Your Name: ______________________

T&AM 203 Final Exam

Tuesday Dec 12, 2000 3:00 — 5:30 PM

Draft March 20, 2007

5 problems, 100 points, and 150 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ `= 18`" instead of, say, "`theta7dot = 18`". Pick generic (not special) numerical values for constants not defined in the problem statement.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/20
Problem	2:	/20
Problem	3:	/20
Problem	4:	/20
Problem	5:	/20
TOTAL:		**/100**

1)(20 pts) Spring mass.

a) (5 pts) Find the equation of motion, a differential equation, for the variable x in the system above. Your differential equation can contain x, its time derivatives, m, c, k, and ℓ_0 (Please read item (b) on the cover page.)

b) (5 pts) Assume $c = 0$, $x(t = 0) = d$, and $\dot{x}(t = 0) = 0$. What is $\dot{x}$ at time t (answer in terms of some or all of m, k, ℓ_0, d, and t.

c) (5 pts) Assume relatively large c ($c^2 > 4km$), $x(t = 0) = d$, and $\dot{x}(t = 0) = 0$. Find $x(t)$ (or write code that would find $x(t)$).

d) (5 pts) Whether or not you have succeeded at part (c) above, make a clear plot of x *vs* t for the conditions in part (c) above.

(work for problem 1, cont'd.)

2)(20 pts) Car on a ramp. A junior level engineering design course asks students to build a cart (mass $= m_c$) that rolls down a ramp with angle θ. A small weight (mass $m_w \ll m_c$) is placed on top of the cart on a surface tipped with respect to the cart (angle ϕ). Assume the small mass does not slide. Assume massless wheels with frictionless bearings

a) (5 pts) Find the acceleration of the cart. Answer in terms of some or all of $m_c, g, \hat{\mathbf{i}}, \theta$ and $\hat{\mathbf{j}}$. (In accordance with the directions on the front cover you may use other convenient coordinates if you like.).

b) (10 pts) What coefficient of friction μ is required (the smallest that will work) to keep the small mass from sliding as the cart rolls down the slope? Answer in terms of some or all of m_c, m_w, g, θ, and ϕ.

c) (5 pts) What angle ϕ will allow a small mass to ride on the cart with the smallest coefficient of friction? Answer in terms of some or all of m_c, m_w, g, and θ. (You get full credit for a correct answer to this question even if the answer to (b) is incorrect. Conversely, an answer based on incorrect work in part (b) is incorrect.)

(Work for problem 2, cont'd.)

3)(20 pts) A swinging disk. A uniform disk of mass m and radius R is hinged at one end and swings in its plane from a hinge on its circumference.

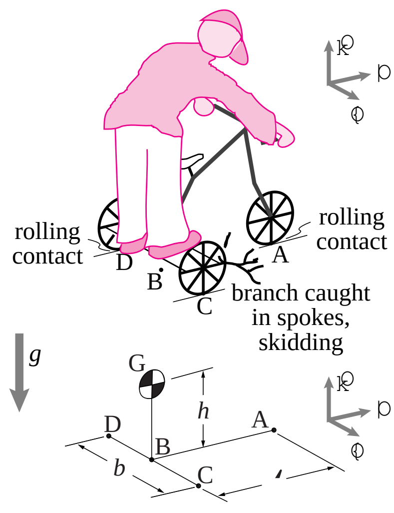
a) (10 pts) Find a differential equation that describes its motion. Describe the motion with an angle θ that is zero when the disk is hanging straight down. (Your equation should have in it some or all of θ, its time derivatives, m, g, and R.

b) (5 pts) What is the period t_p of small oscillations? Answer in terms of some or all of m, R and g.

c) (5 pts) If instead the disk was swinging in the perpendicular direction (with its center moving perpendicular to the plane of the disk) would the frequency of oscillation be higher, lower, or the same? (Correct guess earns one point.)

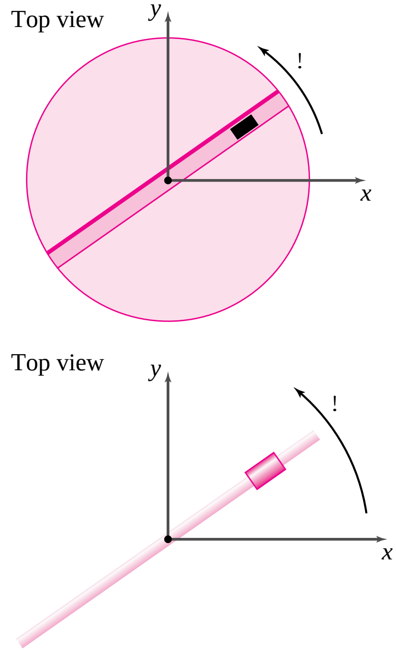
(work for problem 3, cont'd.)

4)(20 pts) Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\mathbf{j}}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch. *Find the acceleration of the tricycle* (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$).
[Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



(Work for problem 4, cont'd.)

5)(20 pts) Mass on a lightly greased slotted turntable or spinning uniform rod. Assume that the rod/turntable in the figure is massless and also free to rotate. Assume that at $t = 0$, the angular velocity of the rod/turntable is $1\,\mathrm{rad/s}$, that the radius of the bead is one meter, and that the radial velocity of the bead, dR/dt, is zero. The bead is free to slide on the rod. Where is the bead at $t = 5\,\mathrm{sec}$?



(Work for problem 5, cont'd.)

Your Name: ____________________

T&AM 203 Prelim 1

Tuesday Sept 26, 2000 7:30 — 9:00$^+$ PM

Draft September 26, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Two pages of formulas, from the front of the text, and a blank page for tentative scrap work are provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

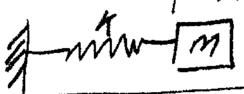
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

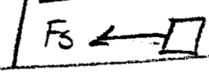
Problem	1:	____/40
Problem	2:	____/30
Problem	3:	____/30
TOTAL:		____/100

1) (40 pts)

1a) (15 pts) A mass m is connected to a spring k and launched from its static equilibrium position at a speed of v_0. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? (Answer in terms of some or all of m, k, and v_0.) [Neglect gravity and friction.] $\mapsto$ X=0 AT EQUILIBRIUM



F.B.D



$F_S = kx$

$\underline{\mathcal{L}}:\; m\ddot{x} = -kx \Rightarrow \ddot{x} + \frac{k}{m}x = 0$

I.C.'s $x(0) = 0$, $\dot{x}(0) = v_0$

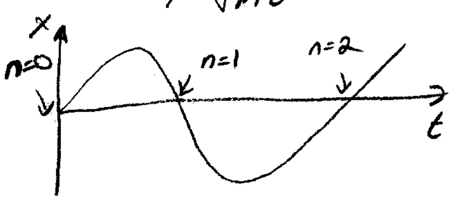
GENERAL SOLUTION $\Rightarrow x(t) = A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$

$\dot{x}(t) = -\sqrt{\tfrac{k}{m}}A\sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + \sqrt{\tfrac{k}{m}}B\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$

$\left.\begin{array}{l} x(0) = A = 0 \Rightarrow A = 0 \\ \dot{x}(0) = \sqrt{\tfrac{k}{m}}B = v_0 \Rightarrow B = v_0\sqrt{\tfrac{m}{k}} \end{array}\right\} \Rightarrow x(t) = v_0\sqrt{\tfrac{m}{k}}\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$

NEED TO FIND WHEN $x(t) = 0 \Rightarrow v_0\sqrt{\tfrac{m}{k}}\sin\left(\sqrt{\tfrac{k}{m}}\,t\right) = 0 \Rightarrow \sin\left(\sqrt{\tfrac{k}{m}}\,t\right) = 0$

$\Rightarrow \sqrt{\tfrac{k}{m}}\,t = n\pi$

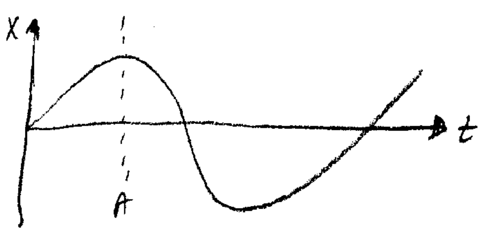


$\sqrt{\tfrac{k}{m}}\,t = 2\pi$, $\boxed{t = 2\pi\sqrt{\tfrac{m}{k}}}$

1b) (10 pts)

For the mass above, how far does the mass move from the launch position before it first reverses its velocity?

