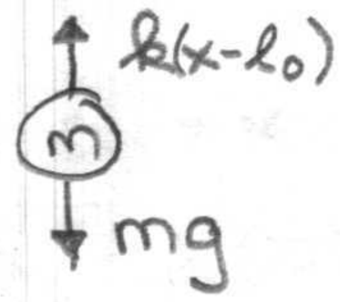
9.53

Mass m hanging from spring of constant k, l_0

a) Free body diagram: 

b) $\sum F_x = ma \Rightarrow mg - k(x-l_0) = ma$

c) $m\ddot{x} + k(x-l_0) = mg \Rightarrow \ddot{x} + \frac{k}{m}(x-l_0) = g$

d) Check $x(t) = l_0 + \frac{mg}{k}$, $\ddot{x} = 0$

$$\Rightarrow 0 + \frac{k}{m}\left(l_0 + \frac{mg}{k} - l_0\right) = \frac{k}{m}\frac{mg}{k} = g \checkmark$$

e) This is the deformed equilibrium position of the system under gravity load.

f) Let $\hat{x} = x - \left(l_0 + \frac{mg}{k}\right) \Rightarrow x = \hat{x} + \left(l_0 + \frac{mg}{k}\right)$

$\therefore$ we have $\ddot{\hat{x}} + \frac{k}{m}\left(\hat{x} + l_0 - l_0 + \frac{mg}{k}\right) = g$

OR $\ddot{\hat{x}} + \frac{k}{m}\left(\hat{x} + \frac{mg}{k}\right) = g \Rightarrow \ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

g) We solve $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$ to get $\hat{x} = c_1 \sin\left(t\sqrt{\frac{k}{m}}\right) + c_2 \cos\left(t\sqrt{\frac{k}{m}}\right)$

$\Rightarrow$ Initial conditions: $\hat{x}(0) = D$, $\dot{\hat{x}}(0) = 0$

$\dot{x} = c_1\sqrt{\frac{k}{m}}\cos\left(t\sqrt{\frac{k}{m}}\right) - c_2\sqrt{\frac{k}{m}}\sin\left(t\sqrt{\frac{k}{m}}\right)$

$\dot{x}(0) = 0 = c_1\sqrt{\frac{k}{m}} \quad \therefore c_1 = 0$

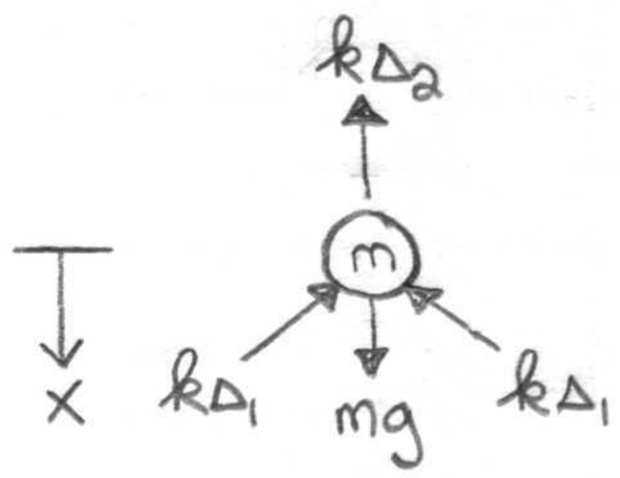
$x(0) = c_2 = D \Rightarrow \boxed{\hat{x}(t) = D\cos\left(t\sqrt{k/m}\right)}$

h) Period $= \frac{2\pi}{\omega}$, where $\omega = \sqrt{k/m}$ $\quad \therefore \boxed{T = 2\pi\sqrt{m/k}}$

i) This would not make sense because the spring would want to oscillate upwards past the top of the spring (at its support).

9.54

Free-body diagram of system:



From this, $\Delta_2 = x$
and $\Delta_1 = x\cos 30° = \frac{1}{2}x$

$$\sum F_x = ma = mg - kx - \frac{1}{2}k\left(\frac{1}{2}x\right) - \frac{1}{2}k\left(\frac{1}{2}x\right) = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + \frac{3}{2}kx = mg$$

OR $\ddot{x} + \frac{3k}{2m}x = g$ (governing equation)

SOLUTION: $x(t) = \underbrace{C_1 \sin\left(t\sqrt{\frac{3k}{2m}}\right) + C_2\cos\left(t\sqrt{\frac{3k}{2m}}\right)}_{x_h(t)} + \underbrace{\frac{2mg}{3k}}_{x_p(t)}$

