Lab #4 - Gyroscopic Motion of a Rigid Body

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INTRODUCTION

Gyroscope is a word used to describe a rigid body, usually with symmetry about an axis, that has a comparatively large angular velocity of spin, $\dot{\psi}$, about its spin axis. Some examples are a flywheel, symmetric top, football, navigational gyroscopes, and the Earth. The gyroscope differs in some significant ways from the linear one and two degrees-of-freedom systems with which you have experimented so far. The governing equations are 3-dimensional equations of motion and thus mathematical analysis of the gyroscope involves use of 3-dimensional geometry. The governing equations for the general motion of a gyroscope are non-linear. Non-linear equations are in general hard (or impossible) to solve. In this laboratory you will experiment with some simple motions of a simple gyroscope. The purpose of the lab is for you to learn the relation between torque, angular momentum, and rate of change of angular momentum. You will learn this relation qualitatively by moving and feeling the gyroscope with your hands and quantitatively by experiments on the precession of the spin axis.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. What is a gyroscope?
2. Where is the fixed point of the lab gyroscope?
3. How will moments (torques) be applied to the lab gyroscope?

THE GYROSCOPE

Our experiment uses a rotating sphere mounted on an air bearing (see Figure 2) so that the center of the sphere remains fixed in space (at least relative to the laboratory room). This is called a *gyroscope with one fixed point*.

As the gyroscope rotates about its spin axis it is basically stable. That is, the spin axis remains fixed in space and resists any externally applied force that would tend to alter its direction. As you should see in the experiment, the larger the spin rate the larger the moment needed to change the direction of the spin axis. When a moment is applied to a gyroscope, the spin axis will itself rotate about a new axis which is perpendicular to both the spin axis and to the axis of the applied moment. This motion of the spin axis is called *precession*.

DYNAMICS OF THE SYMMETRIC TOP

We will now use 3-dimensional rigid-body dynamics to determine the equations of motion for a symmetric top under the influence of gravity. This is a famous mechanics problem equivalent to our experimental set-up. Our analysis requires us to first define 2 different coordinate frames (see Figure 1). The $\left\{\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\right\}$ coordinate system remains fixed in space (in an inertial frame) while the $\{\hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}, \hat{\mathbf{e_3}}\}$ coordinate system is semi-fixed to the rotating rigid-body (in a rotating non-inertial frame). That is it's allowed to only rotate about the $\hat{\mathbf{e_1}}$ and $\hat{\mathbf{e_2}}$ axes (in other words the rotating frame does not spin with the body about its spin axis). Furthermore, the semi-fixed coordinate axis is chosen to be a *principal coordinate axis* of the rigid body. This will simplify our analysis by diagonalizing the inertia tensor.

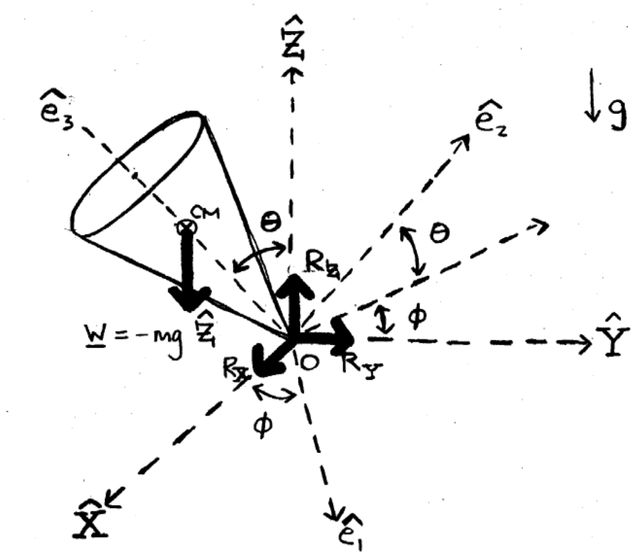

Figure 1: A free-body diagram of the symmetric top including both coordinate frames.