From 1a), $x(t) = v_0\sqrt{\tfrac{m}{k}}\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$



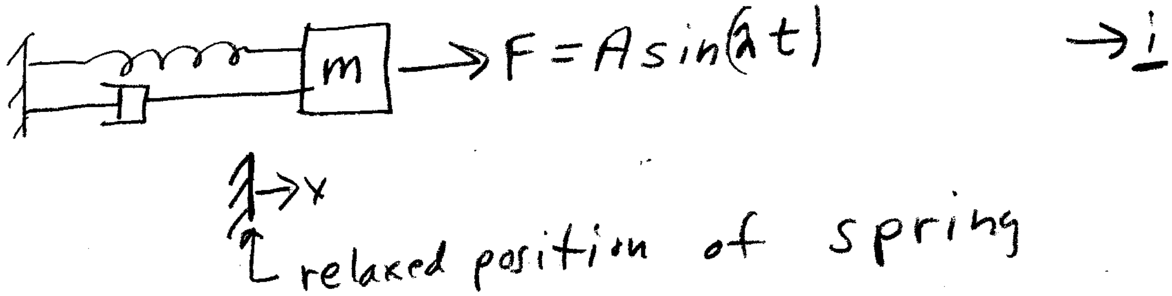
REVERSES VELOCITY AT $t = A$, ITS POSITION CORRESPONDS TO THE <u>AMPLITUDE</u> OF THE MOTION

THE AMPLITUDE OF $x(t)$ IS $v_0\sqrt{m/k}$,

SO $\boxed{d = v_0\sqrt{\tfrac{m}{k}}}$

1c) (15 pts)

A mass $m = 1\,\mathrm{kg}$ is held in place by a spring $k = 1\,\mathrm{N/m}$ and dashpot $c = 1\,\mathrm{N/(m/s)}$. An oscillating force is applied of $F = A\sin(\lambda t)$, with $A = 1\,\mathrm{N}$ and $\lambda = 1/\mathrm{s}$. After any initial transients have died down, how far does the mass go back and forth (the distance from one extreme to the other)?



FBD:

LMB: $\{\sum \underline{F} = \dot{\underline{L}}\}$

$$-T_s - T_D + F = m\ddot{x}$$

$\{\}\cdot\underline{i} \Rightarrow$

$$-kx - c\dot{x} + A\sin\lambda t = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = A\sin\lambda t$$

$$\Rightarrow \ddot{x} + \dot{x} + x = \sin(t) \quad \text{[in consistant m, kg, s units]}$$

Guess steady soln. of form: $x = B\cos t + C\sin t$

why? Because it works

$$\ddot{(B\cos t + C\sin t)} + \dot{(B\cos t + C\sin t)} + (B\cos t + C\sin t) = \sin t$$

$$(-B\cos t - C\sin t) + (-B\sin t + C\cos t) + (B\cos t + C\sin t) = \sin t$$

Collect sine & cosine terms [eg, $\int_0^{2\pi}\{\ \}\cos(t)\,dt$ plucks out cosine]

$-B + C + B = 0$ (cos terms)

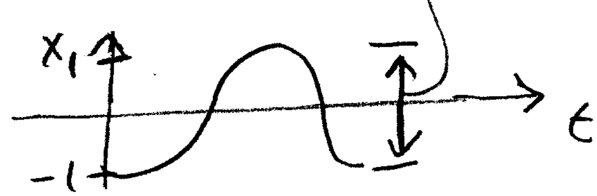
$\Rightarrow C = 0$

$-C - B + C = 1$ (sin terms)

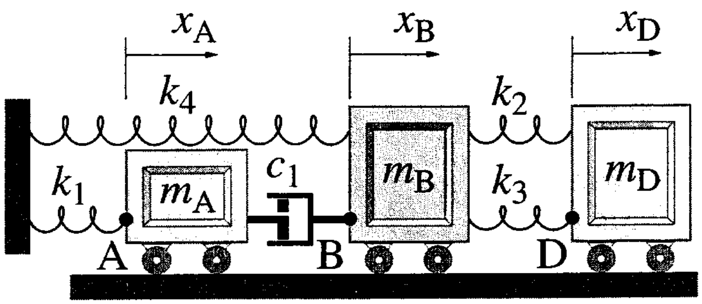
$\Rightarrow B = -1$

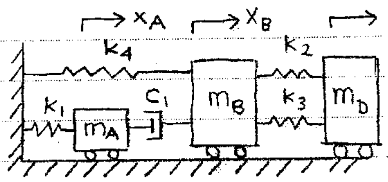
$\Rightarrow x = -\cos(t)$

$$\boxed{X_{dist} = 2\,\mathrm{m}}$$

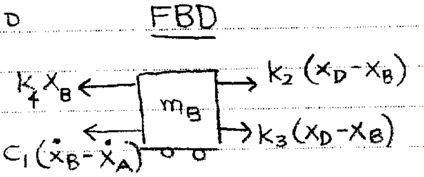


2)(30 pts) A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given k_1, k_2, k_3, k_4, c_1, m_A, m_B, m_D, x_A, x_B, x_D, $\dot{x}_A$, $\dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\underline{\mathbf{a}}_B = \ddot{x}_B\,\hat{\mathbf{i}}$.





FBD



LMB : $\{\Sigma \underset{\sim}{F} = m_B \underset{\sim}{a}_B\} \cdot \hat{i}$

$$\Rightarrow -k_4 x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2+k_3)(x_D - x_B) = m_B \ddot{x}_B$$

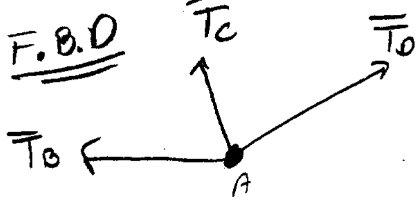
$$\therefore \ddot{x}_B = \frac{1}{m_B}\left[c_1 \dot{x}_A - c_1 \dot{x}_B - (k_2+k_3+k_4)x_B + (k_2+k_3)x_D\right]$$

$$\underset{\sim}{a}_B = \ddot{x}_B\,\hat{i}$$

3) (30 pts)

In three-dimensional space with no gravity a particle with $m = 3\,\mathrm{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\underline{\mathbf{a}} = (a\,\hat{\mathbf{i}})\,\mathrm{m/s^2}$ where a is not yet known. The tension in AB is 4 N The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the code to find a. The last line should read `a = ...` with `a` being assigned to the acceleration in the $\hat{\mathbf{i}}$ direction.

F.B.D



$\Sigma = \vec{T}_B + \vec{T}_C + \vec{T}_D = m\vec{a} = 3a\vec{i}$ m/s

$|\vec{T}_B| = T_B = 4N$

DEFINE UNIT VECTORS $\vec{T}_B = T_B\vec{\lambda}_B$, $\vec{T}_C = T_C\vec{\lambda}_C$, $\vec{T}_D = T_D\vec{\lambda}_D$

DEFINE $\vec{b}$ S.T. $m\vec{a} = a\vec{b}$ $\{\vec{b} = (m\hat{i})\}$

THEN $T_B\vec{\lambda}_B + T_C\vec{\lambda}_C + T_D\vec{\lambda}_D = a\vec{b}$ → NOW PUT ALL UNKNOWNS ON LEFT KNOWNS ON RIGHT

$\Rightarrow T_C\vec{\lambda}_C + T_D\vec{\lambda}_D - a\vec{b} = -T_B\vec{\lambda}_B$

$\Downarrow$

$$\left[\vec{\lambda}_C \,\middle|\, \vec{\lambda}_D \,\middle|\, -\hat{b}\right]\begin{bmatrix} T_C \\ T_D \\ a \end{bmatrix} = -T_B\left[\vec{\lambda}_B\right]$$

```
% a MATLAB script file to find 3 tensions
m   = 3;
a   = [ 1 2 3]';

rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';

uAB = rAB/norm(rAB); % norm gives vector magnitude
% You write the code below (however many lines you need).
% Don't copy any of the numbers above.
% Don't do any arithmetic on the side.
uAC= rAC/norm(rAC);
uAD= rAD/norm(rAD);
b= [m 0 0]';
Tb= 4;
r= -Tb*uAB;
A= [uAC uAD -b];
res = A\r;
a= res(3);
```

"Solutions"

Your Name: ANDY RUINA

T&AM 203 Prelim 2

Tuesday Oct 24, 2000 7:30 — 9:00$^+$ PM

Draft October 16, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
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- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

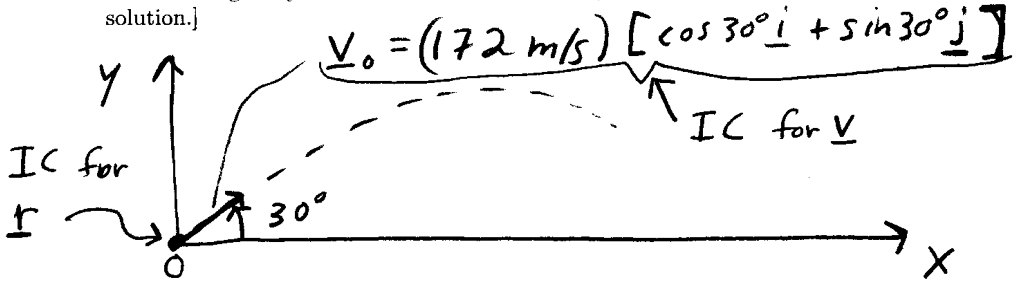
Problem	1:	/40
Problem	2:	/30
Problem	3:	/30
TOTAL:		/100

1)(40 pts) **Projectile motion.** Someone in the mideast shot a projectile at someone else. The basic facts:

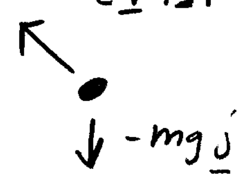
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag proportional to cv^2 with $c = .01$ kg / m.
Gravity $g = 10$ m/s.

a) (25 pts) Write MATLAB code to find the height at $t = 1$ s. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, num setup, find height at 1 s].

b) (15 pts) Estimate the height at $t = 1$ s using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the first second of your solution.]



