"SOLUTIONS"

Your Name: Andy Ruina

TA's name and Section time: ____________________

T&AM 203 Final Exam

Monday May 14, 2007

Draft May 14, 2007

5 problems, 25^{+} points each, and 150.0 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/25
Problem 4:	/25
Problem 5:	/25

1) **(25 pt)** The inadequately commented Matlab code below runs without error. It solves at least one basic mechanics problem.

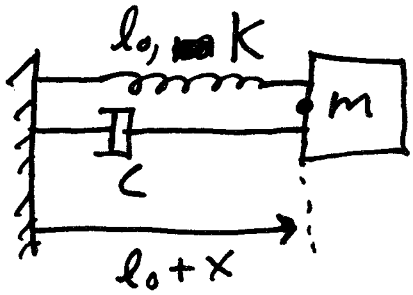
a) Using a sketch, and words if needed, describe a mechanical system to which this code applies.

b) What does A represent? B? C?

c) Pose the specific question that the Matlab code answers.

d) Draw the plot that Matlab draws. Label the 'y' axis appropriately. Your plot needs to be quantitatively correct at $t = 0$. At other points the plot should clearly show the nature of the solution, but need not be quantitatively accurate.

damped spring & mass



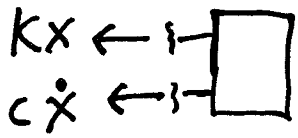
a

```
function finalexam()
A = 5; B = 1; C = 3;
tspan = linspace(0,10,1000);
z0 = [0 5];
[t,soln] = ode45(@rhs,tspan,z0,[],A,B,C);
z1 = soln(:,1);
z2 = soln(:,2);
plot(t,z1);
xlabel('time')
ylabel('Put something meaningful here.')
end

function zdot = rhs(t,z,A,B,C)
z1 = z(1);
z2 = z(2);
z1dot = z2;
z2dot = -(A*z1 + B*z2)/C;
zdot = [z1dot z2dot]';
end
```

$\dot{x} = v$

FBD



$F = ma$

$\Rightarrow -kx - c\dot{x} = m\ddot{x}$

$\Rightarrow \ddot{x} = -(kx + c\dot{x})/m$

$\Rightarrow \dot{v} = -(kx + cv)/m$

A B C

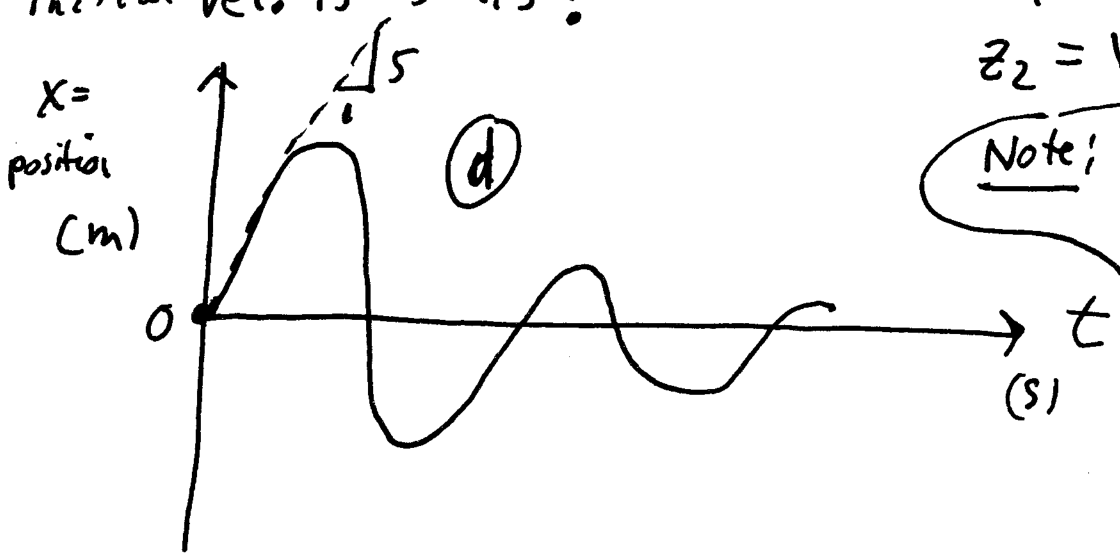
b

For m = 3kg, c = 1 N/(m/s) & k = 5 N/m what is motion if initial position is zero & initial vel. is 5 m/s?

c

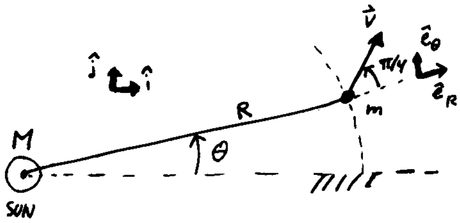
$z_1 = x$

$z_2 = v$



Note: $c^2 - 4km$
$= 1 - 60$
$= -59 < 0$
$\Rightarrow$ underdamped oscillations

2) **(25 pt)** At the moment of interest an asteroid m is a distance R from the sun M and attracted to the sun according to Newton's law of universal gravitation (with constant G). At just this instant the asteroid's velocity $\underline{v}$ (with magnitude $v = |\underline{v}|$) makes an angle of $\pi/4$ with a radial line.