Using the aforementioned coordinate definitions, the frame rotation vector $\underline{\mathbf{\Omega}}$ is

$$\underline{\mathbf{\Omega}} = \dot{\phi}\hat{\mathbf{Z}} + \dot{\theta}\hat{\mathbf{e_1}} = \dot{\theta}\hat{\mathbf{e_1}} + \dot{\phi}\sin\theta\hat{\mathbf{e_2}} + \dot{\phi}\cos\theta\hat{\mathbf{e_3}} \tag{1}$$

while the body rotation vector $\underline{\omega}$ is

$$\underline{\omega} = \underline{\mathbf{\Omega}} + \dot{\psi}\hat{\mathbf{e_3}} = \dot{\theta}\hat{\mathbf{e_1}} + \dot{\phi}\sin\theta\hat{\mathbf{e_2}} + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\hat{\mathbf{e_3}} \tag{2}$$

The angular momentum of the top about the fixed origin, $\underline{\mathbf{H}}_\mathbf{o}$, in the rotating coordinate frame, is

$$\underline{\mathbf{H}}_\mathbf{o} = [I_o]\underline{\omega} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = I\omega_1\hat{\mathbf{e_1}} + I\omega_2\hat{\mathbf{e_2}} + I_{zz}\omega_3\hat{\mathbf{e_3}} \tag{3}$$

where $I_{xx} = I_{yy} = I$ due to the symmetry of the rigid body. Differentiating with respect to time, we find the time rate of change of the angular momentum to be

$$\dot{\underline{\mathbf{H}}}_\mathbf{o} = I\dot{\omega_1}\hat{\mathbf{e_1}} + I\dot{\omega_2}\hat{\mathbf{e_2}} + I_{zz}\dot{\omega_3}\hat{\mathbf{e_3}} + \underline{\mathbf{\Omega}} \times \underline{\mathbf{H}}_\mathbf{o} \tag{4}$$

where the final term arises due to the use of a rotating coordinate frame. Performing the required vector cross-product we get

$$\underline{\mathbf{\Omega}} \times \underline{\mathbf{H}}_\mathbf{o} = \begin{vmatrix} \hat{\mathbf{e_1}} & \hat{\mathbf{e_2}} & \hat{\mathbf{e_3}} \\ \omega_1 & \omega_2 & \Omega_3 \\ I\omega_1 & I\omega_2 & I_{zz}\omega_3 \end{vmatrix} = \left(I_{zz}\omega_2\omega_3 - I\omega_2\Omega_3\right)\hat{\mathbf{e_1}} + \left(I\omega_1\Omega_3 - I_{zz}\omega_1\omega_3\right)\hat{\mathbf{e_2}} + 0\hat{\mathbf{e_3}} \tag{5}$$

Using Figure 1 we find the total applied torque to be

$$\sum \underline{\mathbf{M_o}} = \underline{\mathbf{r_{cm}}} \times \underline{\mathbf{W}} = h\hat{\mathbf{e_3}} \times -mg\hat{\mathbf{Z}} = hmg\sin\theta\hat{\mathbf{e_1}} \tag{6}$$

We now use angular momentum balance about the fixed origin - $\sum \underline{\mathbf{M_o}} = \underline{\dot{\mathbf{H}}_\mathbf{o}}$. Substituting (4), (5), and (6) into the angular momentum balance and "dotting" with all 3 rotating unit vectors, we end up with 3 separate equations:

$$I\dot{\omega}_1 + I_{zz}\omega_2\omega_3 - I\omega_2\Omega_3 = hmg\sin\theta \tag{7a}$$

$$I\dot{\omega}_2 + I\omega_1\Omega_3 - I_{zz}\omega_1\omega_3 = 0 \tag{7b}$$

$$I\dot{\omega}_3 = 0 \tag{7c}$$

Equation (7c) says that $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ is constant. Physically, we interpret this as saying the "total spin" of the rigid body about the $\hat{\mathbf{e_3}}$-axis is constant.

We simplify the analysis of the two remaining equations by restricting ourselves to "steady-precession". Steady-precession occurs when we restrict the kinematics to constant spin rate $\dot{\psi}_\mathbf{o}$, constant precession $\dot{\phi}_\mathbf{o}$, and constant pitch $\theta_\mathbf{o}$. With these restrictions, (7b) is trivially satisfied and we are left with one equation

$$\dot{\phi}_\mathbf{o}\sin\theta_o\left[I_{zz}\left(\dot{\phi}_\mathbf{o}\cos\theta_o + \dot{\psi}_\mathbf{o}\right) - I\dot{\phi}_\mathbf{o}\cos\theta_o\right] = hmg\sin\theta_o \tag{8}$$

There are 3 constants in (8), two of which can be independently fixed in order to solve for the third. **In this lab you will set the spin rate $\dot{\psi}_\mathbf{o}$ and the pitch angle $\theta_\mathbf{o}$ and find the resulting precession speed $\dot{\phi}_\mathbf{o}$ for several different applied torques.**

Taking a look at the special case of $\theta_o = \frac{\pi}{2}$, equation (8) reduces to

$$I_{zz}\dot{\phi}_\mathbf{o}\dot{\psi}_\mathbf{o} = hmg \tag{9}$$

Thus for a gyroscope (or rotor) whose spin axis is orthogonal to the applied torque we find that the product of the moment of inertia, spin rate, and precession rate is equal to the applied torque.