$= x_h(t) + x_p(t)$

a) Vibration frequency $= \frac{\omega}{2\pi} = \boxed{\frac{1}{2\pi}\sqrt{3k/2m}}$

b) Given initial conditions: $x(0) = 0$, $\dot{x}(0) = 0$

$$\dot{x}(t) = C_1\sqrt{\frac{3k}{2m}}\cos\left(t\sqrt{\frac{3k}{2m}}\right) - C_2\sqrt{\frac{3k}{2m}}\sin\left(t\sqrt{\frac{3k}{2m}}\right)$$

$$\Rightarrow \dot{x}(0) = 0 = C_1$$

$$x(0) = C_1\sin(0) + C_2\cos(0) + \frac{2mg}{3k} = 0$$

$$\therefore C_2 = -2mg/3k$$

$$x(t) = \frac{2mg}{3k}\left[1 - \frac{2mg}{3k}\cos\left(t\sqrt{\frac{3k}{2m}}\right)\right]$$

$\therefore$ max deflection relative to equilibrium position $= \boxed{2mg/3k}$

9.55

Given: $k = 200 \frac{lbf}{ft}$, $m = 150$ lbm, $g = 32.2$ ft/s^2

a) If contact with trampoline never breaks,

$$m\ddot{x} + kx = -mg \quad \text{OR} \quad \ddot{x} + \omega^2 x = -g, \text{ where } \omega = \sqrt{k/m}$$

Solution: $x = C_1 \sin(\omega t) + C_2 \cos(\omega t) - g/\omega^2$

$x(0) = 0$ and $\dot{x}(0) = 0$ (INITIAL COND.)

$$\dot{x} = C_1 \omega \cos(\omega t) - C_2 \omega \sin(\omega t)$$

$$\Rightarrow \dot{x}(0) = C_1 \omega = 0 \Rightarrow C_1 = 0$$

$$x(0) = C_2 - g/\omega^2 = 0 \quad \therefore \quad C_2 = mg/k$$

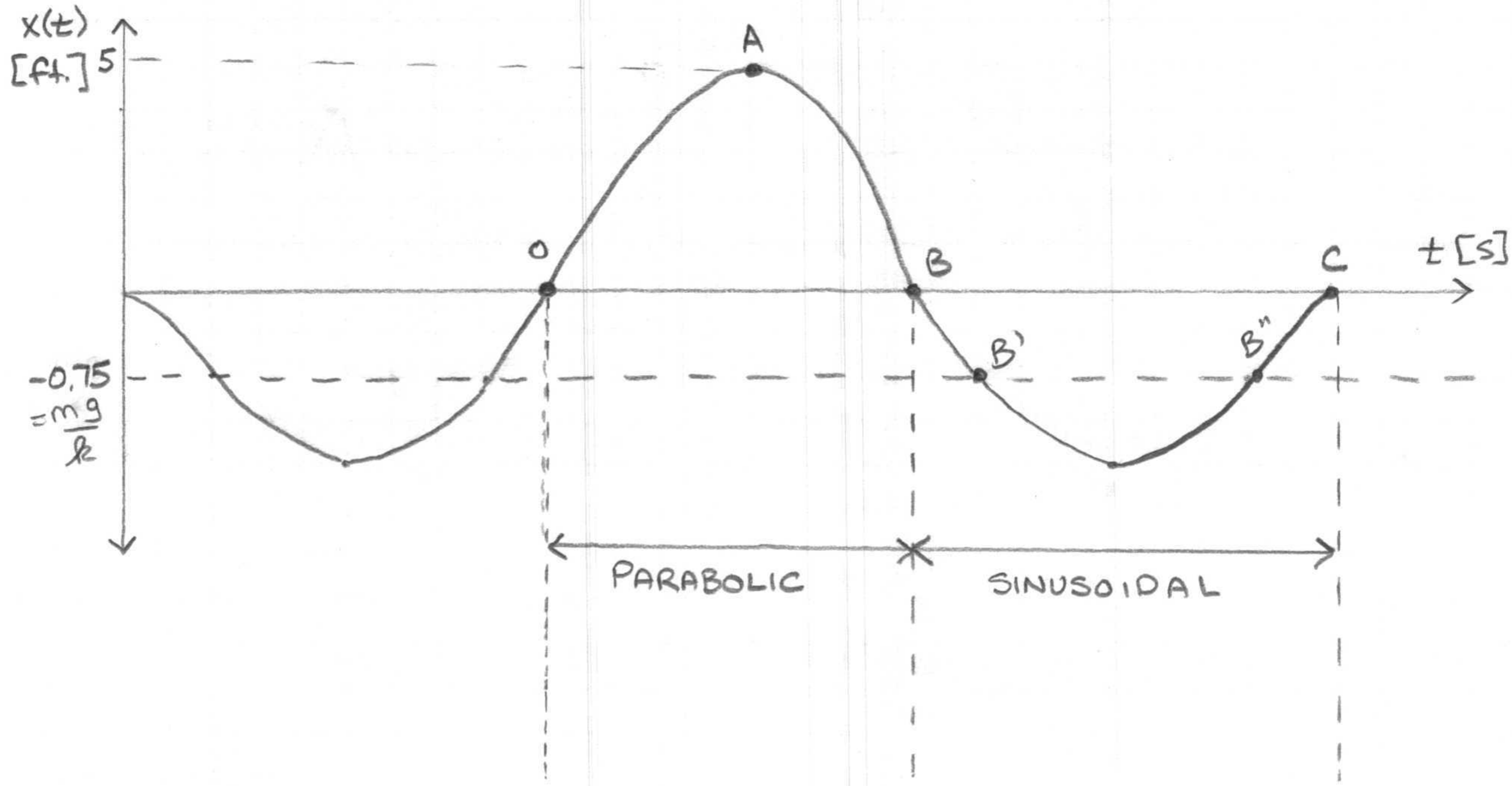
$$\therefore \quad x(t) = \frac{mg}{k}\left[\cos(\omega t) - 1\right]$$

