Lab #2 - Two Degrees-of-Freedom Oscillator

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INTRODUCTION

The system illustrated in Figure (2.1) has two degrees-of-freedom. This means that two is the minimum number of coordinates necessary to uniquely specify the position of the system. The purpose of this laboratory is to introduce you to some of the properties of linear vibrating systems with two or more degrees-of-freedom. You have already seen a one degree-of-freedom vibrating system (the mass-spring-dashpot system) and should have some familiarity with the ideas of *natural frequency* and *resonance*. These ideas still apply to an undamped linear system with two or more degrees-of-freedom.

The new idea for many degrees-of-freedom systems is the concept of *modes* (also called *normal modes*). Each *normal mode* consists of a *mode shape* and corresponding *natural frequency*. The system will exhibit resonance if forced at one of its natural frequencies. The number of modes a system has is equal to the number of degrees-of-freedom. Thus the system below has two modes and two natural frequencies.
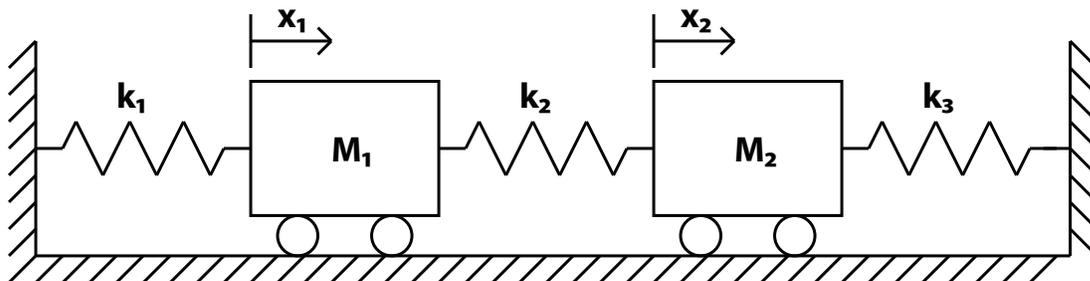

Figure 2.1: A simple two-degree-of-freedom system.

The primary goals of this laboratory are for you to learn the concept of normal modes in a two degrees-of-freedom system – the simplest system which exhibits such modes. You will learn this by experimentation and calculation.

PRE-LAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. Are the number of degrees of freedom of a system and the number of its *normal modes* related? Explain.

2. How can a normal mode be recognized physically?

3. What do you expect to happen when you drive a system at one of its *natural frequencies*?

4. Draw a free body diagram and derive the equations of motion for a three degrees-of-freedom system, with three different masses, four equal springs, and no forcing. Put them in matrix form. *Your result should resemble equation (2.4) except your matrix will be 3x3 and you will have no* $\underline{\mathbf{f}}(\mathbf{t})$ *term.* Substitute in the normal mode solution *(see (2.7))* to get an eigenvalue problem *(see (2.5))*. How would the eigenvalues and eigenvectors of your matrix relate to the *mode shapes* and *natural frequencies*?

5. Using MATLAB, find the eigenvalues and eigenvectors of the following matrix and print the results (HINT: Type `help eig` for assistance).

$$[A] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{2.1}$$

NORMAL MODES

A *normal mode* is a special type of vibration what occurs when all of the points in the system are moving in simple harmonic motion. In addition, In a *normal mode* vibration all points move with the same angular frequency ω and are exactly in-phase or exactly out of phase. An example on the following page (See 2.2) illustrates a *normal mode* vibration for a two degrees-of-freedom-system. Note:

- Both masses are moving in simple harmonic motion.

- The system has a period of $T = 4\pi$ (sec), and thus an angular frequency of $\omega = \frac{2\pi}{T} = \frac{1}{2}$.

- When one mass is at its maximum displacement, the other is at its minimum displacement - thus the masses are totally out of phase.

