

$$\sum \vec{F} = m\vec{a}$$

$$\{ F\hat{e}_\theta = m[(\ddot{r} - \omega^2 r)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta$$

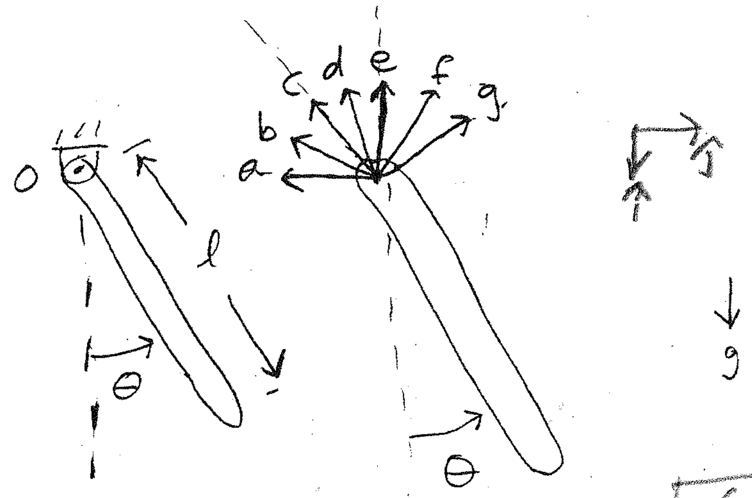
$$\{\}\cdot\hat{e}_r \Rightarrow \quad 0 = \ddot{r} - \omega^2 r$$

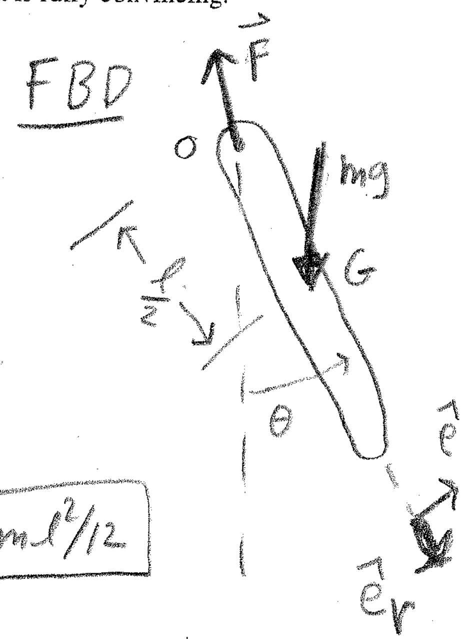
$$\Rightarrow \quad \boxed{\ddot{r} = \omega^2 r}$$

3) A uniform stick with mass m and length ℓ swings from a frictionless hinge at 0.

a) Find the equation of motion. That is, find $\ddot{\theta}$ in terms of some or all of θ, $\dot{\theta}$, ℓ, m and g.

b) The partial free body diagram shows some conceivable reaction forces at O. Which of these are you confident are in the wrong direction (a,b,c,d,e,f,g)? Justify your answer in a manner that is fully convincing.





$$\boxed{I^G = m\ell^2/12}$$

AMB/O:

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{r}_{G/O} \times mg\hat{\imath} = \vec{r}_{G/O} \times m\vec{a}_G + I^G\ddot{\theta}\hat{k}$$

$$\vec{r}_{G/O} = \frac{\ell}{2}\hat{e}_r \qquad \vec{a}_G = \ddot{\theta}\frac{\ell}{2}\hat{e}_\theta - \frac{\ell}{2}\dot{\theta}^2\hat{e}_r$$

$$\Rightarrow \quad \frac{\ell}{2}\hat{e}_r \times mg\hat{\imath} = \frac{m\ell}{2}\hat{e}_r \times \left(\ddot{\theta}\frac{\ell}{2}\hat{e}_\theta - \frac{\ell}{2}\dot{\theta}^2\hat{e}_r\right) + I\ddot{\theta}\hat{k}$$

$$\left\{ -\frac{\ell}{2}\cancel{m}g\sin\theta\,\hat{k} = \cancel{m}\frac{\ell^2}{4}\ddot{\theta}\hat{k} + \frac{\cancel{m}\ell^2}{12}\ddot{\theta}\hat{k} \right\}$$