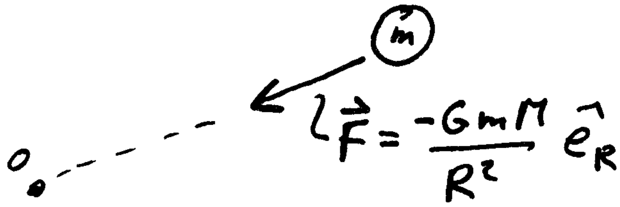


a) Find the acceleration of the asteroid. Your answer must be in terms of v, R, m, M, G, $\hat{\mathbf{e}}_R$, $\hat{\mathbf{e}}_\theta$ and *no* other quantities. If you think you need other quantities (e.g., $\dot{\theta}$) you must find them.

b) Find $\ddot{\theta}$ in terms of v, R, m, M and G.

c) Find $\ddot{R}$ in terms of v, R, m, M and G.

FBD



LMB: $\vec{F} = m\vec{a}$

$\vec{a} = \vec{F}/m$

$$\boxed{\vec{a} = -(GM/R^2)\,\hat{e}_R} \quad (a)$$

Kinematics:

$$\{\vec{V} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta\}$$

$$\{\}\cdot\hat{e}_R \Rightarrow \boxed{V\frac{\sqrt{2}}{2} = \dot{R}} \quad (1)$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow V\frac{\sqrt{2}}{2} = R\dot{\theta}$$

$$\boxed{\dot{\theta} = \frac{V}{R}\frac{\sqrt{2}}{2}} \quad (2)$$

$$\{\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta\} \quad (3)$$

$$\{\}\cdot\hat{e}_R \Rightarrow \ddot{R} - R\dot{\theta}^2 = -GM/R^2$$

$$\Rightarrow \ddot{R} = R\dot{\theta}^2 - GM/R^2$$

$$\dot{\theta} = \frac{V}{R}\frac{\sqrt{2}}{2}$$

$$\boxed{\ddot{R} = \frac{V^2}{2R} - GM/R^2} \quad (c)$$

$$\{(3)\}\cdot\hat{e}_\theta \Rightarrow R\ddot{\theta} + 2\dot{R}\dot{\theta} = 0$$

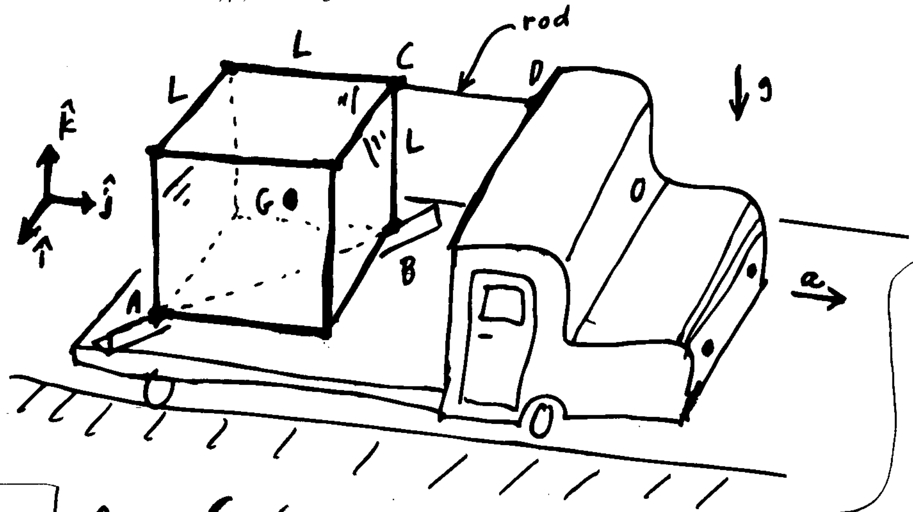
$$\Rightarrow \ddot{\theta} = -2\dot{R}\dot{\theta}/R$$

$$= -2\left(V\frac{\sqrt{2}}{2}\right)\left(\frac{V}{R}\frac{\sqrt{2}}{2}\right)/R \Rightarrow \boxed{\ddot{\theta} = -V^2/R^2} \quad (b)$$

3) **(25 pt)** A uniform cubical block (sides L, density ρ) rides on the back of a truck that has acceleration $\underline{\mathbf{a}} = a\,\hat{\mathbf{j}}$. The block rests on a horizontal but diagonal wedge on the bottom of the truck. Assume the block only touches the wedge at points A and B (marked on sketch). The block does not slide on the wedge. Rod CD keeps the block from tipping.