If we wanted to write out the equation of motion for this system, we would need a state vector $\underline{\mathbf{x}}(t)$ with two elements $x_1(t)$ and $x_2(t)$ - one to represent the position of each mass as a function of time. That equation might look something like this:

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \sin\left(\frac{1}{2}t\right) \tag{2.2}$$

Here, ω is the *natural frequency* of the *normal mode* (the same for all masses), and the vector $\underline{\mathbf{c}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is its *mode shape*. The *mode shape* $\underline{\mathbf{c}}$ tells you phase and relative amplitude of motion of each mass - here both masses have the same relative amplitude ($|c_1| = |c_2|$) *though in general that is not the case.* Since in this example c_1 has the opposite sign of c_2 the two masses are completely out of phase.
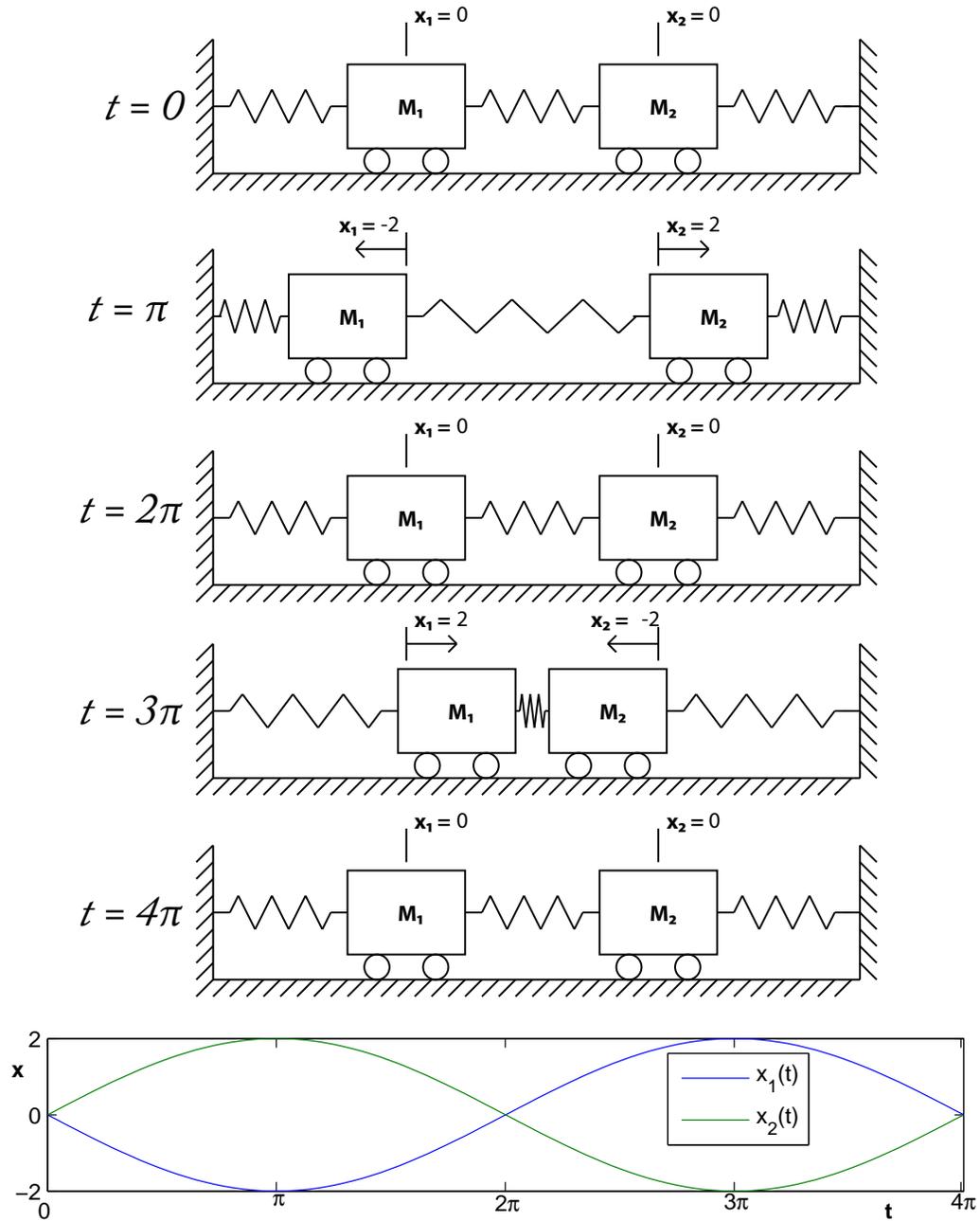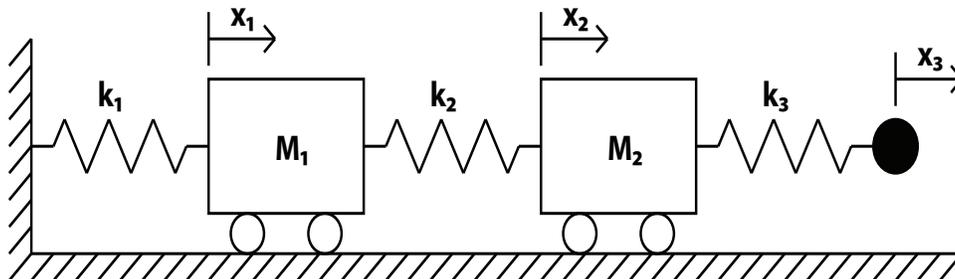
**Figure 2.2:** A normal mode vibration of a two-degree-of-freedom system.