FBD:



$V = |\underline{V}| = (\dot{x}^2 + \dot{y}^2)^{1/2}$

LMB: $\sum \underline{F} = m\underline{a}$

$$\{-c\underline{V}V - mg\underline{j} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j})\}$$

$$\left.\begin{array}{ll} \{\ \}\cdot\underline{i} \Rightarrow \dot{V}_x = -cV_xV/m, & \dot{x} = V_x \\ \{\ \}\cdot\underline{j} \Rightarrow \dot{V}_y = -cV_yV/m - g, & \dot{y} = V_y \end{array}\right\} \text{ODEs}$$

driver

```
x0=0; y0=0;
vx0 = 172*cos(pi*30/180);
vy0 = 172*sin(pi*30/180);
z0=[x0 y0 vx0 vy0]';
tspan= [0 1];
[t z]= ode45('sadam', tspan, z0);
height= z(end,2)
```

(gives ≈ 46.5 m)

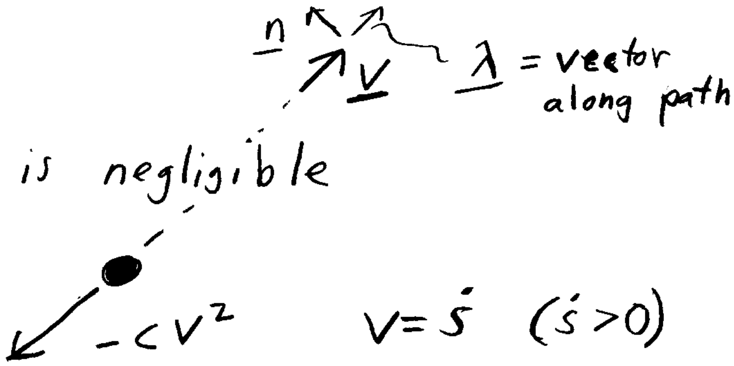
in file sadam.m

```
function zdot= sadam(t,z)
c=.01; m=1; g=10;
x= z(1); y= z(2);
vx= z(3); vy= z(4);
xdot= vx; ydot= vy;
v=(vx^2+vy^2)^.5;
vxdot= -c*vx*v/m;
vydot= -c*vx*v/m - g;
zdot=[xdot ydot...
      vxdot vydot]';
```

(work for problem 1, cont'd.)

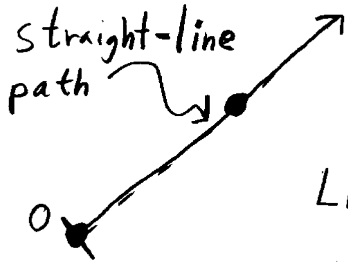
b) Assume gravity is negligible

$\Rightarrow$ FBD:



$V = \dot{s} \quad (\dot{s} > 0)$

No force in $\underline{n}$ dir. ($\perp$ to path) $\Rightarrow$ straight line motion



s = arclength in dir. of motion

LMB: $m\dot{V} = -cV^2$ ①

First solve ①: $\frac{dV}{dt} = \frac{-c}{m}V^2 \Rightarrow \frac{dV}{V^2} = \frac{-c}{m}dt$

$\Rightarrow \quad +V^{-1} = \frac{+c}{m}t + C$

IC: $V(t=0) = V_0 \Rightarrow C = \frac{1}{V_0} \Rightarrow V = \frac{1}{\frac{1}{V_0} + \frac{c}{m}t} = V_0 \frac{1}{1 + \frac{cV_0}{m}t}$

$\Rightarrow \quad \dot{s} = V_0 \frac{1}{1 + \frac{cV_0}{m}t} \Rightarrow ds = V_0 \frac{dt}{1 + \frac{cV_0}{m}t}$

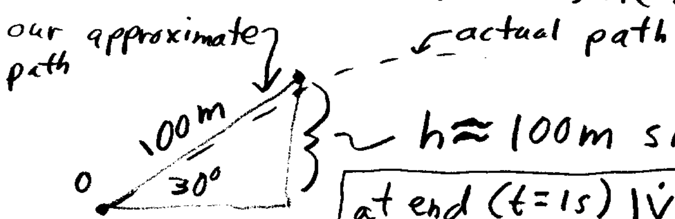
$\Rightarrow \quad s = \frac{V_0 m}{cV_0} \ln\left(1 + \frac{cV_0}{m}t\right) + C$

IC: $s(t=0) = 0 \Rightarrow C = 0 \Rightarrow \boxed{s = \frac{m}{c}\ln\left(1 + \frac{cV_0}{m}t\right)}$

Plug in #s: $s = \frac{1\,kg}{.01\,kg/m} \ln\left(1 + \frac{(.01\,kg/m)(172\,m/s)}{1\,kg}(1s)\right)$

$= 100m \cdot \ln(1 + 1.72) = 100m \ln(2.72)$

$\approx 100m \cdot \ln(e) = 100m$



"exact" Matlab ans. is 46.5m

$h \approx 100m \sin(30°) = \boxed{50\,m}$

at end ($t = 1s$) $|\dot{V}| = cV^2 = .01\left(\frac{172}{2.72}\right)^2 = \left(\frac{1.72}{2.72}\right)^2 172$
(in m/s²) $\quad |\dot{V}| >> 10 = g$ (no big error to neglect g)

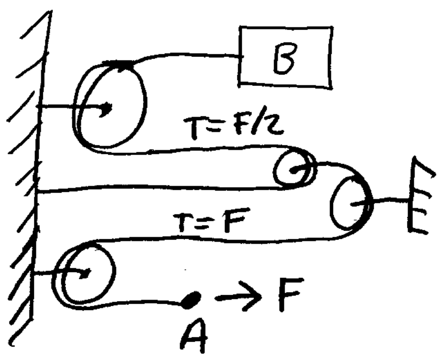
2)(30 pts) **Design a pulley system.** You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\underline{\mathbf{F}} = F\,\hat{\mathbf{i}}$ pulling it to the right. Mass B has mass m_B. You can connect the point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). You must design the system so that the mass B accelerates to the left with $\frac{F}{2m_B}$ (i.e., $\underline{\mathbf{a}}_B = -\frac{F}{2m_B}\,\hat{\mathbf{i}}$).

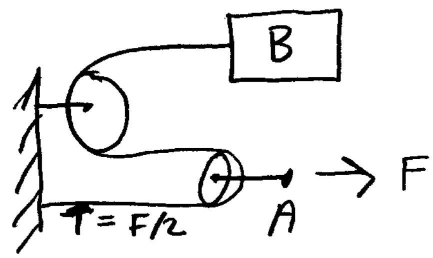
a) (25 pts) Draw the system clearly. Justify your answer with enough words or equations so a reasonable person, say a grader, can tell that you understand your solution.

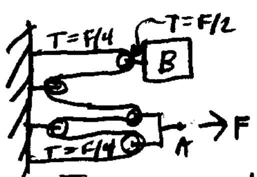
b) (5 pts) Find the acceleration of point A.

$\rightarrow \underline{i}$

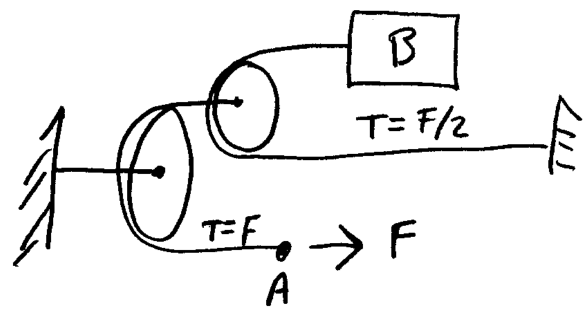
a) <u>Some solns:</u>







etc...



In all cases m_B pulled to left with tension = F/2 $\Rightarrow$ $a_B = -\frac{F}{2m_B}\,\underline{i}$

b) Power balance $\Rightarrow$ $T_B(-V_B) = T_A V_A$

$\boxed{V_B = \text{vel. of B to the right}}$

$$\left\{\frac{F}{2}(-V_B) = F\,V_A\right\}$$

$\frac{d}{dt}(*) \Rightarrow$

$$\frac{F}{2}(-a_B) = F\,a_A$$

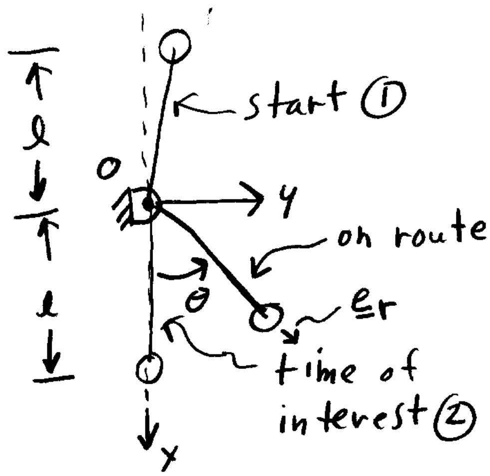
$$a_A = \frac{-a_B}{2} = -\frac{(-F/2m_B)}{2}$$

$$\boxed{a_A = \frac{F}{4m_B}}$$

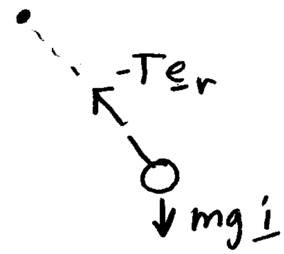
3)(30 pts) Tension in pendulum. 2D. A simple pendulum consists of a point mass m connected by a rigid massless rod with length ℓ to a frictionless hinge at O. The only applied force is from gravity. It is released from a vertical orientation with the mass directly *above* the hinge. It is pushed very slightly to the right (with a velocity that you can assume is arbitrarily small) and thus slowly at first falls, then quickly swings through the vertically down orientation and then back up on the left side.

