Your Name: "SOLUTIONS"

TA's name and Section time: TA,

T&AM 203 Prelim 1

Tuesday February 27, 2007

Draft February 27, 2007

3 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

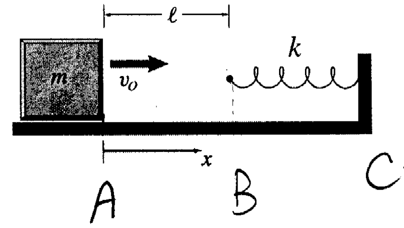
b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25

1) **(25 pt)** A mass m slides on a surface with negligible friction with initial velocity v_0 to the right. It is a distance ℓ from a spring with constant k. Give all answers in terms of some or all of m, v_0, ℓ and k.



a) (8 points) During the collision of the mass with the spring, what is the maximum deflection of the end of the spring?

b) (8 points) What is the total time of travel from the initial position back to the initial position?

c) (7 points) Make a sequence of plots that are vertically aligned and all of which have t on the 'x' axis:

i) x vs t
ii) $\dot{x}$ vs t
iii) $\ddot{x}$ vs t
iv) E_K vs t
v) E_P vs t

For each of the plots put values (in terms of the given variables) at key points on the plot. For example, in the plot of x vs t label the initial value of x as zero and the maximum value of x as ℓ + your answer to part (a) above.

SOLUTION BY:
JING SHI

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a) This is a conservative system

At time 0, $E_k = \frac{1}{2} m V_o^2$, $E_p = 0$

At the maximum deflection: $E_k = 0$, $E_p = \frac{1}{2} k \ell_m^2$

where ℓ_m is the maximum deflection of the spring

$$\frac{1}{2} m V_o^2 + 0 = 0 + \frac{1}{2} k \ell_m^2$$

$$\boxed{\ell_m = \sqrt{\frac{m}{k}}\, V_o}$$

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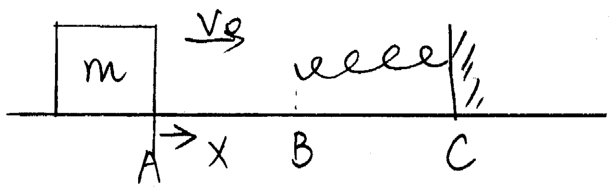
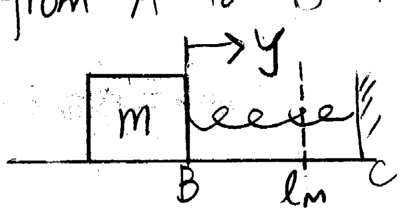
b)



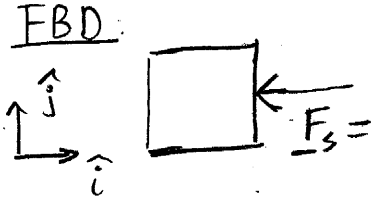
from A to B: $V = V_o$, $a = 0$.

Time spent from A to B: $t_1 = \frac{\ell}{V_o}$

from B to ℓ_m



FBD:



$F_s = -ky\hat{i}$

$$m\ddot{y} = -ky$$

$$y = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t$$

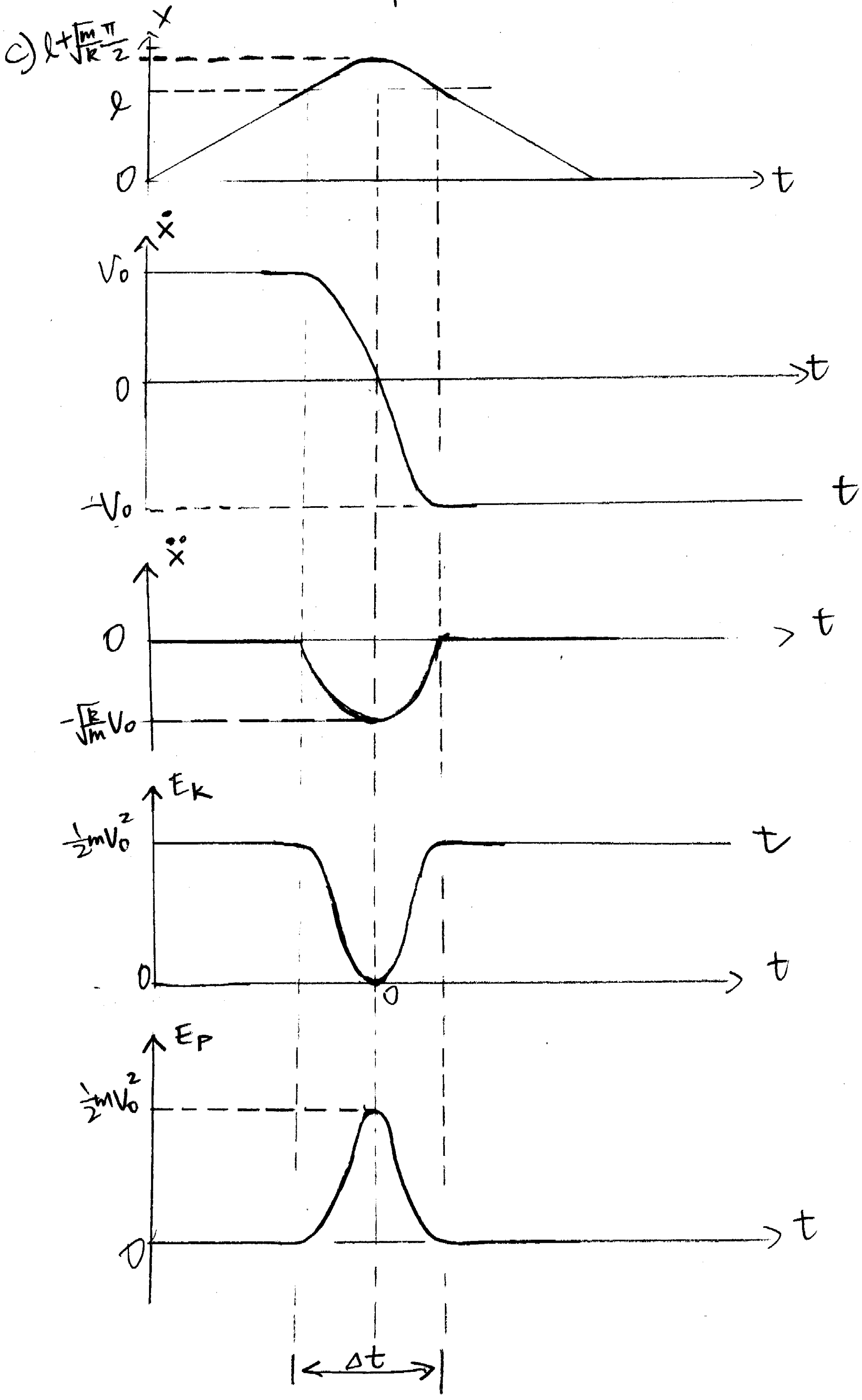
at time 0: $V_B = V_o$, $y_B = 0$

$$y = \sqrt{\frac{m}{k}}\, V_o \sin\sqrt{\frac{k}{m}}\,t$$

when $Y=\sqrt{\frac{m}{k}}V_0$, $t_2=\sqrt{\frac{m}{k}}\frac{\pi}{2}$

$\therefore$ Total time spent : $T=2(t_1+t_2)=2\left(\frac{l}{V_0}+\sqrt{\frac{m}{k}}\frac{\pi}{2}\right)$

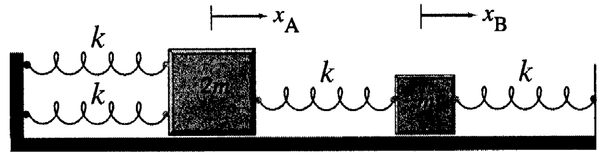
c)