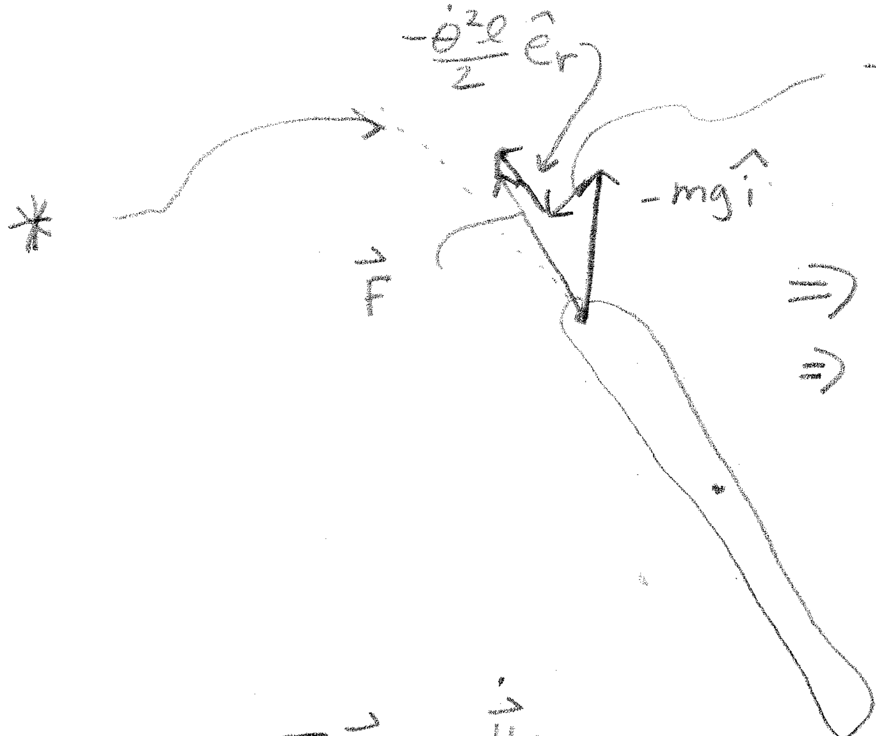
$$\{\}\cdot\hat{k} \Rightarrow \quad -\frac{\ell g}{2}\sin\theta = \frac{\ell^2}{3}\ddot{\theta}$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = \frac{-3g}{2\ell}\sin\theta}$$

LMB: $\sum \vec{F} = m\vec{a}$ $\qquad -\frac{3g}{2\ell}\sin\theta$

$$\vec{F} + mg\hat{i} = m\left(\ddot{\theta}\,\frac{\ell}{2}\hat{e}_\theta - \dot{\theta}^2\frac{\ell}{2}\hat{e}_r\right)$$

$$\Rightarrow \quad \vec{F} = m\left[-g\hat{i} - \frac{3g}{4}\sin\theta\,\hat{e}_\theta + \left(-\dot{\theta}^2\frac{\ell}{2}\right)\hat{e}_r\right]$$



$\Rightarrow \vec{F}$ is up & to left.

$\Rightarrow \vec{F}\cdot\hat{j} < 0$

$\vec{F}\cdot\hat{i} < 0$

AMB$_{/G}$: $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$

$$\underbrace{\left(-\frac{\ell}{2}\hat{e}_r\right)}_{\vec{r}_{O/G}} \times \vec{F} = \ddot{\theta} I \hat{k}$$

$\ddot{\theta} = -\frac{3g}{2\ell}\sin\theta$

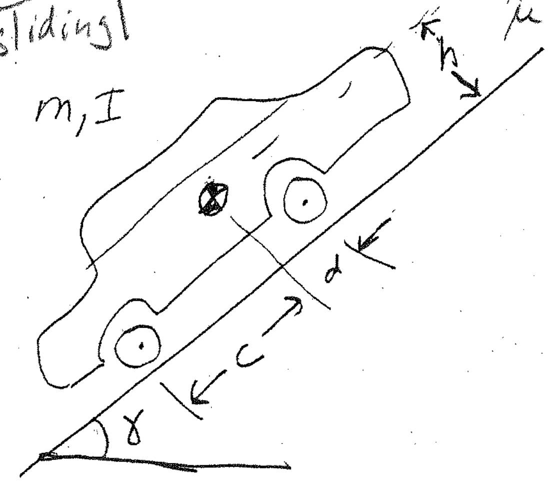
for $\theta > 0 \Rightarrow \ddot{\theta} < 0 \Rightarrow \vec{F}$ causes CW rotation