DERIVING THE EQUATIONS OF MOTION

We will now derive the equations of motion for a driven two degrees-of-freedom system. The diagram and physical setup are shown in Figures 2.3 and 2.5.

Figure 2.3: Illustration of a coupled mass-spring system.

Here, rather than having the rightmost spring attached to a fixed support, we have it attached to a sinusoidally driven support whose position is $x_3(t)$. Do not be fooled into thinking that x_3 counts as a degree of freedom - here we know how we are driving the system and so x_3 is a given. Look back over `Lab 1` if you are confused about this point - we use the same trick there to drive a one degree-of-freedom system. Now, we will draw the free-body diagram for each mass and work out its equation of motion. To help get the signs right, assume that
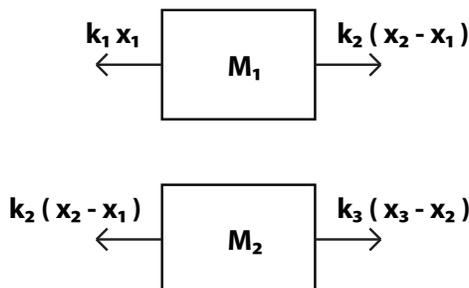
Figure 2.4: The free-body diagrams for masses m_1 and m_2.

the displacements are all positive (i.e. to the right) with $x_1 < x_2 < x_3$. This puts all of the springs into tension relative to their equilibrium condition. The equations of motion for each mass respectively (assuming equal spring constants) are

$$k\left(x_2 - x_1\right) - kx_1 = m_1\ddot{x}_1 \tag{2.3a}$$

$$k\left(x_3 - x_2\right) - k\left(x_2 - x_1\right) = m_2\ddot{x}_2 \tag{2.3b}$$

We can rewrite this in matrix form as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{2k}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{kx_3(t)}{m_2} \end{bmatrix} \tag{2.4}$$

or as

$$\ddot{\underline{\mathbf{x}}} = [A]\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t) \tag{2.5}$$

Where the matrix $[A]$ contains information about the system response to forcing and the vector $\underline{\mathbf{f}}(t)$ contains information about the external forcing.

SOLVING THE EQUATIONS OF MOTION USING NORMAL MODES

To make matters easier, let's consider the case where there is no external forcing, thus $\underline{\mathbf{f}}(\mathbf{t}) = 0$ and our equation of motion (2.5) reduces to:

$$\ddot{\underline{\mathbf{x}}} = [A]\,\underline{\mathbf{x}} \tag{2.6}$$

Now we'll look for the normal mode solutions of the system. Remember - a *normal mode* vibration is when both masses are moving in simple harmonic motion with the same angular frequency ω, but potentially different amplitudes of motion c_i. Before we gave an example of a normal mode solution. Here is the general form of a normal mode solution for a two degrees-of-freedom system:

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} (A\cos(\omega t) + B\sin(\omega t)) \tag{2.7}$$

Once again, ω is the *natural frequency* of the mode which tells you the angular frequency with which every mass vibrates, and $\underline{\mathbf{c}}$ is the *mode shape* which tells you the phase and relative amplitude of motion of each mass. If we plug in our *ansatz* for the solution into the equation of motion, we can solve for the natural frequency and mode shape that will make the equation hold. Substituting (2.7) into (2.6) and canceling out cosine and sin terms yields

$$-\omega^2 \underline{\mathbf{c}} = [A]\,\underline{\mathbf{c}} \tag{2.8}$$

It turns out that we have non-trivial solutions to (2.8) only for certain values of $-\omega^2$ and then only when $\underline{\mathbf{c}}$ is a multiple of a specific vector. Equation (2.8) is in the form of an eigenvalue problem from linear algebra, and these sets of solutions are called the "eigenvalues" (λ_i) and corresponding "eigenvectors" $(\hat{\lambda}_\mathbf{i})$ of the matrix $[A]$. These can easily be solved for by hand, or by using a computer algebra program such as `MATLAB` or `SciLab`. For an n degrees-of-freedom system there will be n such sets of "eigenvalues" and "eigenvectors". **Thus, a n degrees-of-freedom system has n *natural frequencies* and n *mode shapes* given by**

