" SOLUTIONS "

Your Name: STAFF

Your TA: "

T&AM 203 Prelim 3

Tuesday April 25, 2000 7:30 — 9:00$^+$ PM

Draft April 25, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

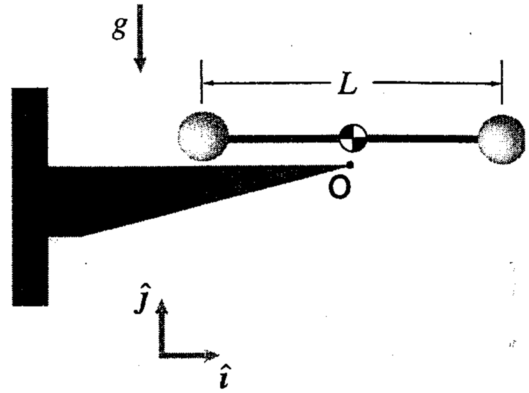
b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/35
Problem	2:	/30
Problem	3:	/35
TOTAL:		/100

1) 35 pts) Bouncing baton. Two equal point masses are connected by a massless rigid rod of length L. While horizontal, the baton falls without rotation until it reaches the speed v and the left ball strikes the rigid surface of a table. At this instant the center of the rod is just over the right edge of the table. The collision is elastic (conserves energy).



a) Immediately after impact, what are the velocity of the rod's center and the angular velocity of the rod? Answer in terms of some or all of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, v, L, g,$ and m.

b) Assuming no other interaction with the table, accurately describe —using equations if appropriate— the subsequent motion and rotation of the baton. Answer in terms of some or all of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, v, L, g,$ and m.

c) What is the minimum value of v for which the left mass will miss the table in its subsequent motion. Assume no subsequent collision of the massless rod with the surface. Answer in terms of some or all of $v, L, g,$ and m.

a) Can use $AMB_{/A}$:

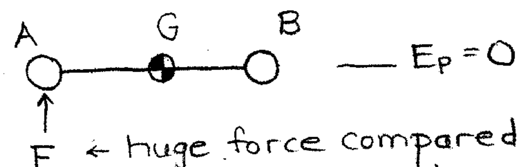
$$\underbrace{\sum \underline{M}_{/A}}_{\underline{0}} = \dot{\underline{H}}_{/A}$$

$\Rightarrow$ $\underline{H}_{/A}$ conserved

$$\underline{H}_{/A}^{①} = \sum \underline{r}_{i/A} \times m_i \underline{v}_i$$
$$= \cancel{\underline{r}_{A/A}} \times m\underline{v}_A + \underline{r}_{B/A} \times m\underline{v}_B$$
$$= L\hat{i} \times m(-v\hat{j}) = -mLv\hat{k}$$

$$\underline{H}_{/A}^{②} = \sum \underline{r}_{i/A} \times m_i \underline{v}_i$$
$$= \underline{0} + \underline{r}_{B/A} \times m\underline{v}_B$$
$$= L\hat{i} \times m(-v_B\hat{j})$$
$$= -mLv_B\hat{k}$$

FBD during collision



← huge force compared to mg ⇒ neglect mg

① just before collision
② " after "

$$\{\underline{H}_{/A}^{①} = \underline{H}_{/A}^{②}\} \cdot \hat{k}$$

$\Rightarrow$ $v_B = v$

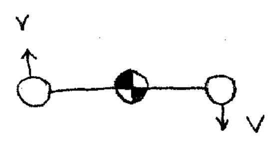
$\Rightarrow$ $\boxed{\underline{v}_B = -v\hat{j}}$

Energy conservation $\Rightarrow$ $E_{K_1} + \cancel{E_{P_1}}_{0} = E_{K_2} + \cancel{E_{P_2}}_{0}$

$$\Rightarrow \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_B^2 + \tfrac{1}{2}mv_A^2$$
$$mv^2 = \tfrac{1}{2}m(-v)^2 + \tfrac{1}{2}mv_A^2$$

$\Rightarrow$ $v_A = v$

$\Rightarrow$ $\boxed{\underline{v}_A = v\hat{j}}$



(Continue work for problem 1 here)

Use relative motion to get $\underline{v}_{CM}$, $\underline{\omega}$:

$$\underline{v}_{CM} = \underline{v}_A + \underline{v}_{CM/A} = v\hat{j} + -\omega\hat{k} \times \tfrac{L}{2}\hat{i} = (v - \tfrac{1}{2}\omega L)\hat{j} \quad (1)$$

$$\underline{v}_{CM} = \underline{v}_B + \underline{v}_{CM/B} = -v\hat{j} + -\omega\hat{k} \times -\tfrac{L}{2}\hat{i} = (-v - \tfrac{1}{2}\omega L)\hat{j} \quad (2)$$