Period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{150 \text{ lbm}}{200 \text{ lbf/ft}} \times \frac{1 \text{ lbf}}{1 \text{ lbm} \cdot g}}$

$= 2\pi\sqrt{\frac{150}{200 \text{ 1/ft} \times 32.2 \text{ ft/s}^2}} = \boxed{0.959 \text{ seconds}}$

b) Max amplitude $= \frac{mg}{k} = \frac{150 \text{ lbm } g}{200 \text{ lbf/ft}} \times \frac{1 \text{ lbf}}{1 \text{ lbm} \cdot g} = \boxed{0.75 \text{ feet}}$

c) If the jumper loses contact with the trampoline, to jump to a height of 5 feet, the motion is sinusoidal while in contact and parabolic (from projectile motion theory) once the jumper is in the air. See next page for a plot of what this looks like.



To analyze, assume we begin at point A, a height of 5 feet above the trampoline. $(x=5, \dot{x}=0)$

From $A \to B$, $x(t) = -\frac{1}{2}gt^2 + h_0 = 5 - \frac{1}{2}gt^2$

$\therefore$ $x(t) = 0$ when $5 - \frac{1}{2}gt^2 = 0$, or $t = \sqrt{\frac{2(5\text{ ft})}{32.2\text{ ft/s}^2}}$

$t = 0.5573$ seconds

$\therefore$ Total time from 0 to B is $2t = 1.115$ seconds

@ B, $x = 0$ and $\dot{x} = -gt\,/_{t=0.5573} = -17.945$ ft/s $= \dot{x}_0$

We use this as an initial condition to define a new sine wave, setting point B as $t=0$.

$$x(t) = c_1 \sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + c_2 \cos\left(\sqrt{\tfrac{k}{m}}\,t\right) - \frac{gm}{k}$$

$$x(0) = 0 = c_1(0) + c_2 - \frac{mg}{k} \quad \therefore \quad c_2 = \frac{mg}{k}$$

$$\dot{x}(0) = \dot{x}_0 = c_1\sqrt{\frac{k}{m}}\cos(0) - c_2\sqrt{\frac{k}{m}}\sin(0)$$

$$\therefore \quad c_1\sqrt{\frac{k}{m}} = \dot{x}_0 \quad \text{OR} \quad c_1 = \dot{x}_0\sqrt{\frac{m}{k}}$$

$$\therefore \; x(t) = \dot{x}_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{mg}{k}\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{mg}{k}$$

We want to find when this expression $= -\frac{mg}{k}$

$\Rightarrow$ This is point B' on previous graph

Using Matlab or a calculator to solve,

$$t = -\tan^{-1}\left(\frac{g}{\dot{x}_0}\sqrt{\frac{m}{k}}\right) \Big/ \sqrt{\frac{k}{m}}$$

$$= -\tan^{-1}\left(\frac{32.2\ ft/s^2}{-17.945\ ft/s}\sqrt{\frac{.150\ lbm}{200\ lbf/ft \times 32.2\ ft/s^2}}\right) \Big/ \sqrt{\frac{200 \times 32.2}{.150}} = 0.04079\,s$$