$$\omega_i = \sqrt{-\lambda_i}; \qquad \underline{\mathbf{c}} = \hat{\lambda}_\mathbf{i} \tag{2.9}$$

where (λ_i) and $(\hat{\lambda}_\mathbf{i})$ are the eigenvalues and eigenvectors (respectively) of the matrix $[A]$.

The general motion of the system can then some combination of the normal modes:

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{c_1}}(A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)) + \underline{\mathbf{c_2}}(A_2\cos(\omega_2 t + \phi_2) + B_2\sin(\omega_2 t + \phi_2)) \tag{2.10}$$

Here the coefficients A_i and B_i would depend on the initial conditions. In the language of linear algebra, we say that the *normal modes* span the space of possible solutions.

LABORATORY SET-UP

- **Air Track**
  The lab set-up consist of an air-track hooked up to the lab's air system, four or more air track gliders, four plug-in springs, a mechanical oscillator (for external forcing), a photogate timer, and a digital stopwatch. Please note that there are two somewhat incompatible styles of glider which should only be used on the appropriate air tracks. Each glider has a label listing its approximate mass (including spring) and the air tracks on which it will work. **You should remeasure the masses of the gliders and springs at the start of your lab.**
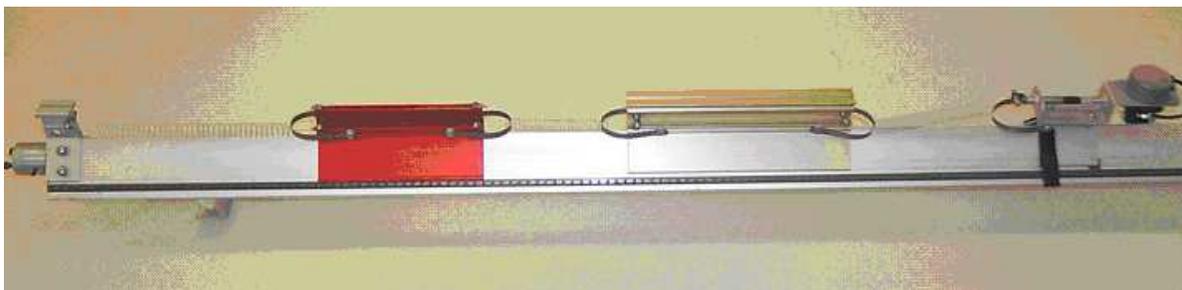
**Figure 2.5:** The laboratory set-up you will be working with.

- **Using the *SciLab* Software**

  To open *SciLab* click on its icon located on the desktop of your computer. This program is a freeware program similar to *MatLab* and should look quite similar.

  To find the eigenvalues and eigenvectors of a matrix you must use the function `spec()` as shown below. Input the matrix $[A]$, call `spec()` on it, and the program will return the eigenvalues along the diagonal of a square matrix and the eigenvectors as the columns of the second returned matrix (just like `eig()` in MATLAB).

PROCEDURE

1. Play with the air track, gliders, and timer. Adjust the mechanical driver left or right so that each spring, at equilibrium, has a total length of about 20 cm (the driver is attached with a Velcro strap). Have the TA turn on the main air supply, if it is not already on, and turn on the valve at the end of the air track.

2. Measure the mass of each of your carts, and one of your springs (you may assume that all springs have the same mass).

3. Next you will find the spring constant for your springs.

    - Attach a small weight (40 to 50 grams) to one end of a spring and hold the other end solidly against the tabletop.
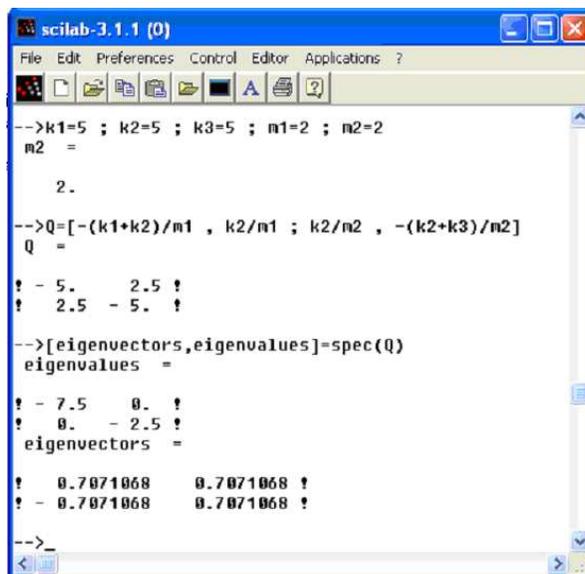
Figure 2.6: Screenshot of *Scilab* in use.

