"SOLUTIONS"

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Section day and time: Wed. 11:15

# T&AM 203 Prelim 2

## Tuesday April 16, 2002

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3 problems, 100 points, and 90$^{+}$ minutes.

**Please follow these directions to ease grading and to maximize your score.**

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I. ) neat,
  II. ) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

| | |
|---|---|
| Problem 1: | 30/30 |
| Problem 2: | 40/40 |
| Problem 3: | 30/30 |
| TOTAL: | 100/100 |

1) (30 pt) A front-wheel drive car has an engine with arbitrarily large power. It has a stiff suspension and light well-lubricated front wheels. It rides on level ground. Given:

$m =$ total car mass;

$I^G =$ the polar moment of inertia about an axis through the center of mass G and in the ussual $zz$ direction (perpendicular to the plane on which a side view of the car is drawn);

$h =$ the height of G above the ground;

$d =$ the distance G is forward of the rear ground contact at A;

$e =$ the distance G is behind the front ground contact at B;

$\mu =$ the coefficient of friction between the wheels and ground;

$g =$ the gravity constant. ~~In terms of some or all of $m, I^G, g, h, d, e,$ and $\mu$, find the maximum possible forward acceleration on level ground.~~

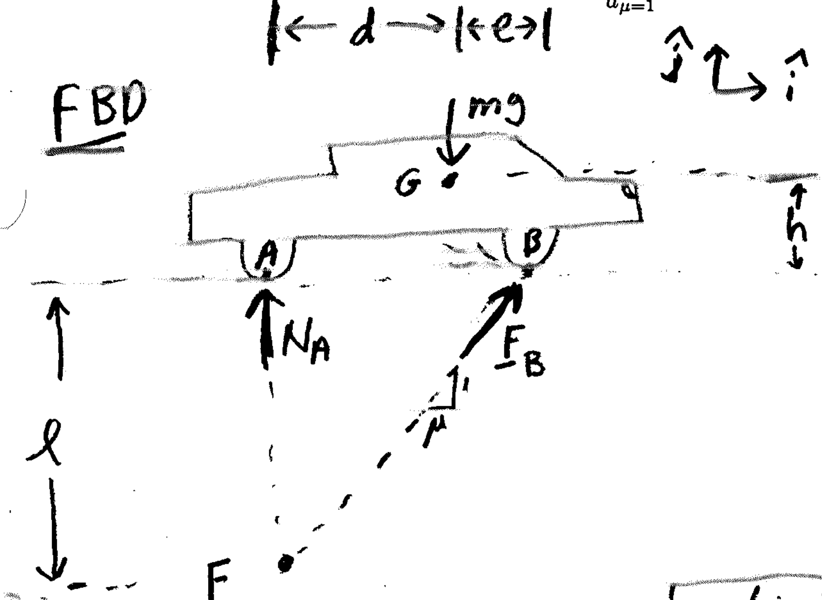
a) (20 points) Assuming the car does not tipe over, find the maximum possible forward acceleration $a$ ($\mathbf{a}_G = a\hat{\mathbf{i}}$). Answer in terms of some or all of $m, I^G, g, h, d, e,$ and $\mu$.

b) (5 points) (Good reasoning may or may not depend on part (a)) Assuming $d = e = h$, what is the largest value of $\mu$ that is possible without violating reasonable assumptions (assume that rubber with arbitrarily large $\mu$ could be made reasonably)? [Clearly state the reasonable assumptions that you are checking]. The answer is one of these

$$\mu_{max} = 0, .5, 1, \sqrt{2}, \sqrt{3}, 2, e, 2\sqrt{2}, 3, \pi, 4, 8, \text{ or } \infty.$$

c) (5 points) (Good reasoning may or may not depend on part (a)) Assuming $d = e = h$ and that the car does not tip over, by what ratio does the peak acceleration increase if $\mu$ is increased from $\mu = 1$ to $\mu = \infty$ (infinite coefficient of friction). The answer is one of these

$$\frac{a_{\mu=\infty}}{a_{\mu=1}} = 0, .5, 1, \sqrt{2}, \sqrt{3}, 2, e, 2\sqrt{2}, 3, \pi, 4, 8, \text{ or } \infty.$$