$\Delta t=\pi\sqrt{\frac{M}{K}}$

SOLUTION BY:
MEGHAN SUCHORSKY

2) (25 pt) Normal modes etc. Mass A is $2m$, mass B is m. All springs have constant k and are relaxed when $x_A = x_B = 0$. Give all answers in terms of m and k.



a) (8 points) Find the differential equations ('the equations of motion') for this system. Reduce to the simplest tidy form you can.

b) (7 points) One normal mode for this system has the two masses moving oppositely with the B moving twice as much as A. Find the period of this normal mode vibration (no need for eigen-values etc).

c) (5 points) By intuition, guessing, or whatever, find another normal mode and its period of oscillation.

d) (5 points) The minimal (and inadequately commented) Matlab code below simulates the motion of this system for the first normal mode above. Missing is the function `frhs`. Write it.

```
function normalmodes
m = 5; k = 2;
xA0 = 1; xB0 = -2;
xAdot0 = 0; xBdot0 = 0;
z0 = [ xA0 xAdot0 xB0 xBdot0 ];
tspan = linspace(0,10,1000);
[t z] = ode45(@frhs, tspan, z0, [], m,k);
xA = z(:,1);
xB = z(:,3);
plot (t,xA,t,xB); title('First normal mode')
end
```

ⓐ For FBD assume $x_B > x_A > 0$ $\xrightarrow{x}$

$2kx_A \leftarrow$ [2m] $\rightarrow k(x_B - x_A) \leftarrow$ [m] $\leftarrow kx_B$

block A block B

block A: $\Sigma F = 2m\ddot{x}_A$

$$k(x_B - x_A) - 2kx_A = 2m\ddot{x}_A$$

$$\boxed{\frac{k}{2m}(x_B - 3x_A) = \ddot{x}_A} \quad (1)$$

block B: $\Sigma F = m\ddot{x}_B$

$$-k(x_B - x_A) - kx_B = m\ddot{x}_B$$

$$\boxed{\frac{k}{m}(x_A - 2x_B) = \ddot{x}_B} \quad (2)$$

in matrix form

$$\boxed{\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} = \begin{bmatrix} -3k/2m & k/2m \\ k/m & -2k/m \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}}$$

Parts ⓑ & ⓒ on next page...

ⓓ

```
function zdot = frhs(t,z,m,k);
XA = z(1) ;
XB = z(3) ;
VA = z(2);
VB = z(4);
XAdot = VA;
VAdot = k/2m(XB-3XA);
XBdot = VB;
VBdot = k/m(XA-2XB);
zdot = [XAdot; VAdot;
        XBdot; VBdot];
```

(b) $x_B = -2x_A$ (3)

plug (3) into (1)

$$\frac{k}{2m}(-2x_A - 3x_A) = \ddot{x}_A$$

$$\frac{-5k}{2m}x_A = \ddot{x}_A$$

$$\omega_A = \sqrt{5k/2m} \quad (4)$$

Or, plug (3) into (2)

$$\frac{k}{m}\left(-\frac{1}{2}x_B - 2x_B\right) = \ddot{x}_B$$

$$-\frac{5kx_B}{2m} = \ddot{x}_B$$

$$\omega_B = \sqrt{5k/2m} \quad (5)$$

note (5) agrees with (4), $\omega_B = \omega_A$ so masses are moving in a normal mode

$$\boxed{T = 2\pi/\omega = 2\pi\sqrt{2m/5k}}$$

(c) when else does $\omega_B = \omega_A$?