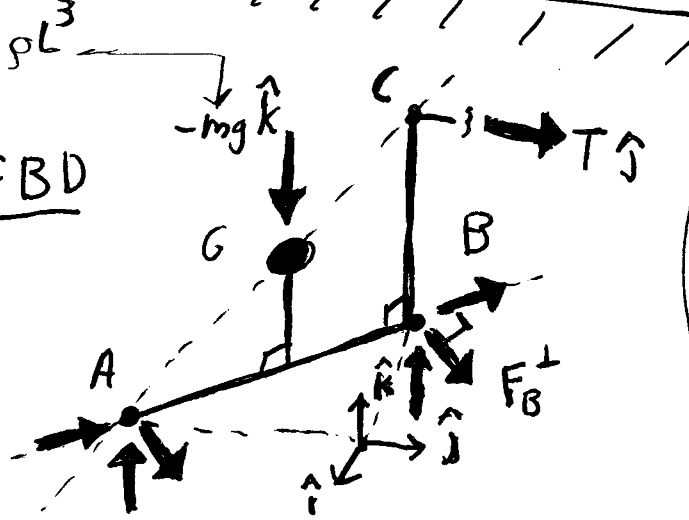
a) What is the tension in rod CD (in terms of some or all of L, ρ, a and g?

b) The horizontal part of the reaction at B can be resolved into components parallel and orthogonal to the line AB. What is orthogonal component of the force acting at the block at B (in terms of some or all of L, ρ, a and g?



* gravity forces equiv. to a force at C.O.M

FBD



Notes:

* 7 unknown reactions (the forces along AB at A & B cannot be found separately, only their sum)
* gravity force could have been dropped, by inspection.

AMB/AB: $\left\{ \sum \vec{M}_{/B} = \dot{\vec{H}}_{/B} \right\} \cdot \vec{r}_{AB}$

$\vec{r}_{AB} = L(-\hat\imath + \hat\jmath)$

$$\left[\vec{r}_{BG} \times (-mg\hat{k})\right]\cdot\vec{r}_{AB} + \left[\vec{r}_{BC} \times T\hat\jmath\right]\cdot\vec{r}_{AB} + \vec{0} + \vec{0} + \dots = \left[\vec{r}_{BG} \times ma\hat\jmath\right]\cdot\vec{r}_{AB}$$

$\vec{r}_{BC} = L\hat{k}$

$\vec{r}_{BG} = \frac{L}{2}(\hat\imath - \hat\jmath + \hat k)$

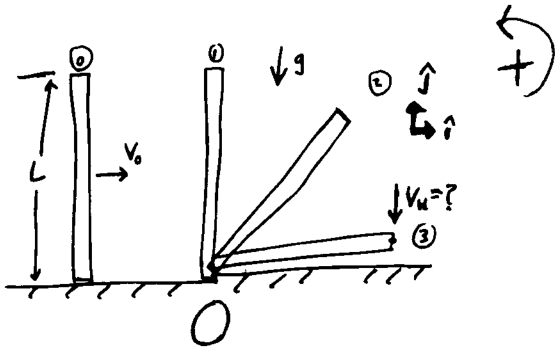
$$\Rightarrow \underbrace{\tfrac{1}{2}(+\hat\jmath + \hat\imath)\cdot(-\hat\imath + \hat\jmath)}_{0}\, mg + (-T\hat\imath)\cdot(-\hat\imath + \hat\jmath) = \tfrac{1}{2}(\hat k - \hat\imath)\cdot(-\hat\imath + \hat\jmath)\, ma$$

$$\boxed{+T_{CD} = ma/2} \quad \text{(a)} \qquad \boxed{\rho L^3 = m}$$

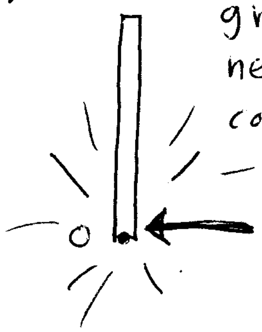
AMB/AC: All forces pass through this axis but for $F_B^\perp$.

Also $m\vec{a}$ acts at G on this axis. $\Rightarrow \boxed{F_B^\perp = 0}$ (b)

4) **(25 pt)** A running person trips and falls on his or her face. Imagine an American football player tackled at the ankles and, from the instant of the tackle on, the ankles act as a hinge. Model the running person in state (0) as an upright uniform rod of length L and mass m running at speed v_0. From just after being tripped, call this state (1), until just before the head hits the ground at state (3) the person falls like a pendulum hinged at the feet. Find, in terms of some or all of v_0, m, g and L, the speed of the top of the head when it hits the ground.