- Pull the weight down a few centimeters and release it, and then measure the period of oscillation (*average over 10 periods*). Use $\omega = \sqrt{\frac{k}{m}}$ to find k. Remember to include part of the spring as well as the plug mass in "m" – half is a good approximation in this case.

- Repeat this calculation for each spring. *Note: there will be variability in k from spring to spring.*

4. Choose two gliders of different sizes to work with, and set them up on the track.

   - Using the measured masses and spring constants, calculate the *mode shapes* and *natural frequencies* of the system - *see (2.9)*. You may do the calculations by hand or use *SciLab* on the computer. Remember to add the the mass of a plug-in spring when calculating the cart mass.

   .

5. The system is set into a normal mode oscillation by applying the appropriate initial conditions. Choose one of your normal modes to use.

   - First, place the system in equilibrium. One simple method is to turn the air track on and off repeatedly until the gliders stop moving.

   - With the air off, displace m_1 by an amount $A*c_1$cm, and displace m_2 by $A*c_2$cm, where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is the appropriate "eigenvector" or *mode shape*. Pick A large enough that neither mass is displaced by less than 1 cm, but small enough that neither mass slides on the track with the air off.

- Turn on the air track valve abruptly. The system should oscillate in the normal mode which your chose.

- Measure the angular frequency (ω) of oscillation, and verify that it is approximately equal to the natural frequency calculated in `SciLab`. The angular frequency of the masses is found by timing a number of oscillations (i.e. 10) and then converting the resulting period to ω (which has units of rad/sec). Digital stopwatches are available at the air track.

- Note the phase difference between the two masses - are they in phase or out of phase?

- Repeat the above procedure for the other normal mode.

6. Use some arbitrary initial conditions and set the system into a non-normal mode oscillation. Observe the motion. (It should be difficult to see that it is the sum of normal mode vibrations.)

7. Next we will attempt to obtain normal mode vibrations by driving the system at each natural frequency - the driving frequency of oscillation is obtained by timing the motion of the driving rod connected to the motor, using either a stopwatch or a photogate timer.

   - Choose one of the normal modes you have calculated. With the air off, set the driving frequency to the corresponding natural frequency which you have calculated. Now turn on the air.

   - Does the system resonate when you turn the air on? Be patient, it may take a try or two to get resonance. Start the system from rest every time you change the motor speed.

   - When you get resonance, measure the frequency of oscillation and compare it to the natural frequency which you calculated in SciLab. Note, as best you can, the relative phase between the scotch yoke (driving) and the masses (response) at resonance.

   - Repeat this procedure for the other normal mode.

8. Drive the system at a frequency close to (but not equal to) one of the natural frequencies. Is the amplitude of motion of either mass constant? Now drive the system at a frequency much higher than either natural frequency. Note the amplitude of motion of each mass.

9. Set up the air track with three (approximately) equal masses and four (approximately) equal springs. Adjust the mechanical oscillator to give an equilibrium spring length of about 20 cm. Verify by observation that $[1 \quad -1.414 \quad 1]^T$ is approximately a normal mode for this system.

10. Find another normal mode for this system by observation. Find another still. Are there any more? Use *SciLab* to find the normal modes and natural frequencies.

11. Using *SciLab*, find the normal modes and natural frequencies for a system with three unequal masses and four equal springs, and test them on the air track (free vibration only).

LAB REPORT QUESTIONS

1. List your values of k for the springs and a sample calculation. What is the average value of k, and what was the largest variation from the average (in percent)?

2. List your mode shapes and natural frequencies calculated in (4). Did you obtain normal mode oscillations using initial conditions based on your eigenvectors? How could you tell? How close were the natural frequencies what you measured in (5) to the ones you calculated in (4)? Was the phase correct? How does this experiment deviate from theory?