LABORATORY SET-UP

Our lab gyroscope is a steel ball on an air bearing (see Figure 2). On one side of the ball a rod is mounted for reference and for touching. This side of the ball has also been bored out so that the rod side is lighter and the center of mass can be adjusted to either side of the center of the sphere by sliding a balance weight in or out. The balance weight is black, with reflective tape, to make rotation rate measurements easier. The sphere is supported in a spherical cup into which high pressure air is supplied so that the sphere is actually supported by a thin layer of air (similar to the air track).

To experimentally measure the spin rate $\dot{\psi}$ of the gyroscope you will use a tachometer (measures in rotations per minute, or rpm). To measure the precession rate $\dot{\phi}$ you will use a stop-watch. Finally, the metric scale will be used to measure the torques you will be applying to the gyroscope.

As a final example of the gyroscopic effect you will play around with a bicycle wheel and rotating platform for hands-on experience and a demonstration of the conservation of angular momentum.

PROCEDURE

1. Turn on the air source.

2. Place the black balance weight on the rod so that if the sphere is released with no spin the rod does not tend to fall down or pop upright from a horizontal position. Note that this is easier said than done, so try to get it as close to motionless as possible. *Where is the center of mass of the system (sphere, rod, and disk) after the gyroscope is balanced? What effect does gravity have on the motion of the balanced gyroscope? If you don't perfectly balance the gyroscope it will result in an error in the calculation of what quantity?*

3. Without spinning the ball, point the rod in some particular direction (up, or towards the door, for example). Carefully release the rod and watch it for several seconds. *Does it keep pointing in the same direction?* Touch the rod lightly with a small strip of paper. *How much force is required to change the orientation of the rod? In which direction does the rod move?* Rotate the table underneath the air bearing. *Does the rod move?*

4. Get the ball spinning and repeat step #4. One good way to do this is to roll the rod between your hands. Stop any wobbling motion by holding the tip lightly and briefly. Avoid touching the ball itself. **Do not allow the rod to touch the base and do not jar the ball while it is spinning.** *What is the effect of spin on the gyroscope motion? Why are navigation gyroscopes set spinning?*

5. While the ball is spinning, apply forces to the end of the rod using one of the pieces of Teflon on a string. The ball should continue to rotate freely as you apply the force because of the low friction of the Teflon. Gently move the end of the rod (keep the rod from touching the bearing cup, or the rod may spin wildly). *What is the relationship between the force you are applying and the velocity of the tip of the rod (estimated magnitude and direction)?* Remember that tension is always in the direction of the string.

6. For a more quantitative look at the motion of a gyroscope:

   (a) Add another weight to the rod so that the gyroscope is no longer balanced. Record its mass and position on the rod for use in calculations later (see Figure 2).

   (b) Get the ball spinning, but not wobbling, and point the rod towards one of the three support screws on the air bearing platform. With the rod horizontal, simultaneously release the rod and start the handheld digital stopwatch. The spinning ball and rod will begin to precess in a horizontal plane. Depending upon the precession rate you may want to stop the timer after one full revolution, or after only one-third or two-thirds of a revolution.

   (c) Halfway through the timing interval use the optical tachometer to measure the spin rate of the ball (this gives an average). The light beam from the tachometer should be aimed at the reflective tape on the black balance weight. The tachometer measures the rate of the pulses of light returning from the tape, and displays the result in r.p.m. Hold the tachometer at a distance of 10 cm or so. For higher accuracy, try to follow the precession of the rod with the tachometer. This may require practice and patience. If you find it more convenient, measure the spin rate at the start of the precession period and again at the end, and then find the average.

(d) Repeat the procedure for at least two additional spin rates. Try to use a wide range of spin rates; e.g., 200, 400, and 600 r.p.m.

7. Remove the weight and repeat step #6 with at least two more weights for a total of at least three different weights and three different spin rates per weight. The spin rates need not be the same as the ones you used before, but they should cover a similarly wide range of r.p.m.