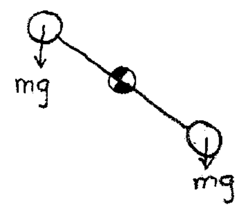
$\{(1) = (2)\} \cdot \hat{j} \quad \Rightarrow \quad v - \frac{1}{2}\omega L = -v - \frac{1}{2}\omega L$

$\Rightarrow \quad \omega = \frac{2v}{L}, \quad \boxed{\underline{\omega} = -\frac{2v}{L}\hat{k}}$

$\Rightarrow \quad \boxed{\underline{v}_{CM} = 0}$

b) LMB: $\Sigma \underline{F} = (m+m)\underline{a}_{CM}$

FBD after collision



$-2mg\hat{j} = 2m\underline{a}_{CM}$

$\Rightarrow \underline{a}_{CM} = -g\hat{j}$

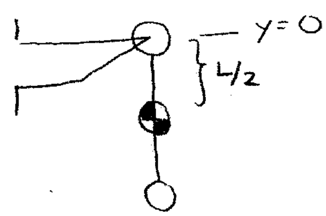
$\Rightarrow$ The center of mass falls straight down with acceleration of magnitude g.

AMB/CM : $\underbrace{\Sigma \underline{M}_{/CM}}_{\underline{0}} = \dot{\underline{H}}_{/CM}$ gravity forces cancel each other

$\Rightarrow \underline{H}_{/CM}$ conserved $\Rightarrow$ $\underline{\omega}$ constant

$\Rightarrow$ The baton rotates at rate $\underline{\omega} = -\frac{2v}{L}\hat{k}$.

c) For the minimum v, the baton should rotate by 90° when the center of mass has fallen a distance $L/2$.



$\omega = \frac{2v}{L}$ = constant

$\Rightarrow \quad \omega = \frac{\Delta\theta}{\Delta t} \leftarrow$ find this

$$\left.\begin{aligned} \underline{a}_{CM} &= -g\hat{j} \\ \Rightarrow \underline{v}_{CM} &= -gt\hat{j} \\ \underline{r}_{CM} &= -\tfrac{1}{2}gt^2\hat{j} \end{aligned}\right\}$$

When $\underline{r}_{CM} = -\frac{L}{2}\hat{j}$

$\frac{L}{2} = \frac{1}{2}g(\Delta t)^2$

$\Rightarrow \Delta t = \sqrt{L/g}$

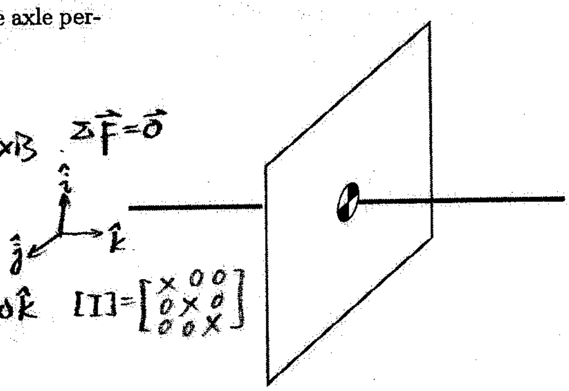
$\therefore \quad \omega = \frac{2v}{L} = \frac{\pi/2}{\sqrt{L/g}}$

$\therefore \quad \boxed{v = \frac{\pi}{4}\sqrt{gL}}$

2)(30 pts) Static and Dynamic Balance A series of bodies, each of uniform density and each with total mass m, rotate at a constant angular speed ω about a fixed horizontal axis. Ignore gravity. For each body state whether the body is (i) *statically* balanced and whether it is (ii) *dynamically* balanced. Give clear arguments using words or equations to support your claims.

(iii, iv) For each body you must *either* (a) add one point mass m or (b) add two point masses each of mass $m/2$ (your choice) that *maintain* static and dynamic balance if they are balanced, or that *make* the bodies statically and dynamically balanced. Justify your placement with words and/or equations. The masses need not be added to the bodies, but could be attached off the bodies by structures with negligible mass. [Hint: none of the placements are unique. You may draw a side view if that helps clarify your placement.]

a) A rectangular plate (height h, length ℓ) mounted with the axle perpendicular to the plate and through its center.



i) Statically balanced? (yes/no) Why?

yes. center of the mass is on the axis. $\Sigma\vec{F}=\vec{0}$

ii) Dynamically balanced? (yes/no) Why?