3. Describe what you observed when you forced the system at the following 3 frequencies:

   (a) forcing frequency = a natural frequency

   (b) forcing frequency close to a natural frequency (Was the amplitude of the oscillations constant in this case? If not, how did it vary?)

   (c) forcing frequency far from a natural frequency.

4. How many normal modes are there in the three equal mass system (*theoretically*) and what are they? How many were you able to find experimentally and how did you recognize them as normal modes? How do the modes you found compare with those you calculated?

5. What were your calculated normal modes and natural frequencies for the system with three unequal masses? Did normal mode oscillations occur with these ratios and frequencies on the air track?

CALCULATIONS & NOTES

APPENDIX: SOLVING THE EQUATIONS OF MOTION VIA A CHANGE OF BASIS

So far we have discussed how normal modes are the simplest oscillatory functions from which *all* motions of the two degrees-of-freedom system can be thought to be comprised of. Mathematically, the normal modes y_1 and y_2 satisfy the equations of motion for simple harmonic oscillators with natural frequencies ω_1 and ω_2 respectively.

$$\ddot{y}_1 + \omega_1^2 y_1 = 0 \tag{2.11a}$$

$$\ddot{y}_2 + \omega_2^2 y_2 = 0 \tag{2.11b}$$

Since the equations of motion for the normal modes are simple in terms of the y_1, y_2 coordinates, it would be nice if we could find some transformation between the physical coordinates x_1, x_2 and these new variables, i.e. $\underline{\mathbf{x}} = f(\underline{\mathbf{y}})$, so that we can solve the problem in terms of the easier coordinates and then transform back into the original ones. We can accomplish this mathematically by performing a change-of-basis from the original basis into the *eigenbasis* of $[A]$. We define our new normal mode coordinates by

$$\underline{\mathbf{x}} = [S]\,\underline{\mathbf{y}} \tag{2.12}$$

where the change-of-basis matrix $[S]$ is defined as

$$[S] = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.13}$$

Plugging this change of variables into (2.5) we get the new equation

$$[S]\,\underline{\ddot{\mathbf{y}}} = [A]\,[S]\,\underline{\mathbf{y}} + \underline{\mathbf{f}}\,(t) \tag{2.14}$$

Left-multiplying both sides by $[S^{-1}]$ gives us

$$\underline{\ddot{\mathbf{y}}} = \left[S^{-1}\right][A]\,[S]\,\underline{\mathbf{y}} + \left[S^{-1}\right]\underline{\mathbf{f}}\,(t) = [\Lambda]\,\underline{\mathbf{y}} + \underline{\tilde{\mathbf{f}}}(t) \tag{2.15}$$

where

$$[\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & -3k \end{bmatrix} \tag{2.16}$$

Looking at the unforced case, $\underline{\tilde{\mathbf{f}}}(t) = \underline{\mathbf{0}}$, we see from (2.15) that in the new normal mode coordinates we now have two uncoupled second-order ODEs,

$$\ddot{y}_1 + k y_1 = 0 \tag{2.17a}$$

$$\ddot{y}_2 + 3k y_2 = 0 \tag{2.17b}$$

the solutions of which are

$$y_1 = A_1 \cos\sqrt{k}t + B_1 \sin\sqrt{k}t \tag{2.18a}$$

$$y_2 = A_2 \cos\sqrt{3k}t + B_2 \sin\sqrt{3k}t \tag{2.18b}$$

Using (2.12) we can now transform back into the original x_1, x_2 coordinates giving

$$\underline{\mathbf{x}} = [S]\,\underline{\mathbf{y}} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} =$$
$$\underline{\mathbf{v_1}}\,(A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)) + \underline{\mathbf{v_2}}\,(A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)) \qquad (2.19)$$

where we have substituted $\omega_1 = \sqrt{k}$ and $\omega_2 = \sqrt{3k}$. This is the same result we found before in (2.10), so you might not think much was gained by performing this change-of-basis. However, the real advantage of this method appears when we consider the forced case.