Geometry:

$$\frac{d+e}{\ell} = \frac{\mu}{1}$$

$$\Rightarrow \ell = \frac{d+e}{\mu}$$

$$\sum \underline{M}_{/E} = \dot{\underline{H}}_{/E}$$

no tipping

$$-dmg\hat{k} = \underline{r}_{G/E} \times m\underline{a}_G + I^G \cancel{\alpha}\hat{k}$$

$$= ((\ell + h)\hat{j} + d\hat{i}) \times m a \hat{i}$$

$$= -\left(\frac{d+e}{\mu} + h\right) a \hat{k} m$$

$\{\}\cdot\hat{k} \Rightarrow$

$$d mg = \left(\frac{d+e}{\mu} + h\right) a m$$

$$\boxed{a = \frac{dg}{\left(h + \frac{d+e}{\mu}\right)} \qquad \underline{a}_G = \frac{dg\hat{i}}{\left(h + \frac{d+e}{\mu}\right)}}$$

(a)

(prob 1 cont'd)

(b) $\sum \underline{M}_{/B} = \dot{\underline{H}}_{/B}$

$$\left\{ \left(-N_A(d+e) + mge\right)\hat{k} = -mah\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow N_A = \frac{m(ah+ge)}{d+e} > 0$$

so rear wheels have no lift off problem. Front wheels can't have a lift off problem because the drive ~~force~~ goes to zero as lifting proceeds.

$\Rightarrow$ $\boxed{\mu = \infty \text{ is o.k.}}$ (b)

c) Could redo problem using assumptions, or just plug in soln. to (a): $(d=e=h)$

$$\boxed{\frac{a_{\mu=\infty}}{a_{\mu=1}} = \frac{\dfrac{dg}{(d+\frac{2d}{\infty})}}{\dfrac{dg}{d+\frac{2d}{1}}} = \frac{\frac{g}{1}}{\frac{g}{3}} = 3}$$ !

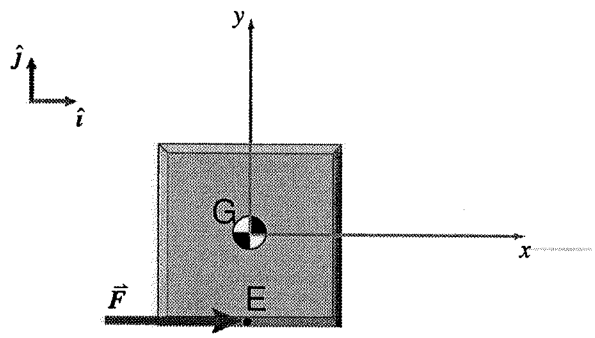
(Front wheel drive car (w/ $d=e=h$) can have accel. of $g/3$ if $\mu=1$ and $g$ if $\mu=\infty$.)

2) (40 pt) A uniform 1kg plate that is one meter on a side is initially at rest in the position shown. A constant force $\underline{\mathbf{F}} = 1\,\mathrm{N}\,\hat{\imath}$ is applied at $t = 0$ and maintained henceforth. [If you need to calculate any quantity that you don't know, but can't do the calculation to find it, assume that the value is given.]

a) Find the position of G as a function of time (the answer should have numbers and units).

b) Find a differential equation, and initial conditions, that when solved would give $\theta$ as a function of time. $\theta$ is the counterclockwise rotation of the plate from the configuration shown.

c) Write MATLAB commands that would generate a drawing of the outline of the plate at $t = 1\,\mathrm{s}$. You can use hand calculations or Matlab for as many of the intermediate commands as you like. Add enough hand work so your MATLAB reasoning is clear.