$\therefore$ Distance from B to B' $= 0.04079$ s

We know distance from B' to B'' $= \frac{T}{2} = \frac{0.959\,s}{2}$

$= 0.4795$ seconds

From B'' to C is the same as B to B' $= 0.04079$ s

$\therefore$ Total period $T = 1.115\,s + 2(0.0408\,s) + 0.4795\,s$

$$= \boxed{1.676 \text{ seconds}}$$

9.57

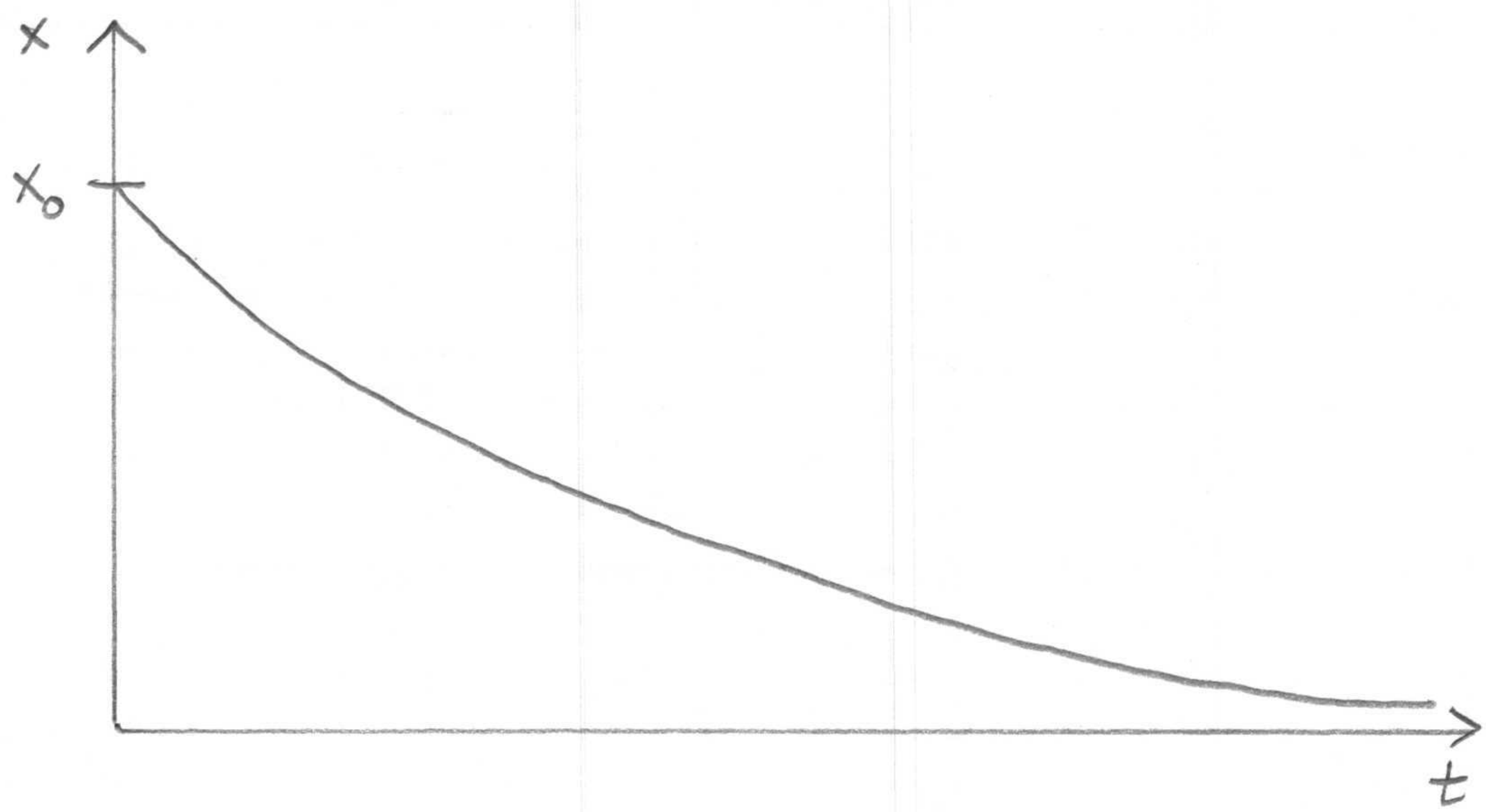
$m\ddot{x} + c\dot{x} + kx = 0$

Given: $m = 0.4$ kg, $c = 10$ kg/s, $k = 5$ N/m

a) $C_{cr} = 2\sqrt{mk} = 2\sqrt{0.4\,kg(5\,N/m)} = 2.82$ kg/s

Since $c > C_{cr}$, system is overdamped

b) For an overdamped system, a typical solution looks like:



c) Either use Matlab or the general eqn. for an overdamped system (see p. 473 of text)

$$x(t) = Ae^{\left(\frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t} + Be^{\left(\frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t}$$