APPENDIX: FORCED TWO-DEGREE-OF-FREEDOM SYSTEM

We now reconsider equation (2.15) when $\underline{\tilde{\mathbf{f}}}(t) \neq \underline{\mathbf{0}}$.

$$\underline{\ddot{\mathbf{y}}} = [\Lambda]\,\underline{\mathbf{y}} + \underline{\tilde{\mathbf{f}}}(t) \qquad (2.20)$$

The two resulting equations are

$$\ddot{y}_1 + \omega_1^2 y_1 = \frac{kx_3}{2m_1} \qquad (2.21a)$$

$$\ddot{y}_2 + \omega_2^2 y_2 = -\frac{kx_3}{2m_1} \qquad (2.21b)$$

where $x_3(t) = F \cos \omega t$ and ω is the forcing frequency. Solving both of these non-homogeneous second-order ODEs yields

$$y_1(t) = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t - \frac{Fk}{2m_1}\left(\frac{1}{\omega^2 - \omega_1^2}\right)\cos \omega t \qquad (2.22a)$$

$$y_2(t) = A_2 \cos \omega_2 t + B_2 \sin \omega_2 t + \frac{Fk}{2m_1}\left(\frac{1}{\omega^2 - \omega_2^2}\right)\cos \omega t \qquad (2.22b)$$
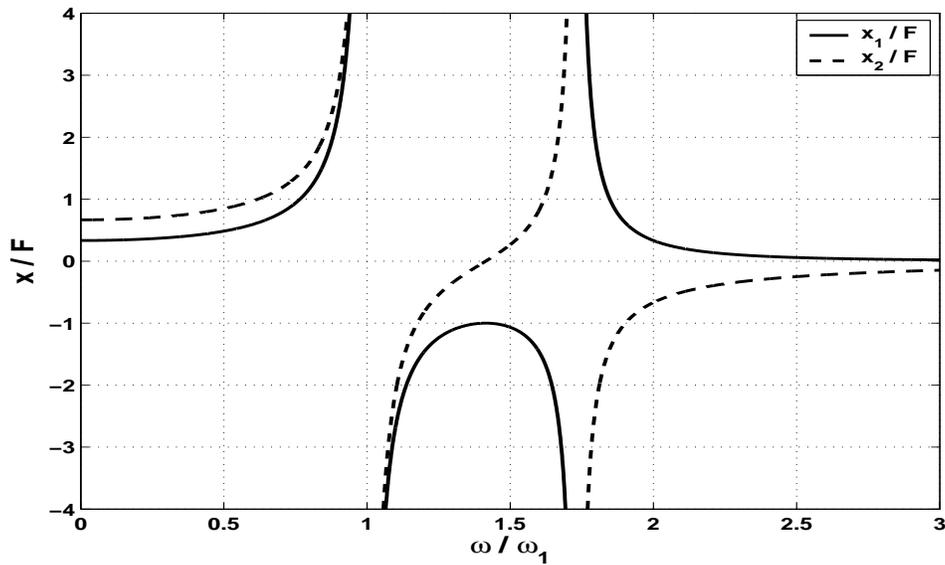
Once again we use (2.12) to transform back into the original coordinates to get

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_{\mathbf{c}}(t) + \frac{Fk}{2m_1}\begin{bmatrix} \frac{1}{\omega^2 - \omega_2^2} - \frac{1}{\omega^2 - \omega_1^2} \\ -\frac{1}{\omega^2 - \omega_2^2} - \frac{1}{\omega^2 - \omega_1^2} \end{bmatrix}\cos \omega t \qquad (2.23)$$

where we have suppressed the homogeneous (or complementary) part of the solution. We note that the particular solution becomes unbounded as the forcing frequency approaches either $\omega = \omega_1$ or $\omega = \omega_2$. In other words, *resonance* occurs when we force the two degrees-of-freedom system at one of the normal modes' natural frequencies. (Obviously the oscillations you will observe in the lab will not be unbounded as the lab set-up is not entirely frictionless.)

We now rewrite the particular solution as

$$\underline{\mathbf{x}}_{\mathbf{p}}(t) = \frac{F}{2}\begin{bmatrix} \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 3} - \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 1} \\ -\frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 3} - \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 1} \end{bmatrix}\cos \omega t \qquad (2.24)$$

Figure 2.7: Plot of the response amplitude to forcing amplitude ratio for the forced two degrees-of-freedom system.

where we have written it in terms of the ratio of the forcing frequency to the smaller normal mode frequency ω_1. Figure 2.7 graphically shows how the amplitudes of the particular (or steady-state) solutions change as the forcing frequency ω is varied.

The plot graphically illustrates what we found earlier – that when the forcing frequency is near the natural frequency of a normal mode, that mode resonates. As $\omega \to \omega_1$ the two masses move in-phase and when $\omega \to \omega_2$ the masses move out-of-phase.