know $x_A = cx_B$* $c = \text{const}$ (6)

plug (6) into (1)

$$\frac{k}{2m}\left(\frac{1}{c}x_A - 3x_A\right) = \ddot{x}_A$$

$$\frac{k}{2m}\left(\frac{1}{c} - 3\right)x_A = \ddot{x}_A$$

$$\omega_A = \sqrt{\frac{k(3-1/c)}{2m}} \quad (7)$$

* definition of normal mode

plug (6) into (2)

$$\frac{k}{m}(cx_B - 2x_B) = \ddot{x}_B$$

$$\frac{k}{m}(c-2)x_B = \ddot{x}_B$$

$$\omega_B = \sqrt{\frac{(2-c)k}{m}} \quad (8)$$

when does (7) = (8)

$$\frac{k(3-1/c)}{2m} = \frac{k(2-c)}{m}$$

$$\frac{3-1/c}{2} = 2-c$$

$$2c^2 - c - 1 = 0$$

$$(2c+1)(c-1) = 0$$

$c_1 = -1/2$, $c_2 = 1$

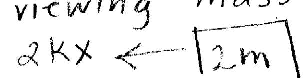
c_1 corresponds to part (b)

$c_2 = 1 \Rightarrow \omega = \sqrt{k/m}$

$$\boxed{T = 2\pi\sqrt{m/k}}$$

Intuitively the other mode should be the masses moving in phase with one another. When this occurs the middle spring is unstretched. With the middle spring unstretched the FBD becomes, using $x_A = x_B = x$

viewing mass A

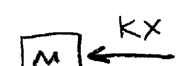


$$2m\ddot{x} = -2kx$$

$$\omega = \sqrt{k/m}$$

$$\boxed{T = 2\pi\sqrt{m/k}}$$

mass B



$$m\ddot{x} = -kx$$

$$\omega = \sqrt{k/m}$$

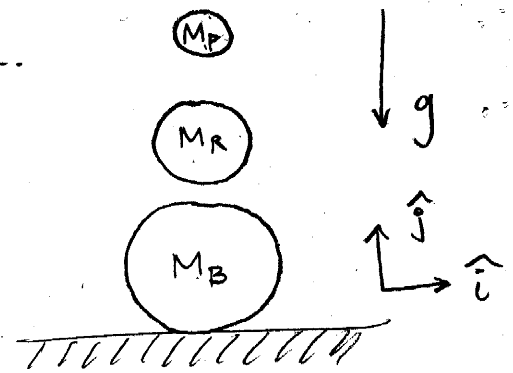
$$\boxed{T = 2\pi\sqrt{m/k}}$$

SAME

3) (25 pt) Three balls are vertically stacked and dropped vertically a distance h on to a hard surface. The gravity constant is g. Think of a basketball m_b, a rubber ball m_r and a ping-pong ball m_p. Consider the most ideal case: all coefficients of restitution are perfectly elastic with $e = 1$, each ball is much more massive than the ball above it $m_b \gg m_r \gg m_p$, and air friction is negligible. Assume that the collisional event is actually a sequence of collisions: the big ball hits the ground and bounces up and then collides with the middle ball which then collides with the top ball. In this most ideal case, how high will the top ping-pong ball bounce up, measuring from its height at the collision point.

SOLUTION BY:
KEVIN
ROMPALA

Given that $M_B \gg M_R \gg M_P$ & $e = 1$.



Taking the event as a series of collisions, the first collision is between M_B & the ground.

From CONSERVATION OF ENERGY, the velocity of M_B before its first collision, $\underline{V}_B$, is

$$M_B g h = \frac{1}{2} M_B (\underline{V}_B \cdot \underline{V}_B) \Rightarrow \boxed{\underline{V}_B = -\sqrt{2gh}\,\hat{j}}$$

For a perfectly elastic collision we have

$$e = 1 = \frac{\overbrace{\cancel{V_{GROUND/FINAL}}}^{0\,\frac{m}{s}} - V_{B/FINAL}}{\underbrace{\cancel{V_{B/INITIAL}}}_{-\sqrt{2gh}} - \underbrace{\cancel{V_{GROUND/INITIAL}}}_{0\,\frac{m}{s}}} \Rightarrow V_{B/FINAL} = -V_{B/INITIAL} = \sqrt{2gh}$$