At the instant when the mass passes through the vertically down position (mass directly below hinge) what is the tension in the rod? (i.e., find T in terms of m, ℓ and g).

If you choose a MATLAB solution instead of pencil and paper (not required, just an option) use $m = 3\,\text{kg}$, $\ell = 2\,\text{m}$ and $g = 10\text{m/s}^2$



FBD:



[just like example in lecture]

$$\underbrace{\Delta E_K}_{\text{increase in kin. energy from state ① to state ②}} = W$$

work of gravity force (tension does not work because $T\underline{e}_r \perp \underline{v} = \dot\theta r \underline{e}_\theta$)

$$\Rightarrow \underbrace{\tfrac{1}{2} m v_2^2}_{\text{K.E. in state 2}} = 2mg\ell$$

$$\boxed{v_2^2/\ell = 4g} \; *$$

Instead of using energy one could do this:

$$\ddot\theta + (g/\ell)\sin\theta = 0$$
$$\ddot\theta\dot\theta + (g/\ell)\dot\theta\sin\theta = 0$$
$$\frac{d}{dt}(\dot\theta^2/2) + \frac{g}{\ell}\frac{d}{dt}(-\cos\theta) = 0$$
$$\dot\theta^2/2 - \frac{g}{\ell}\cos\theta = \text{Const}$$
$$\left.\begin{matrix}\text{at } \theta = \pi \\ \dot\theta = 0\end{matrix}\right] \Rightarrow \dot\theta^2/2 - (g/\ell)\cos\theta = g/\ell$$

LMB in state ②:

$$\Sigma \underline{F} = m\underline{a}$$

$$\left\{-T\underline{i} + mg\underline{i} = m\left[\frac{v_2^2}{\ell}(-\underline{i}) + \ddot\theta \ell \underline{j}\right]\right\}$$

$$\{\ \}\cdot\underline{i} \Rightarrow T = mg + \frac{m v_2^2}{\ell} = mg + 4mg = \boxed{T = 5mg}$$

Tension is 5 times the weight

(Work for problem 3, cont'd.)

Alternative Matlab soln. to 3

(Use FBD from before)

AMB/$_O$: $\sum \underline{M}_{/O} = \dot{\underline{H}}_{/O} \Rightarrow \ell \underline{e}_r \times (mg\underline{i}) = \ell \underline{e}_r \times m[\ell\ddot{\theta}\underline{e}_\theta - \dot{\theta}^2 \ell \underline{e}_r]$

$\Rightarrow -\ell mg \sin\theta = \ell^2 m \ddot{\theta}$

$\Rightarrow \ddot{\theta} = -(g/\ell) \sin\theta$

$\Rightarrow$ 2) $\dot{\omega} = -(g/\ell) \sin\theta$
1) $\dot{\theta} = \omega$
2 first order ODEs

```
m= 3; l = 2; g=10;
theta0 = -pi;
omega0 = -.001;
z0 =[theta0   omega0]';
tspan= [0:.001:10]; % long enough, lots of pts.
[t z]= ode23('pendrhs', tspan, z0);
omegamax = max(z(:,2));
Tmax = m*(g + l*omegamax^2)
```

solve ODEs for long enough time

max $\dot{\omega}$ will be near bottom

calculate tension

pendrhs.m

```
function  zdot = pendrhs (t, z)
m=3; l=2; g=10
theta= z(1); omega= z(2);
thetadot = omega;
omegadot = -(g/l)*sin(theta);
zdot=[thetadot  omegadot]';
```

pend ODEs.

[This gives an answer of 149.7, close to 5·3·10]

"SOLUTIONS" Your Name: ANDY RUINA

T&AM 203 Final Exam

Friday December 17, 2004

Draft December 17, 2004

5 problems, 150 minutes (no extra time).

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors. If you cannot answer a problem with pencil and paper, you can get partial credit for a good Matlab solution.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

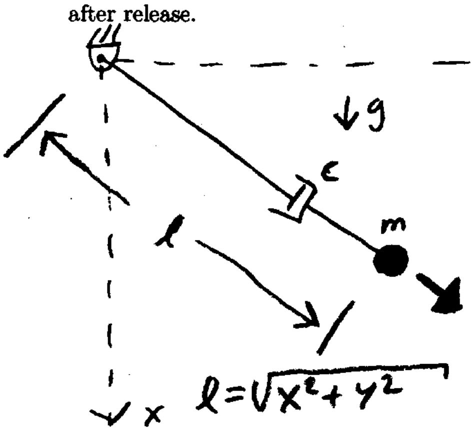
d) Even if not asked for, you can get partial credit by showing a Matlab solution to a problem you can't solve with pencil and paper.

Problem	1:	/20
Problem	2:	/20
Problem	3:	/20
Problem	4:	/20
Problem	5:	/20
TOTAL:		**/100**

1) **(20 pt)** In the almost new sport of spongy jumping a spring is replaced by a dashpot c. Assume $m = 3\,\text{kg}$, $g = 10\,\text{m/s}^2$, and $c = 7\,\text{kg/s}$. The mass is released from rest at $x = 2\,\text{m}$, $y = 3\,\text{m}$.

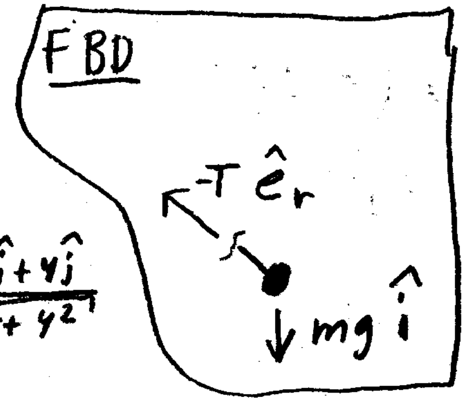
a) (15 points) What are the equations of motion for this system (differential equations involving x and y and their derivatives)? (For this part of the problem please use m, c and g rather than their numerical values.

b) (5 points) *This part will only be graded if part (a) is almost entirely correct.* Write Matlab code that would give the arc-length of the center of mass trajectory over the first 5 seconds after release.



$\ell = \sqrt{x^2+y^2}$

$\hat{e}_r = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$



LMB: $\sum \underline{F}_i = m\underline{a} \Rightarrow -T\hat{e}_r + mg\hat{j} = m\underline{a}$, $\underline{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

$T = c\dot{\ell} = c\left(\dfrac{\partial \ell}{\partial x}\dot{x} + \dfrac{\partial \ell}{\partial y}\dot{y}\right)$

$= c\dfrac{1}{\sqrt{x^2+y^2}}(x\dot{x} + y\dot{y})$

$$\Rightarrow \left\{\left(-c\frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}\right)\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} + mg\hat{i} = (\ddot{x}\hat{i} + \ddot{y}\hat{j})m\right\}$$

$$\{\ \}\cdot\hat{i} \Rightarrow \boxed{\ddot{x} = \frac{-cx}{m}\frac{(x\dot{x}+y\dot{y})}{(x^2+y^2)} + g}$$

$$\{\ \}\cdot\hat{j} \Rightarrow \boxed{\ddot{y} = \frac{-cy}{m}\frac{(x\dot{x}+y\dot{y})}{(x^2+y^2)}}$$

$\dot{s} = |\underline{v}|$

$\dot{V}_x = \ddot{x}$ (from above)

$\dot{V}_y = \ddot{y}$ (from above)

$\dot{x} = V_x$

$\dot{y} = V_y$

$z_1 = x$

$z_2 = y$

$z_3 = V_x$

$z_4 = V_y$

$z_5 = s$

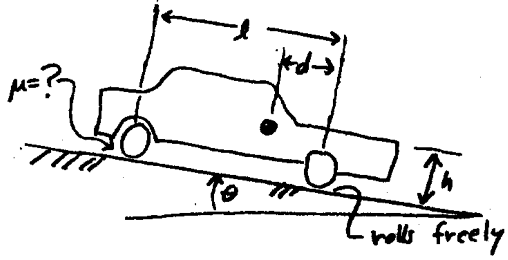
```
      x0  y0   v0    s0
Z0 = [2   3   0 0   0]'.
tspan= [0  5];

[t z] = ode23('spongy', tspan, z0);
arclength= z(end,5);


function  zdot = spongy(t,z)
m=3;  c=7;  g=10;
x= z(1); y=z(2);
Vx=Z(3); Vy= Z(4);

xdot=Vx;
ydot= Vy;
  L2= x^2 + y^2;
  D  =    c*(x*Vx +  y*Vy)/(m*L2);
Vxdot= -x*D + g;
Vydot= -y*D;
sdot=sqrt(Vx^2+Vy^2);
zdot= [xdot, ydot, Vxdot, Vydot, sdot]';
```

2) (20 pt) A car going down a hill of slope θ (measured from the horizontal) puts on its rear brakes, causing the rear wheels to skid. The negligible-mass front wheels roll freely. The car moment of inertia about its center of mass G is I and its mass is m. The wheels are a distance ℓ apart (front to back distance). On level ground G is a height h above the ground and a distance d behind the front wheel. In terms of some or all of m, I, g, d, h, ℓ and θ find the minimum coefficient of friction μ needed to slow the car down.