$\Rightarrow \vec{F}$ points to right of line *

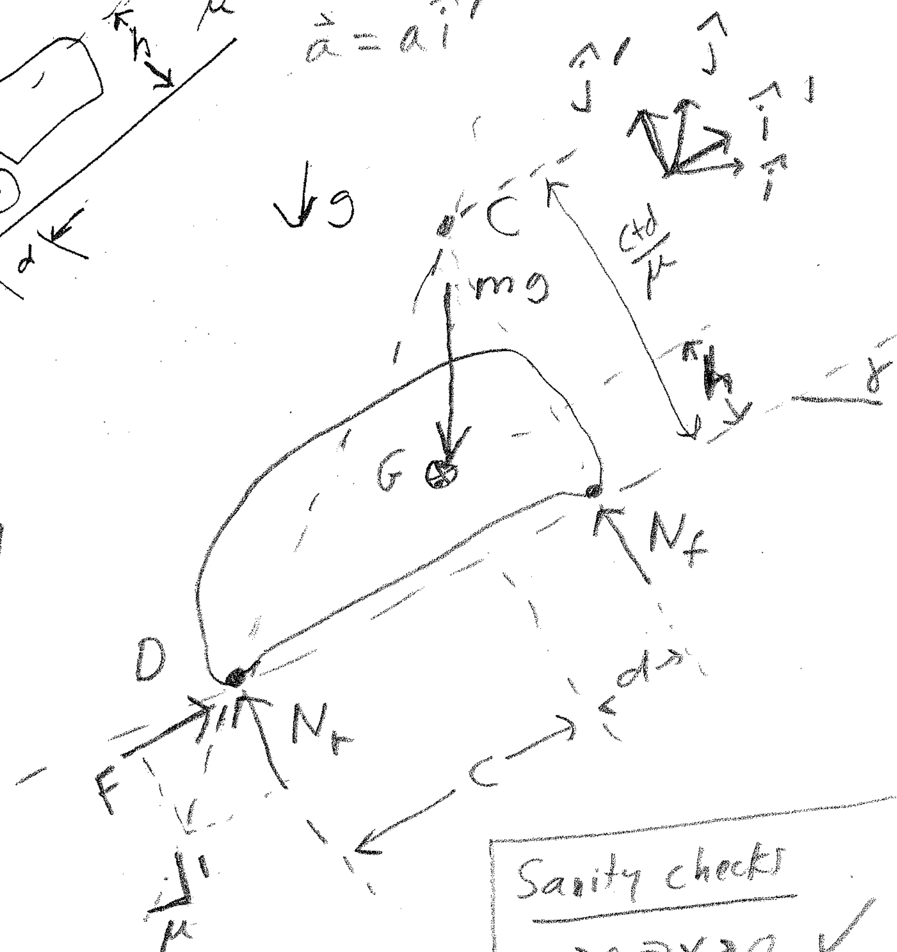
$$\boxed{\text{Only (d) is possible.}}$$

4) A rear-wheel drive car attempts to drive uphill. Assume it does not tip over. What is the steepest slope γ it can go up? Answer in terms of some or all of m, c, d, h, g, μ and the moment of inertia about the center of mass I.

without sliding



$\vec{a} = a\hat{i}'$



Borderline case:

$\vec{a} = \vec{0} \Rightarrow$ statics!

AMB/C:

$$\sum \vec{M}_{/C} = \vec{0}$$

$$\vec{r}_{G/C} \times (-mg\hat{j}) = \vec{0}$$

$$\left[\left(\frac{-(c+d)}{\mu} + h\right)\hat{j}' - d\,\hat{i}'\right] \times (-mg\hat{j}) = \vec{0}$$

$$\Rightarrow \left\{ \left(\frac{-(c+d)}{\mu} + h\right)\underbrace{\hat{j}'\times\hat{j}}_{-\sin\gamma\hat{k}} + d\,\underbrace{\hat{i}'\times\hat{j}}_{\cos\gamma\hat{k}} = \vec{0} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \boxed{\tan\gamma = \frac{d}{\frac{c+d}{\mu} - h}}$$

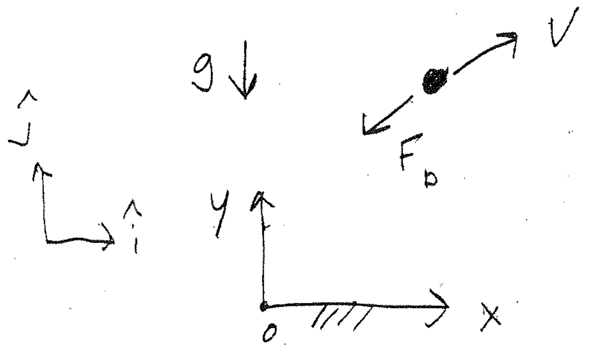
Sanity checks

$\mu \to 0 \Rightarrow \gamma \to 0$ ✓

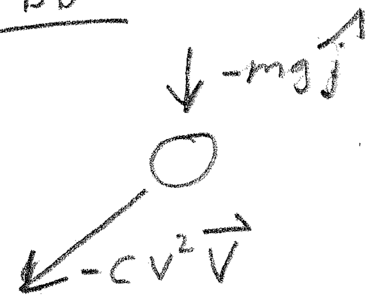
$c = h = 0 \Rightarrow \tan\gamma = \mu$ ✓

$d = 0 \Rightarrow \gamma = 0$ ✓ (interesting)

5) A particle m is acted on by gravity and a cubic drag force $F_D = cv^3$ that opposes its motion. Find $\ddot{x}$ in terms of some or all of $x, y, \dot{x}, \dot{y}, m, g$ and c.



FBD



LMB

$$\sum \vec{F} = m\vec{a}$$

$$\{-mg\hat{j} - cv^2\vec{v} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})\}$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\dot{x}^2 + \dot{y}^2$$

$$\{\}\cdot\hat{i} \Rightarrow \ddot{x} = \frac{-c}{m} v^2 \dot{x}$$

$$\boxed{\ddot{x} = \frac{-c}{m}(\dot{x}^2 + \dot{y}^2)\dot{x}}$$