$\boxed{\text{So after the } 1^{st} \text{ collision } M_B \text{ has velocity } \underline{V}_B = \sqrt{2gh}\,\hat{j}.}$

For the second collision, between M_B & M_R, we need to use the fact it's a perfectly elastic collision plus the linear momentum for the system is conserved in the $\hat{j}$-direction.

$$e = 1 = \frac{V_{B/FINAL} - V_{R/FINAL}}{\underbrace{\cancel{V_{R/INITIAL}}}_{-\sqrt{2gh}} - \underbrace{\cancel{V_{B/INITIAL}}}_{\sqrt{2gh}}} \Rightarrow \boxed{V_{B/FINAL} = -2\sqrt{2gh} + V_{R/FINAL}}$$

$$M_B \underbrace{\cancel{V_{B/INITIAL}}}_{\sqrt{2gh}} + M_R \underbrace{\cancel{V_{R/INITIAL}}}_{-\sqrt{2gh}} = M_B V_{B/FINAL} + M_R V_{R/FINAL}$$

Combining the 2 equations yields

$$\sqrt{2gh}\,(M_B - M_R) = M_B\left(-2\sqrt{2gh} + V_{R/FINAL}\right) + M_R V_{R/FINAL}$$

$$\Rightarrow \sqrt{2gh}\,(3M_B - M_R) = (M_B + M_R)\,V_{R/FINAL}$$

$$\Rightarrow V_{R/FINAL} = \frac{\sqrt{2gh}\,(3M_B - M_R)}{M_B + M_R} \approx 3\sqrt{2gh}$$

$\boxed{\text{since } M_B >> M_R}$

However, instead of find the exact solution & then approximating, we could note that since $M_B >> M_R$, almost all the momentum before & after the collision belongs to M_B. Thus the collision will have negligible effect on the velocity of M_B. Using equations,

$$M_B V_{B/INITIAL} + M_R V_{R/INITIAL} = M_B V_{B/FINAL} + M_R V_{R/FINAL}$$

$$\Rightarrow M_B V_{B/INITIAL} \approx M_B V_{B/FINAL} \Rightarrow \boxed{V_{B/INITIAL} \approx V_{B/FINAL}}$$

With this approximation we now have only 1 equation to solve, namely

$$e = 1 = \frac{\overset{\sqrt{2gh}}{V_{B/FINAL}} - V_{R/FINAL}}{\underset{-\sqrt{2gh}}{V_{R/INITIAL}} - \underset{\sqrt{2gh}}{V_{B/INITIAL}}} \Rightarrow \boxed{V_{R/FINAL} = 3\sqrt{2gh}}$$

$\boxed{\text{Thus after the second collision } M_R \text{ has velocity } \underline{V}_R = 3\sqrt{2gh}\ \hat{j}}$

For the 3rd collision we have

$$e = 1 = \frac{\overset{3\sqrt{2gh}}{V_{R/FINAL}} - V_{P/FINAL}}{\underset{-\sqrt{2gh}}{V_{P/INITIAL}} - \underset{3\sqrt{2gh}}{V_{R/INITIAL}}} \Rightarrow \boxed{V_{P/FINAL} = 7\sqrt{2gh}}$$

$\boxed{\text{After the 3rd collision } M_P \text{ has velocity } \underline{V}_P = 7\sqrt{2gh}\ \hat{j}.}$

To find how high M_P travels after its collision we use CONSERVATION OF ENERGY

$$\cancel{M_P} g Y = \frac{1}{2} \cancel{M_P} (\underline{V}_P \cdot \underline{V}_P) \Rightarrow Y = \frac{1}{g}(49gh) = 49h$$

In conclusion, we find that M_P bounces to $49h$, or 49 times as high as its starting height relative to its collision point.