$\Rightarrow$

$\underline{r}_G = \frac{F}{m} t^2/2 \ \hat{i}$

$= \frac{1N}{1Kg} \frac{t^2}{2} \hat{i}$

$\boxed{\underline{r}_G = \frac{t^2}{2} \hat{i} \ \frac{m}{sec^2} \hat{i}}$

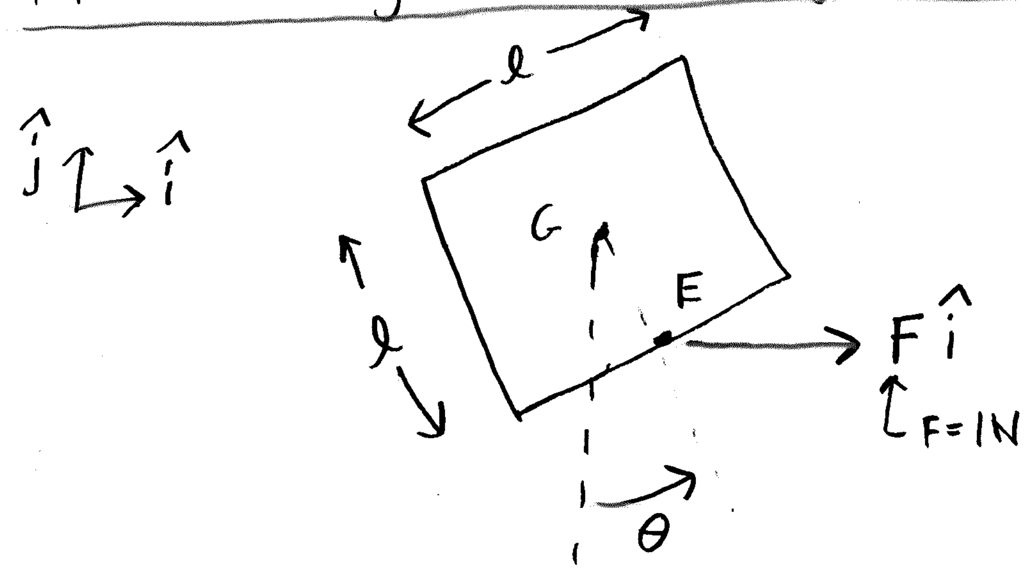
(a)

or

$\underline{r}_G = \frac{t^2}{2} \hat{i}$

if all quantities in meters, kg, sec.

FBD in general configuration




LMB: $\sum \underline{F}_i = m\underline{a}_G$

$\{F\hat{i} = m a_{Gx} \hat{i} + m a_{Gy} \hat{j}\}$

$\{\ \}\cdot\hat{i} \Rightarrow a_{Gx} = F/m \Rightarrow V_{Gx} = \frac{F}{m} t \Rightarrow X_G = \frac{F}{m} t^2/2$

$\{\ \}\cdot\hat{j} \Rightarrow a_{Gy} = 0 \Rightarrow Y_G = 0$

b) $\sum \underline{M}_{/G} = \dot{\underline{H}}_{/G}$ (Prob. 2 cont'd)

$$\underline{r}_{E/G} \times F\hat{i} = I^G \alpha \hat{k}$$

$$\frac{\ell}{2}\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right) \times F\hat{i} = I^G \ddot{\theta}\hat{k}$$

$$\Rightarrow \frac{\ell F}{2}\cos\theta = I^G \ddot{\theta}$$

$$\boxed{\ddot{\theta} = \frac{\ell F}{2 I_G}\cos\theta}$$

$\ell = 1\,m$, $F = 1\,N$, $m = 1\,kg$

$$I^G = \int_{-\ell/2}^{\ell/2}\int_{-\ell/2}^{\ell/2} (x^2+y^2)\frac{m}{\ell^2}\,dx\,dy$$

$$I^G = \left( \frac{x^3}{3}\ell \Big|_{-\ell/2}^{\ell/2} + \frac{y^3}{3}\ell \Big|_{-\ell/2}^{\ell/2} \right)\frac{m}{\ell^2}$$

$$I^G = \left(\frac{\ell^4}{12} + \frac{\ell^4}{12}\right)\frac{m}{\ell^2} = \frac{m\ell^2}{6}$$

$$\Rightarrow \ddot{\theta} = \left(\frac{1\cdot 1}{2\cdot\frac{1}{6}}\cos\theta\right) s^{-2} \Rightarrow \boxed{\ddot{\theta} = 3\cos\theta\ s^{-2}}\ (b)$$