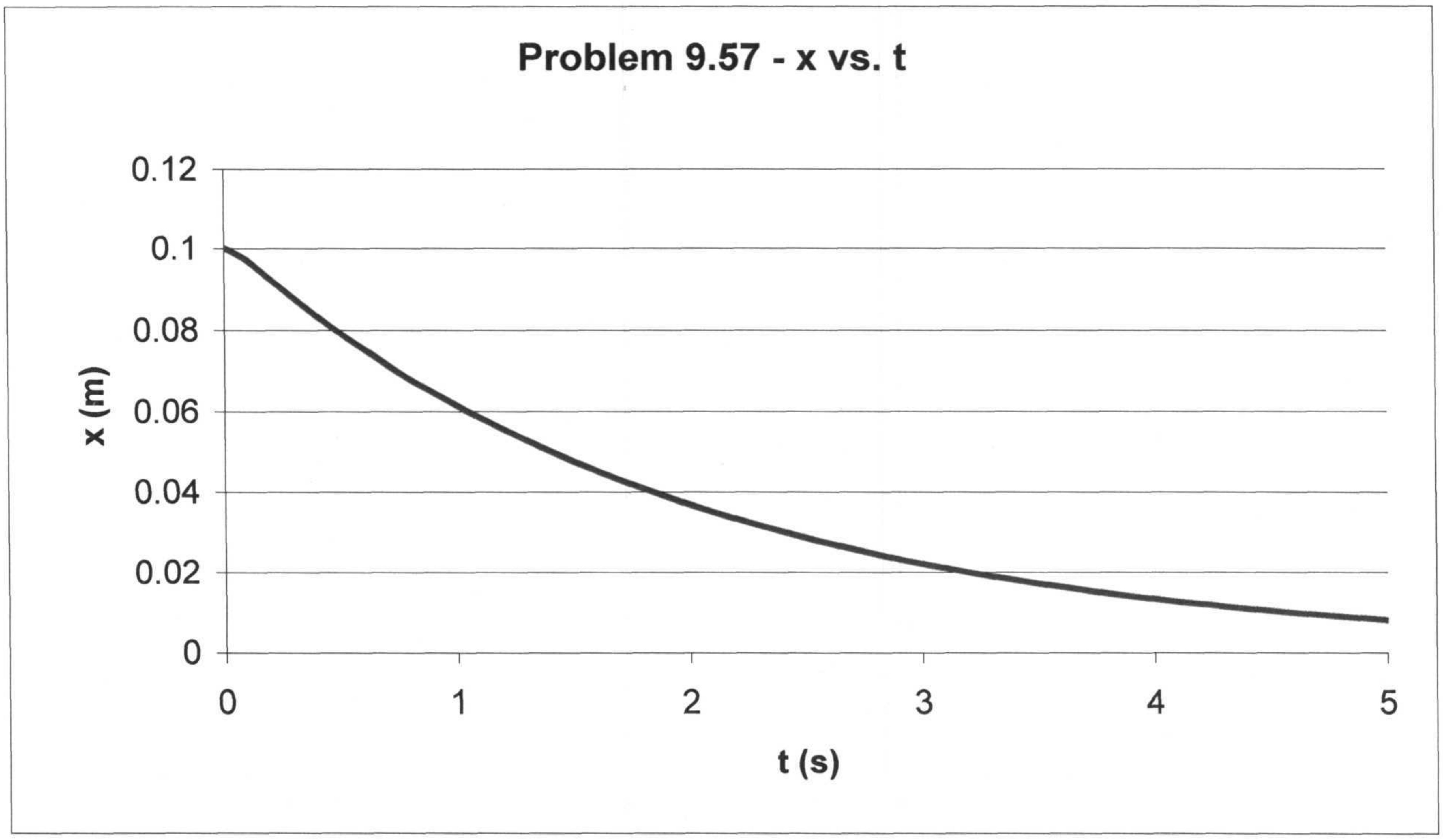
$$= Ae^{-.5104t} + Be^{-24.49t}$$

$$\dot{x}(t) = -0.5104Ae^{-.5104t} - 24.49Be^{-24.49t}$$

$x(0) = 0.1 = A + B$ $\qquad A = 0.1021$

$\dot{x}(0) = 0 = -0.5104A - 24.49B$ $\qquad B = -0.002128$

SEE GRAPH ON NEXT PAGE


Problem 9.57 - x vs. t
0.12
0.1
0.08
0.06
0.04
0.02
0
x (m)
0
1
2
3
4
5
t (s)