8. Turn off the air source and clean up your lab station.

9. Hold the bicycle wheel while someone else gets it spinning. Twist it different ways. Hold your hands level and turn your body in a circle. *How do the forces you apply depend on the direction you twist the axle and on the rotation speed and sense?*

10. Repeat #9 while standing on the rotatable platform.

LAB REPORT QUESTIONS

1. Answer all of the questions given in the procedure above using full self-contained sentences.
2. Suppose that the rod on one spinning air gyroscope is pointed north, at an angle of 42.5 degrees from the horizontal (i.e. along the earth's axis of rotation). A second air gyroscope is pointed east, with its rod horizontal. Assume that the ball is perfectly balanced and that air friction is negligible. How does the orientation of each spinning gyroscope change over a period of several hours?
3. Use your recorded data from parts 6 and 7 of the lab procedure for the following questions.
   (a) Plot the precessional period τ vs. the spin rate $\dot{\phi}$ for your different applied torques. Make sure to use a different color and/or symbol for each data point.
   (b) From your plot derive the relationship between the precessional period τ and the spin rate $\dot{\phi}$?
   (c) For a fixed torque show that the product of the precessional rate and the spin rate $\dot{\phi}$ is a constant.
   (d) The torque should be proportional to the product of the spin rate and the precession rate. Find the constant of proportionality and plot the relationship between torque and the product of spin rate and precession rate (i.e. M_o vs. $\dot{\phi}\dot{\psi}$).
   (e) You have now found a simple formula relating torque, spin rate and precession rate. What is the meaning of the numerical constant in the formula?
4. Explain in words why when you stand on the platform with a spinning bicycle wheel and proceed to rotate the wheel, the platform begins to rotate.

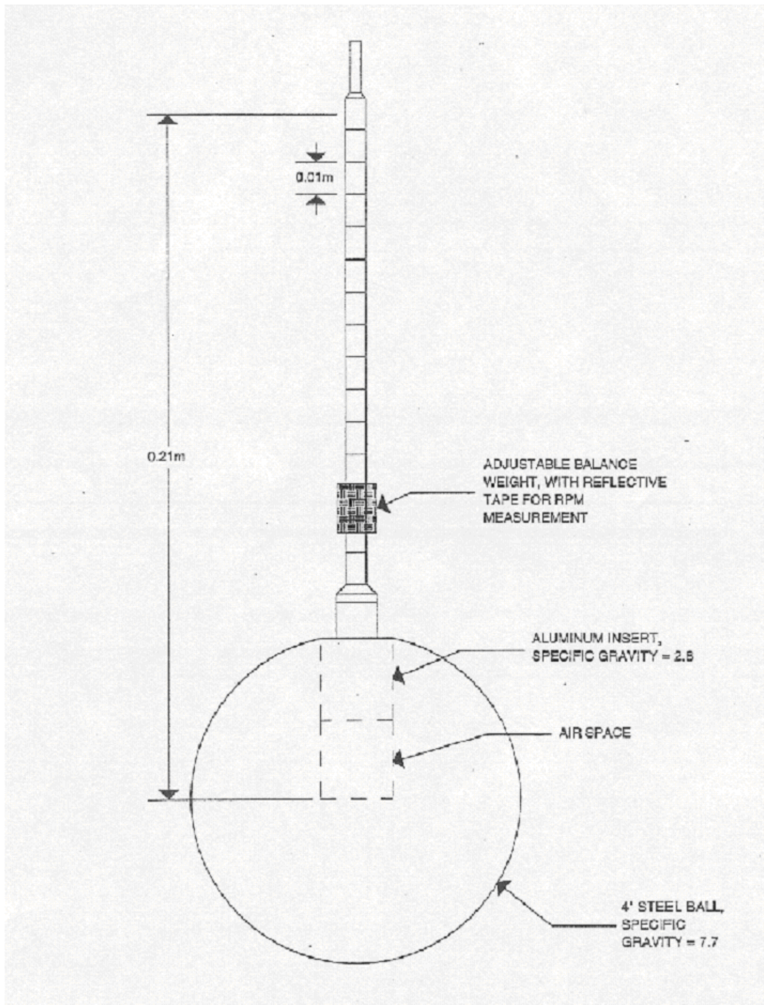

Figure 2: A diagram of the lab gyroscope.

CALCULATIONS & NOTES