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c)

all in consistent mKS units

```
pic = [1  -1  -1   1  1;
       1   1  -1  -1  1 ]/2;
θ0 = 0;  ω0 = 0;
[t z] = ode23('gosh', [0 1], [θ0 ω0])
θ1 = z(end,1);   % final rotation
R = [cos θ1   -sin θ1;
     sin θ1    cos θ1 ];
disp = [1/2  0]';   % displ. from part (a)
homo = [ R  disp;
         [0 0 1] ];   % homog. transf.
newpic = homo * [pic; ones(1,5)];
plot (newpic(1,:), newpic(2,:));
axis ('equal')
```

$z_0$ — $[\theta_0\ \omega_0]$; tspan — $[0\ 1]$

ODE from (b)

```
function zdot = gosh(t,z)
θ = z(1);  ω = z(2);
θ̇ = ω;  ω̇ = 3 cos θ;
zdot = [θ̇  ω̇]';
```

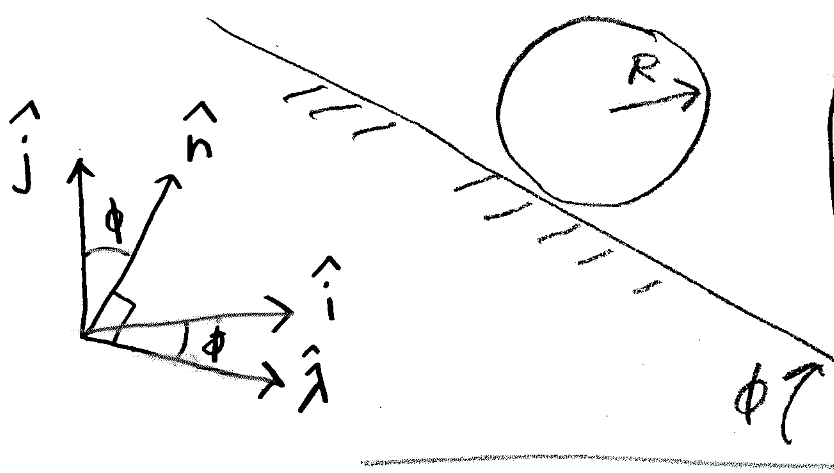
gosh.m

**3) (30 pt)** A uniform disk with mass $m$ and radius $R$ is released from rest to roll down a slope that is tipped $\phi$ from the horizontal. The local gravity constant is $g$.

**a)** (10 pt) Assume that slope is high, or friction coefficient small, so the disk slides down the slope. What is the acceleration of the center of the disk?

**b)** (10 pt) Assume the disk rolls, what is the angular acceleration of the disk?

**c)** (10 pt) If $\mu = .5$, what is the biggest angle $\phi$ for which the disk will roll and not slide.




$\downarrow g$

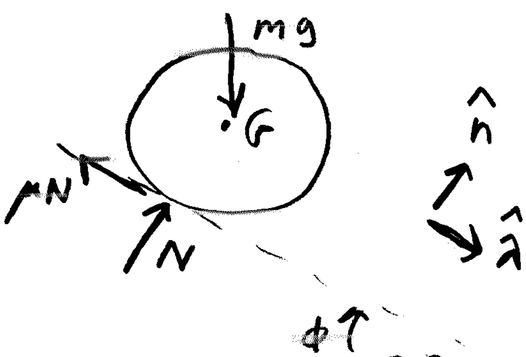
$$I^G = \int r^2 dm = \int_0^R \int_0^{2\pi} r^2 \frac{m}{\pi R^2} r\,dr\,d\theta$$