yes. Look at $\vec{H}$, since $\vec{\omega}$ = const

$\therefore \dot{\vec{H}}=\vec{\omega}\times\vec{H}$, $\vec{H}=[I]\cdot\vec{\omega}$ $\quad \vec{\omega}=\omega\hat{k}$ $\quad [I]=\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$

Since $\vec{\omega} \parallel [I]\cdot\vec{\omega}$ $\quad \therefore \dot{\vec{H}}=\vec{0}=\Sigma\vec{M}$

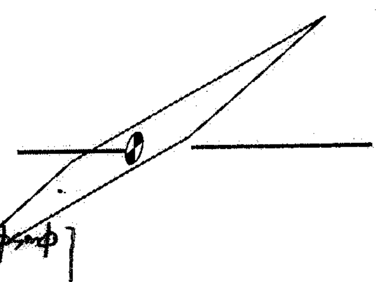
iii) Are you adding one mass or two?

One

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.

Center of the rectangle , it won't affect the previous answer since for this point mass, $[I]=[0]$.

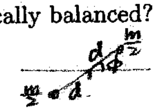
b) The same plate as in (a) above but mounted at an angle $\phi \neq \pi/2$ from the shaft.



i) Statically balanced? (yes/no) Why?

Yes. mass center is on the axis. $\Sigma\vec{F}=\vec{0}$

ii) Dynamically balanced? (yes/no) Why?

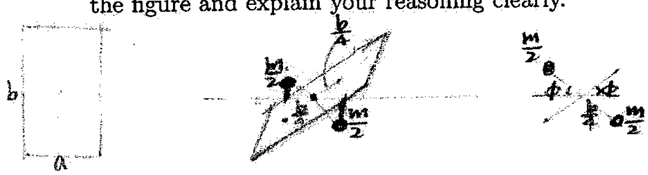
No.  $\vec{\omega}=\omega\hat{k}$. $[I]=\begin{bmatrix} md^2\cos^2\phi & 0 & md^2\cos\phi\sin\phi \\ 0 & md^2 & 0 \\ md^2\cos\phi\sin\phi & 0 & md^2\sin^2\phi \end{bmatrix}$

$[I]\cdot\vec{\omega}=md^2\sin\phi(\cos\phi\hat{i}+\sin\phi\hat{k}) \nparallel \vec{\omega}$. $\therefore \Sigma\vec{M}\neq\vec{0}$

iii) Are you adding one mass or two?

Two.

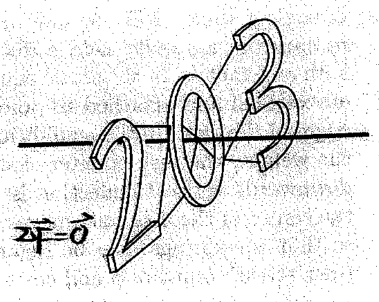
iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



it will make the whole system symmetric. then $I_{xy}=I_{yz}=I_{xz}=0$!

So $[I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \Sigma\vec{M}=\vec{0}$

c) The numerals '203' cut out of a plate and connecteed by massless rods. Each letter has mass $m/3$ and the three center-of-mass points of the individual letters are colinear and equally spaced. The shaft goes through the center of the '0' and is perpendicular to the plane of the letters.



i) Statically balanced? (yes/no) Why?

yes. center of the mass is on the axis. $\Sigma \vec{F} = \vec{0}$

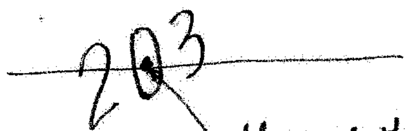
ii) Dynamically balanced? (yes/no) Why?

yes, since $\hat{k}$ is the eigenvector of the [I], and it's perpendicular to the other two eigenvectors which lie in the x-y plane. $\therefore [I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \dot{\vec{H}} = \vec{\omega} \times ([I]\cdot\vec{\omega}) = \vec{0} = \Sigma\vec{M}$

iii) Are you adding one mass or two?

One.

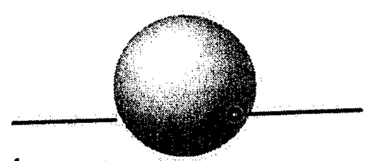
iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



add m at the center of 'O'.

it won't affect the [I].

d) A sphere with radius R where the shaft passes a distance $d < R$ from the center.



i) Statically balanced? (yes/no) Why?

No, center of the mass isn't on the axis.
$\therefore$ need extra force to provide centripetal acceleration.

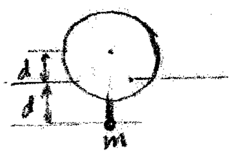
ii) Dynamically balanced? (yes/no) Why?