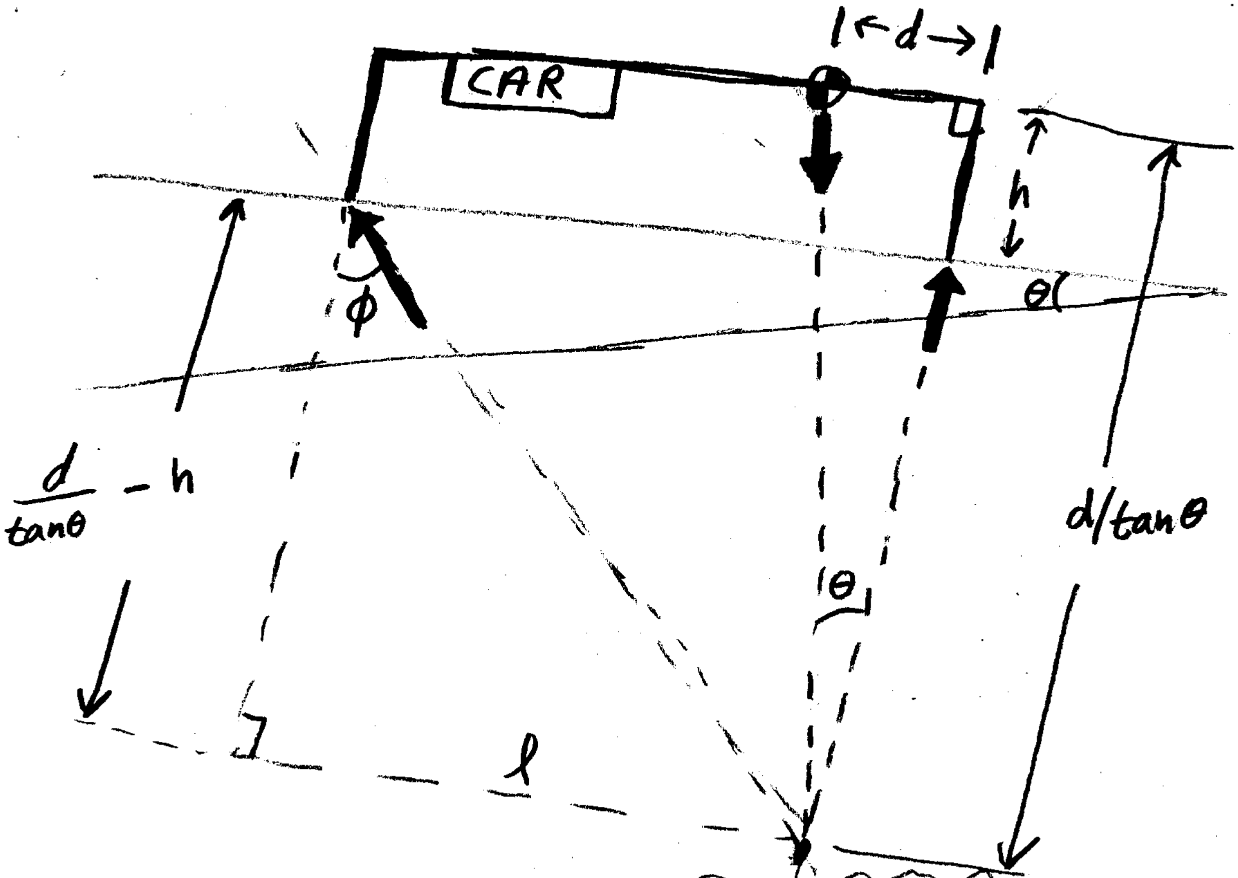
at critical μ

$\underline{a = 0} \Rightarrow \underline{\text{statics}}$

3-force body

$\Rightarrow$ all forces intersect at one point

FBD



$$\tan\phi = \boxed{\mu = \frac{\ell}{\frac{d}{\tan\theta} - h}}$$

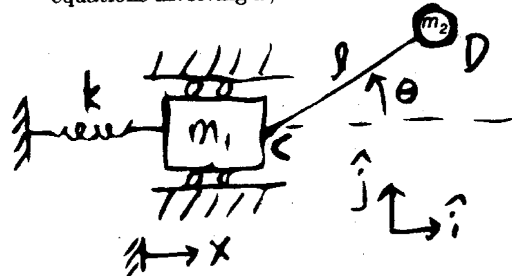
Sanity checks

$\theta \to 0 \Rightarrow \mu \to 0$ ✓

$h = 0, d = \ell \Rightarrow \mu = \tan\theta$ ✓

$d \to (\tan\theta) h \Rightarrow \mu \to \infty$ ✓

3) (20 pt) m_1 slides horizontally with x measuring the stretch of the spring k from its unstretched length. Point mass m_2 is at the end of a massless rod of length ℓ the other end of which is hinged on m_1. Neglect gravity. Find differential equations that govern the motion of the two masses (differential equations involving x, θ and some or all of m_1, m_2 and ℓ, K.

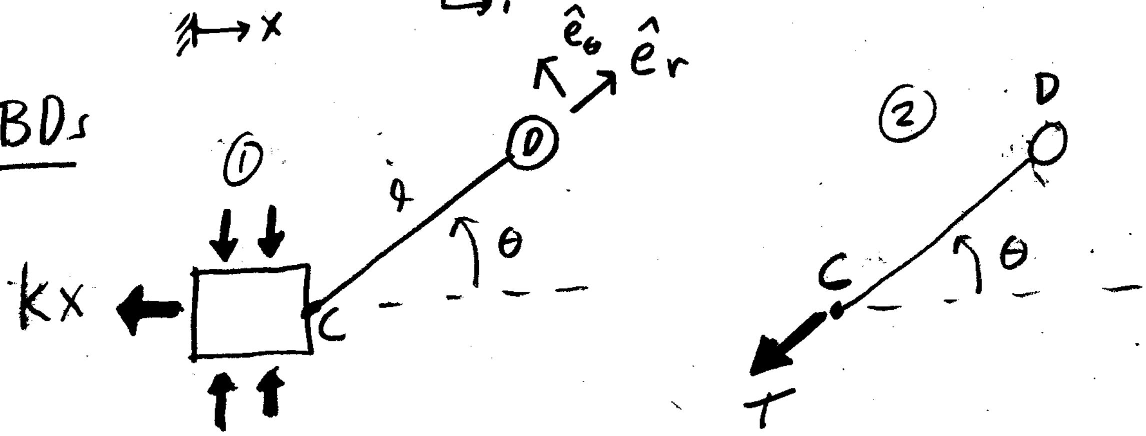


$\hat{e}_\theta \cdot \hat{i} = -\sin\theta$

$\hat{e}_r \cdot \hat{i} = \cos\theta$

$\hat{e}_r \times \hat{i} = -\sin\theta$

FBDs



FBD1 $\quad \left\{ \sum \underline{F}_i = m \underline{a}_i \right\} \cdot \hat{i}$

$$-kx = m_1 \ddot{x} + m_2 \left(\ddot{x} + \left(\ell \ddot{\theta} \hat{e}_\theta - \ell \dot{\theta}^2 \hat{e}_r \right) \cdot \hat{i} \right)$$

$$\boxed{-kx = (m_1 + m_2) \ddot{x} - \ell \ddot{\theta} \sin\theta - \ell \dot{\theta}^2 \cos\theta} \quad ①$$

FBD 2 $\quad AMB_{/C} \Rightarrow$

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$= \underline{a}_C + \underline{a}_{D/C}$

$$\underline{0} = \underline{r}_{D/C} \times (m \underline{a}_D)$$

$$= \ell \hat{e}_r \times m \left[\left(\ell \ddot{\theta} \hat{e}_\theta - \ell \dot{\theta}^2 \hat{e}_r \right) + \ddot{x} \hat{i} \right]$$