$$= \frac{2\pi m}{\pi R^2} \int_0^R r^3 dr$$

$$= mR^2/2$$

a) FBD

sliding




Kinematics

$$\underline{a}_G = a_G \hat{\lambda}$$

LMB

$$\sum \underline{F}_i = m\underline{a}_G$$

$$\{N\hat{n} - \mu N\hat{\lambda} - mg\hat{j} = m a_G \hat{\lambda}\}$$

$$\{\}\cdot\hat{n} \Rightarrow N - mg\underbrace{\hat{j}\cdot\hat{n}}_{\cos\phi} = 0$$

$$\Rightarrow N = mg\cos\phi$$

$$\{\}\cdot\hat{\lambda} \Rightarrow$$

$$-\mu N - mg\underbrace{\hat{j}\cdot\hat{\lambda}}_{-\sin\phi} = m a_G$$

$$\Rightarrow$$

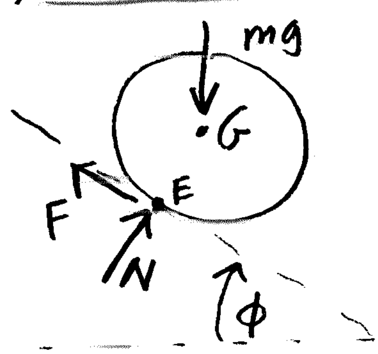
$$-\mu \cancel{m} g\cos\phi + \cancel{m} g\sin\phi = \cancel{m} a_G$$

$$a_G = g(\sin\phi - \mu\cos\phi)$$

$$\Rightarrow \boxed{\underline{a}_G = g(\sin\phi - \mu\cos\phi)\hat{\lambda}}$$

(a)

b) FBD rolling (prob 3 cont'd)




**Kinematics**

$$\underline{a}_G = -R\alpha\hat{\lambda}$$

$$R\alpha = -a_G \qquad \underline{\alpha} = \alpha\hat{k}$$

$$\underline{\text{AMB}}_{/E} \Rightarrow \underline{r}_{G/E} \times (-mg\hat{j}) = \underline{r}_{G/E} \times m\underline{a}_G + I^G\alpha\hat{k}$$

$$\Rightarrow R\hat{n} \times (-mg\hat{j}) = R\hat{n} \times (m(-R\alpha\hat{\lambda})) + I^G\alpha\hat{k}$$

$$\Rightarrow \{-Rmg\sin\phi\,\hat{k} = R^2 m\alpha\hat{k} + I^G\alpha\hat{k}\}$$

$$\{\ \}\cdot\hat{k} \qquad \Rightarrow \alpha = \frac{-Rmg\sin\phi}{\underbrace{I^G}_{mR^2/2} + mR^2} = \frac{-2Rg\sin\phi}{3R^2}$$

$$\Rightarrow \boxed{\text{ang. accel.} = \frac{-2g\sin\phi}{3R}\hat{k}} \quad (b)$$

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c) LMB

$$\sum \underline{F}_i = m\underline{a}_G$$

$$\left\{N\hat{n} - F\hat{\lambda} - mg\hat{j} = m\underbrace{a_G}_{= -\alpha R = 2g\sin\phi/3}\hat{\lambda}\right\}$$

$$\{\ \}\cdot\hat{n} \Rightarrow N - mg\cos\phi = 0$$

$$N = mg\cos\phi \quad \text{(as for sliding)}$$

$$\{\ \}\cdot\hat{\lambda} \Rightarrow -F + mg\sin\phi = 2mg\sin\phi/3$$

$$\Rightarrow F = mg\sin\phi(1 - 2/3)$$

$$= mg\sin\phi/3$$

critical slope when $\mu = \dfrac{F}{N} = \dfrac{mg\sin\phi/3}{mg\cos\phi}$

$$\Rightarrow \tan\phi = 3\underbrace{\mu}_{0.5 \text{ given}} \Rightarrow \tan\phi = 3/2 \Rightarrow \boxed{\phi = \tan^{-1}(3/2)}$$