No, center of the mass is not on the axis.
$I_{xz} \neq 0$. $\therefore [I]\cdot\vec{\omega} \nparallel \vec{\omega}$. $\Sigma\vec{M} \neq \vec{0}$

iii) Are you adding one mass or two?

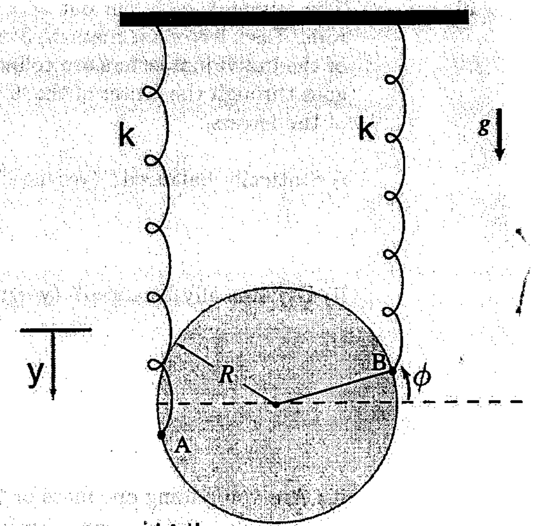
One.

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



After adding this point mass, $I_{xz} = 0$.
and $I_{yz} = I_{xy} = 0$
$\therefore [I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \Sigma\vec{M} = \vec{0}$

3)(35 pts) **Hanging disk, 2-D.** A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k. In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating thier stretch ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$).



a) Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of $\phi, \dot{\phi}, y, \dot{y}, k, m, R$, and g.

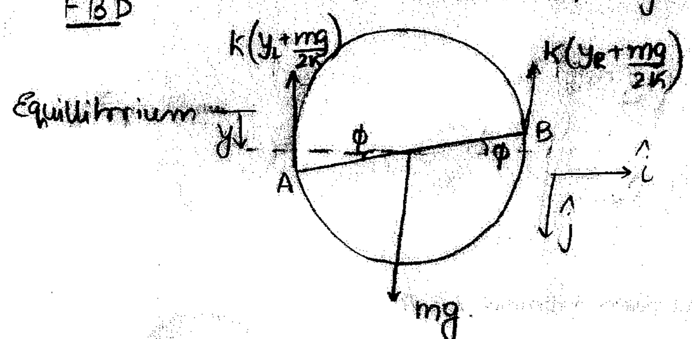
b) Find the natural frequencies of vibration in terms of some or all of k, m, R, and g.

Ans :- Note that 'y' is measured from the equillibrium. (and represents the displacement of the CM).

(a)

So at equillibrium springs are stretched by $l_0 = \frac{mg}{2K}$

FBD



$y_R = y - R\phi$

$y_L = y + R\phi$

LMB in '$\hat{j}$' direction

$$m\ddot{y} = -k\left(y - R\phi + \frac{mg}{2k}\right) - k\left(y + R\phi + \frac{mg}{2K}\right) + mg$$

$$\Rightarrow m\ddot{y} = -ky + kR\cancel{\phi} - \cancel{mg} - ky - k\cancel{R\phi} + \cancel{mg}$$

$$\Rightarrow \boxed{\ddot{y} + \frac{2K}{m}y = 0} \quad (1) \quad \Rightarrow \ddot{y} = -\frac{2K}{m}y$$

AMB about CM :

$$k\left(y_R + \frac{mg}{2K}\right)\cdot R - k\left(y_L + \frac{mg}{2K}\right)\cdot R = I\ddot{\phi}$$

(Continue work for problem 3 here)

$$\Rightarrow K(y-R\phi)R - K(y+R\phi)R = I\ddot{\phi}$$

$$\Rightarrow \ddot{\phi} + \frac{2KR^2}{I}\phi = 0$$

$$I \text{ for a disk} = \frac{mR^2}{2}.$$

$$\Rightarrow \ddot{\phi} + \frac{2K\cancel{R^2}\cdot 2}{m\cancel{R^2}}\phi = 0$$

$$\Rightarrow \boxed{\ddot{\phi} + \frac{4K}{m}\phi = 0} \quad (2)$$

$$\ddot{\phi} = -\frac{4K}{m}\phi.$$

b) Since the system has 2 dof it has two natural frequencies.

which from equations (1) & (2) are

$$\boxed{\omega_1 = \sqrt{\frac{2K}{m}}, \quad \omega_2 = \sqrt{\frac{4K}{m}}}$$

The modes of vibration look like