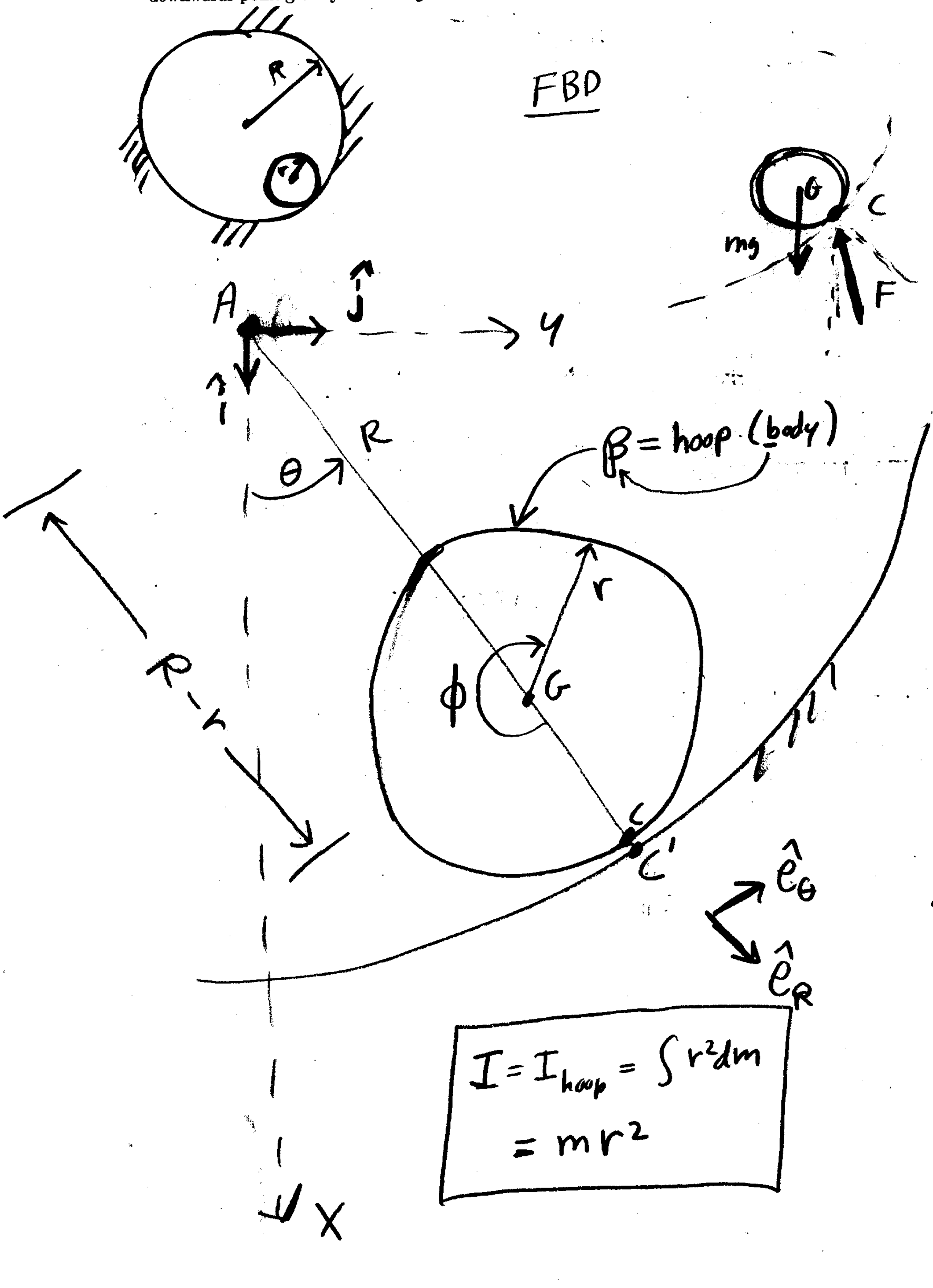
$$\left\{ \quad = \ell m \ddot{x} (-\sin\theta) \hat{k} + \ell^2 m \ddot{\theta} \hat{k} \right\}$$

$\{\ \}\cdot\hat{k} \Rightarrow \quad \boxed{\ddot{\theta} - \frac{\ddot{x}}{\ell}\sin\theta = 0} \quad (2)$

[looks just like the pendulum eqn. but with $-\ddot{x}$ instead of g. Why? pendulum can't distinguish g from an accelerating frame.]

(1) & (2) are 2 coupled 2nd order ODEs for θ, x.

4) (20 pt) Consider a hollow thin-walled pipe with mass m and radius r. The friction coefficient μ is large enough so the pipe rolls without slip inside a rigid stationary hollow pipe with radius R. Find the period of small oscillations (near the bottom) in terms of some or all of m, r, R, μ and the downwards-point gravity constant g.



$$I = I_{hoop} = \int r^2 dm = mr^2$$

Kinematics:

rolling contact $\Rightarrow$ $r\phi = R\theta$ ①

$$\underline{\omega} = \underline{\omega}_{hoop} = \underline{\omega}_{\beta} = (\dot{\theta} - \dot{\phi})\hat{k} = \left(\dot{\theta} - \frac{R}{r}\dot{\theta}\right)\hat{k}$$

$$\underline{\omega}_{\beta} = (1 - R/r)\dot{\theta}\hat{k}$$

$$\underline{a}_G = \ddot{\theta}(R-r)\hat{e}_\theta - \dot{\theta}^2(R-r)\hat{e}_R$$

AMB$_{/C}$:

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$\underline{r}_{G/C} \times mg\hat{i} = \underline{r}_{G/C} \times m\underline{a}_G + I\dot{\underline{\omega}}_\beta$$

$$r\sin\theta\, mg\hat{k} = (-r\hat{e}_R) \times [\ddot{\theta}(R-r)\hat{e}_\theta - \dot{\theta}^2(R-r)\hat{e}_R]$$

$$+ I\ddot{\theta}(1-R/r)\hat{k}$$

$$\{ \quad = -r(R-r)\ddot{\theta}\hat{k}m + I\ddot{\theta}(1-R/r)\hat{k} \}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad rmg\sin\theta = -\left[r(R-r)m + \left(\frac{R}{r}-1\right)I\right]\ddot{\theta}$$

$$= -\underset{I = mr^2}{(I + mr^2)}\left(\frac{R}{r}-1\right)\ddot{\theta}$$

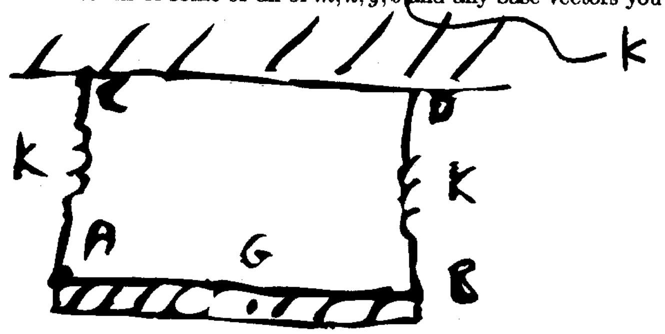
$$\Rightarrow \quad \ddot{\theta} + \frac{\cancel{r}\cancel{m}g}{2\cancel{m}r^{\cancel{2}}(R/r-1)}\sin\theta = 0$$

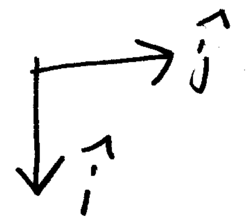
$$\underset{\theta\to 0}{\Rightarrow} \quad \ddot{\theta} + \underbrace{\left[\frac{g}{2(R-r)}\right]}_{\lambda^2}\theta = 0$$

$$\Rightarrow \theta = A\sin(\lambda t) + B\cos\lambda t$$

$$\lambda t_p = 2\pi \Rightarrow \quad \boxed{t_p = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2(R-r)}{g}}}$$

5) (20 pt) A uniform bar AB with length ℓ and mass m is hanging (gravity constant $= g$) in stationary equilibrium from two identical springs k. Suddenly but gently spring AB is cut by a laser beam. Immediately after the cut, what is the acceleration of the rod center at G. Answer in terms of some or all of m, k, g, ℓ and any base vectors you define with clear sketches.

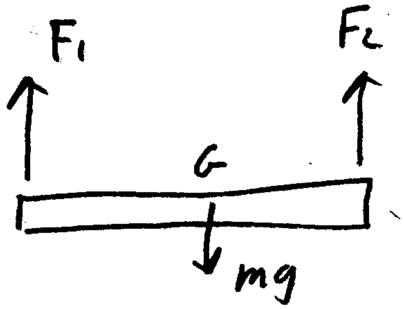




FBDs

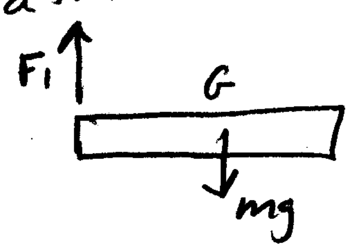
before

(statics



$\Rightarrow F_1 = mg/2$

after



$\sum \underline{F}_i = m\underline{a}_G$

$-F_1\hat{i} + mg\hat{i} = m\underline{a}_G$

$\left(-\frac{mg}{2} + mg\right)\hat{i} = m\underline{a}_G$

$$\boxed{\underline{a}_G = \frac{g}{2}\hat{i}}$$

Your Name: ______________________

T&AM 203 Makeup Prelim

Monday December 6, 2004 noon - 1:30$^+$ PM

Draft December 6, 2004. From the Fall 1996 Final exam

Do any 3 problems.