FBD $(0 \to 1)$



gravity is negligible during collision $\Rightarrow \vec{H}^1_{/O} = \vec{H}^0_{/O}$

$$I^O \omega_1 \hat{k} = \frac{L}{2}\hat{j} \times (m v_0 \hat{i})$$

$$\Rightarrow \quad \omega_1 = -m v_0 L / 2 I^O$$

$I^O = mL^2/3$

$$\omega_1 = \frac{3}{2}(v_0/L)$$

$$\boxed{\text{Kinematics: } V_H = \omega_3 L}$$

$1 \to 3$: Energy is conserved

$$E_{p_1} + E_{k_1} = E_{p_3} + E_{k_3}$$

$$\Rightarrow \quad \frac{1}{2} mgL + \frac{1}{2} I^O \omega_1^2 = 0 + \frac{1}{2} I_O \omega_3^2$$

$$\Rightarrow \quad \omega_3 = \pm\sqrt{\omega_1^2 + mgL/I^O}$$

use "−" so $V_H > 0$

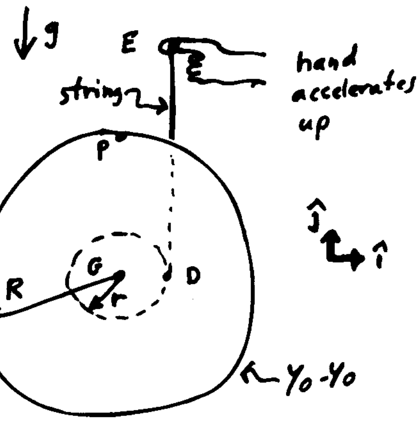
$$\Rightarrow \quad V_H = -L\omega_3 = \sqrt{L^2\omega_1^2 + mgL^3/I_O}$$

$$= \sqrt{L^2\left(\frac{3v_0}{2L}\right)^2 + mgL^3/(mL^2/3)}$$

$$= \sqrt{\frac{3^2}{2^2}v_0^2 + 3gL} \quad \Rightarrow \quad \boxed{V_H = 3\sqrt{(v_0/2)^2 + gL/3}}$$

$$\cancel{V_H = \sqrt{3}\sqrt{v_0^2/2 + gL}}$$

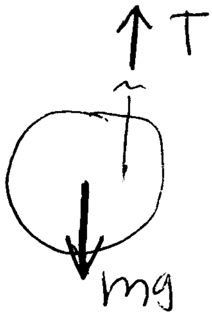
5) **(25 pt)** A yo-yo is dropped. Simultaneously the hand at the other end of the string is accelerated up. The hand is directly over the edge of the yo-yo spool so that the string is vertical. The hand is accelerated up at just the acceleration needed to keep the yo-yo from falling. That is, the yo-yo spins faster and faster but stays in place. You are given the radius r of the spool on which the string is wrapped, the outer radius R of the yo-yo, the mass m of the yo-yo, the yo-yo moment-of-inertia I^G about its center of mass G, and the gravitational constant g.



- a) (1 point) Write "I have read the directions on the cover and understand them."
- b) (8 points) What is the tension in the string?
- c) (8 points) What is the acceleration of the hand?
- d) (8 points) Consider a point P on the yo-yo that is on its outer boundary a distance R directly above G when the yo-yo is released from rest. What is the acceleration of P after one revolution? (Clearly define any unit vectors you use in your work and answer.)

a) I hope I understand the directions on the cover. I wrote them.

b FBD



LMB

$\sum \vec{F} = m\cancel{\vec{a}}^{\,\vec{0}}$

$T - mg = 0$

$\boxed{T = mg}$ (b)

AMB/G :

$T r = I^G \dot{\omega}$

$\dot{\omega} = T r / I^G$

$= mgr/I^G = \text{const}$ (1)

Kinematics

$\vec{v}_D = \vec{v}_E$

$\omega r \hat{j} = v_E \hat{j}$

$\dot{v}_E = \dot{\omega} r$

$= \left(\frac{mgr}{I^G}\right) r$

$\boxed{\vec{a}_E = \frac{mgr^2}{I^G}\hat{j}}$ (c)

Kinematics

$\vec{a}_P = -\omega^2 R \hat{j} + \dot{\omega} R(-\hat{i})$ (2)

Work-Energy (1 revolution)

$2\pi r T = \frac{1}{2} I^G \omega^2$

$\omega^2 = \frac{4\pi r T}{I^G} = \frac{4\pi r m g}{I^G}$ (3)

Apply (3) & (1) to (2)

$\Rightarrow \vec{a}_P = -\frac{4\pi r R m g}{I^G}\hat{j} + \frac{-mgrR}{I^G}\hat{i}$

$\boxed{\vec{a}_P = -\frac{mgrR}{I^G}\left(\hat{i} + 4\pi \hat{j}\right)}$ (d)