Please follow these directions to ease grading and to maximize your score.

a) No ca1cu1at0rs allowed. Ask for extra scrap paper if you need it.

b) Do any 3 problems.

c) Full credit if

- work is

  I.) neat,
  II.) clear, and
  III.) well organized;

± all signs and directions are well defined with sketches and/or words;

↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;

- correct units and correct vector notation are used, when appropriate;
- to the extent that a problem seems ambiguous or not perfectly defined, you clearly state any reasonable assumptions that you make;

→| reasonable justification, enough to distinguish an informed answer from a guess, is given;

- ⟶free body diagrams⟵ (FBD's) are drawn when appropriate;
- your answers are $\boxed{\text{boxed}}$ in; and
- your answers are TIDILY REDUCED.

d) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

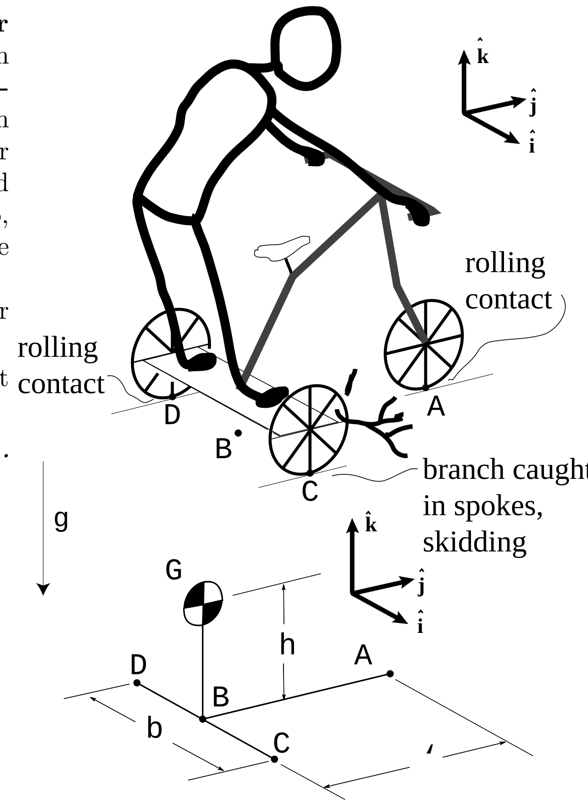
≫ MATLAB commands which would generate the desired answer count as a correct answer for all problems. *Some problems may only be only practically solvable with a computer.* You must be clear about how to interpret the MATLAB output as the answer to the question. If the problem statement is in terms of variables instead of numbers, MATLAB we will assume that the variables have been assigned values prior to the MATLAB commands you write.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	4:	/25

1) (25 pt) Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\mathbf{j}}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch.

Find the accereration of the tricycle (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$).

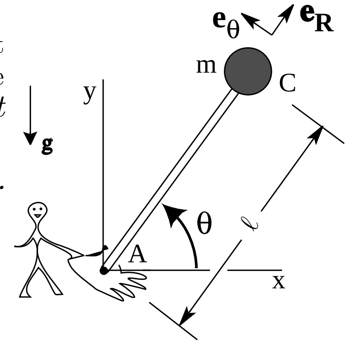
[Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



$\Leftarrow$ *Please put scrap work for problem 1 on the page to the left* $\Leftarrow$.

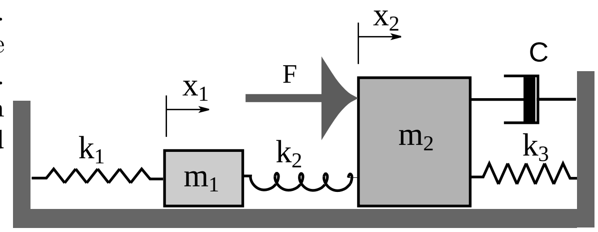
$\Downarrow$ *Put neat work to be graded for problem 1 below* $\Downarrow$.

2) (25 pt) Balancing a broom. Assume the hand is accelerating to the right with acceleration $\mathbf{a} = a\,\hat{\mathbf{i}}$. What is the force of the hand on the broom in terms of m, ℓ, θ, $\dot{\theta}$, a, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and g? *(You may not have any $\hat{\mathbf{e}}_R$ or $\hat{\mathbf{e}}_\theta$ in your answer.)*



$\Leftarrow$ *Please put scrap work for problem 2 on the page to the left* $\Leftarrow$.
$\Downarrow$ *Put neat work to be graded for problem 2 below* $\Downarrow$.

3) (25 pt) Equations of motion. Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be writen in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.



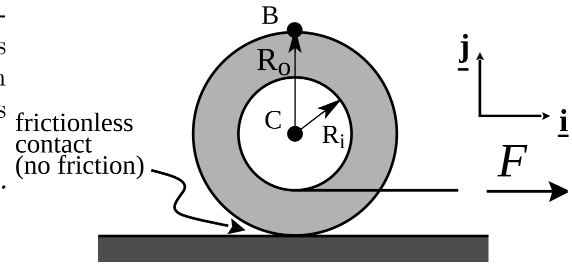
a) (10 points) Fill in the 16 terms of the 4×4 matrix below and the 4 terms of the blank column vector so that the equations are the correct equations for the system shown. Your answer should be in terms of any or all of the constants m_1, m_2, k_1, k_2,k_3, C, the constant force F, and t. Getting the signs right is important.

b) (10 points) Write MATLAB commands in appropriate functions and script files to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions. If you need to use the big matrix you have defined at the bottom of the page indicate its place in your code, you need not copy it in for MATLAB term by term.

$\Leftarrow$ *Please put MATLAB code for problem 3 on the page to the left* $\Leftarrow$.
$\Downarrow$ *Put other neat work to be graded for problem 3 on this page* $\Rightarrow$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad\qquad \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} \qquad\qquad \end{bmatrix}$$

4) (25 pt) **The film**, *Heat Treatment of Aluminum*, is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i. What, in terms of R_i, R_o, m, g, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and F are the accelerations of points C and B at the instant shown (the start of motion)?



$\Leftarrow$ *Please put scrap work for problem 4 on the page to the left* $\Leftarrow$.
$\Downarrow$ *Put neat work to be graded for problem 4 below* $\Downarrow$.

"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: ____________

T&AM 203 Prelim 1

Tuesday Sept 28, 2004

Draft September 27, 2004

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

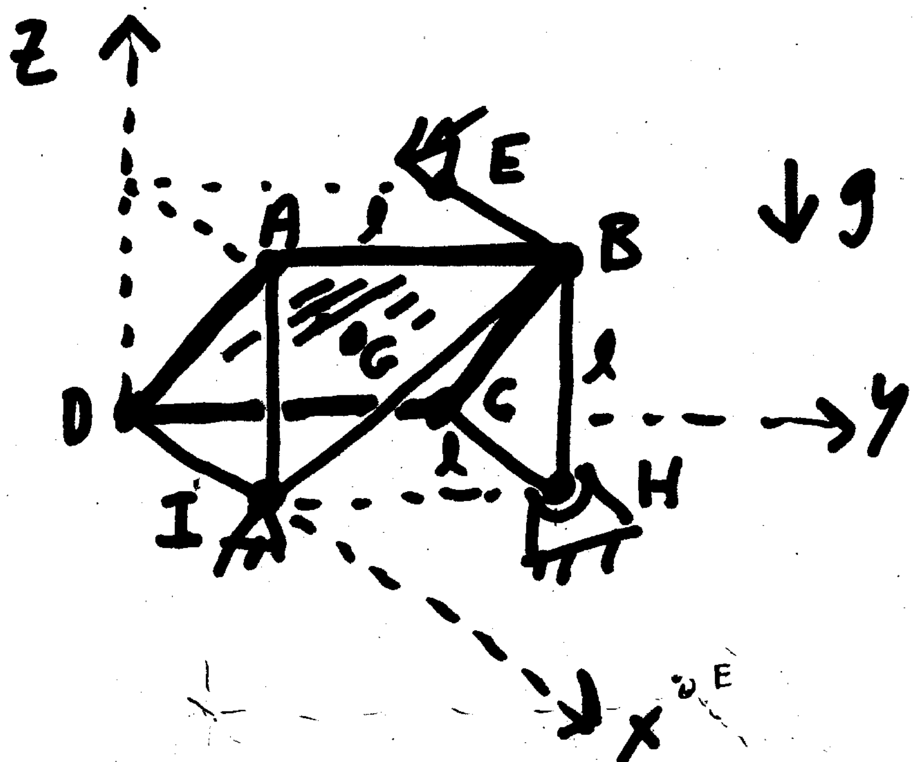
b) Full credit if:

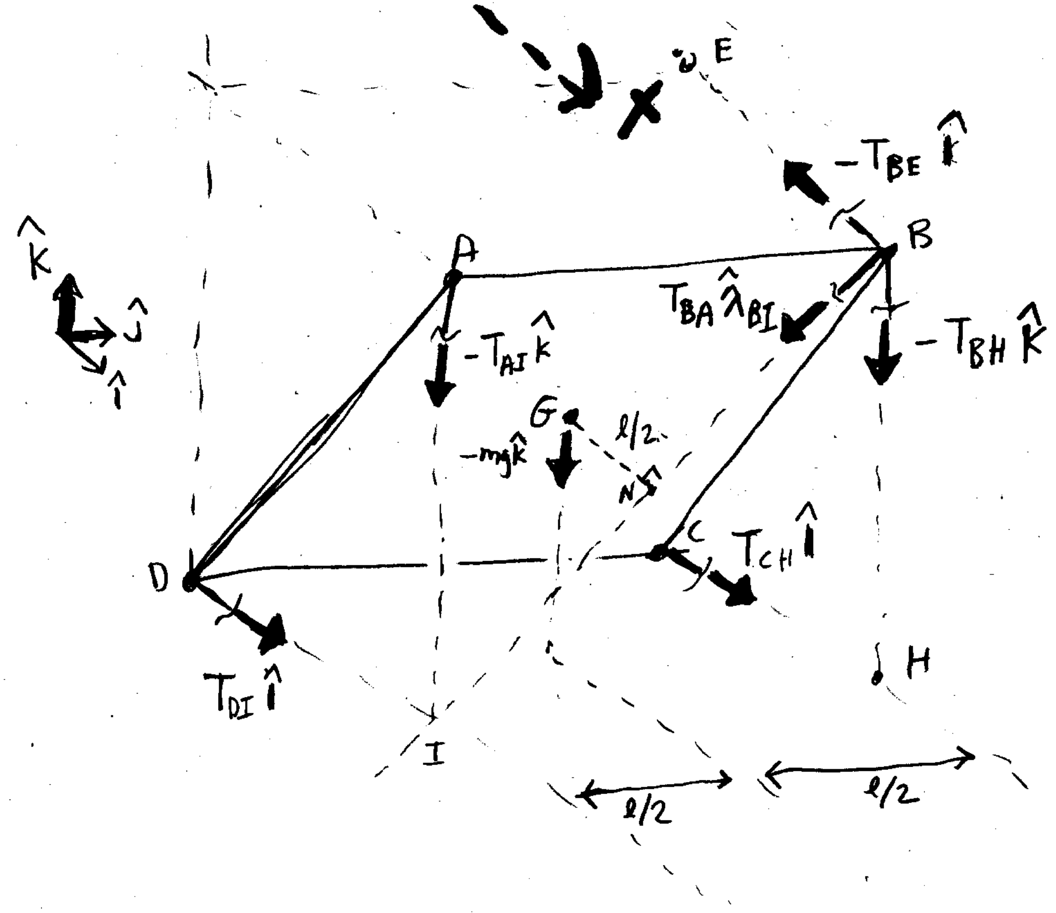
- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25

1) (25 pt) Statics. The uniform plate ABCD with mass m is held up by 6 bars (EB, HB, HC, IB, IA & ID). Find the tension in *any three* of these bars. Answer in terms of some or all of m, g, and ℓ.





$\{\sum \underline{F}_i = \underline{0}\} \cdot \hat{j} \Rightarrow \quad T_{BI}\,\hat{\lambda}_{BI} \cdot \hat{j} = 0 \Rightarrow \boxed{T_{BI} = 0}$

(no other forces have $\hat{j}$ components)

$\sum M_{/DI} = 0 \quad \Rightarrow \quad -mg\,\ell/2 + (-T_{BH})\,\ell = 0 \Rightarrow \boxed{T_{BH} = -mg/2}$

$\uparrow$ axis

(all tensions besides T_{BH} either intersect axis DI or are parallel to it)

$\sum M_{/EB} = 0 \quad \Rightarrow \quad -mg\,\ell/2 + (-T_{AI})\,\ell = 0 \Rightarrow \boxed{T_{AI} = -mg/2}$

(all other tensions don't contribute)

$\sum M_{/IB} = 0 \Rightarrow \quad \underbrace{\left(\frac{2}{\sqrt{2}}\,mg\right)}_{\text{force} \perp \text{ to axis \& lever arm}} \underbrace{\frac{\ell}{2}}_{\text{lever arm}} + T_{CH} \overbrace{\frac{\sqrt{2}\,\ell}{2}}^{d_{NH}} = 0 \Rightarrow \boxed{T_{CH} = -mg/2}$

(No other tensions contribute)

$\sum M_{/IC} = 0 \Rightarrow \quad \underbrace{\left(T_{BE}\frac{\sqrt{2}}{2}\right)\ell}_{\text{slide } T_{BE} \text{ to E to see this}} - \underbrace{\overset{-mg/2}{T_{BH}}\left(\frac{\sqrt{2}}{2}\,\ell\right)}_{\text{slide } T_{BH} \text{ to H to see this.}} = 0 \Rightarrow \boxed{T_{BE} = -mg/2}\,*$

(gravity & other tensions drop out)

$\sum M_{/BH} = 0 \Rightarrow \quad T_{DI}\,\ell = 0 \quad \Rightarrow \boxed{T_{DI} = 0}$

$*\sum M_{/IH} = 0$ gives T_{BE} in one shot.

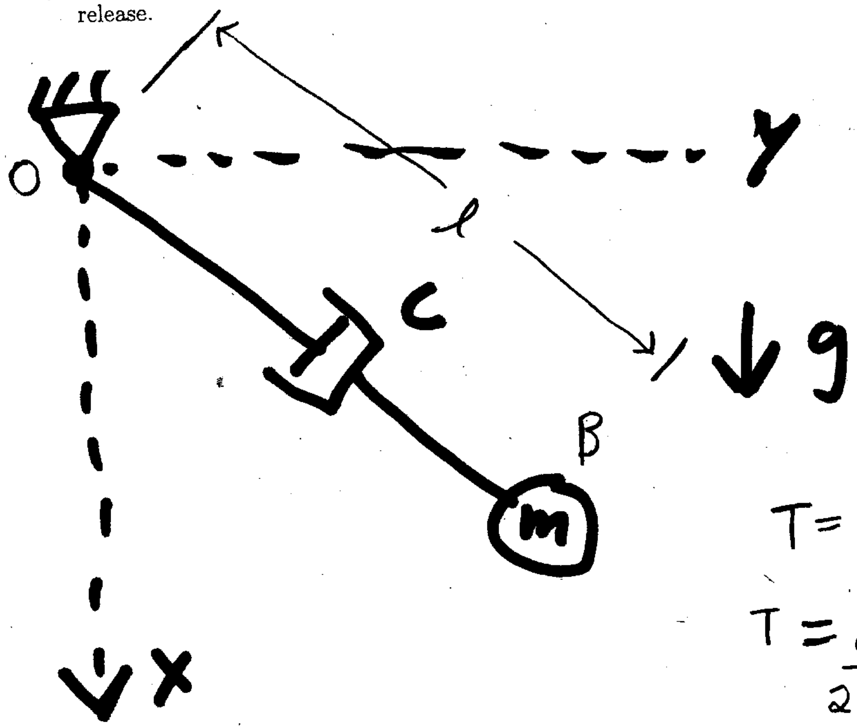
<u>Note</u>: all 6 tensions can be found one at a time, never using values of other tensions.

2) **(25 pt)** In the new sport of spongy jumping a spring is replaced by a dashpot c. Assume $m = 7\,\text{kg}$, $g = 10\,\text{m/s}^2$, and $c = 13\,\text{kg/s}$. The mass is released from rest at $x = 0\,\text{m}$, $y = 5\,\text{m}$.

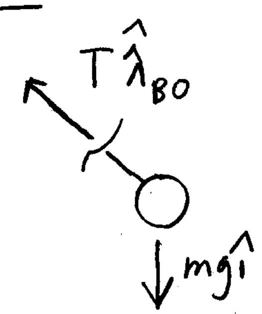
a) Just after (one milli-micro second after) release what is the acceleration of the mass?

b) What are the equations of motion for this system (differential equations involving x and y and their derivatives)?

c) Write Matlab code that would give the x coordinate of the center of mass 16 seconds after release.



FBD



$$T = c\,\dot{\ell}_{OB} = c\frac{d}{dt}\sqrt{x^2+y^2}$$

$$T = \frac{c\,(2x\dot{x} + 2y\dot{y})}{2\sqrt{x^2+y^2}}$$

$$T = c\,(x\dot{x} + y\dot{y})/\ell$$

$$\hat{\lambda}_{MB} = \underline{r}_{BO}/|\underline{r}_{BO}| = (-x\,\hat{\imath} - y\,\hat{\jmath})/\ell$$

$$\Rightarrow T\hat{\lambda}_{BO} = \frac{-c(x\dot{x}+y\dot{y})}{\ell^2}(x\hat{\imath}+y\hat{\jmath})$$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\left\{ mg\,\hat{\imath} - \frac{c(x\dot{x}+y\dot{y})}{\ell^2}(x\,\hat{\imath} + y\,\hat{\jmath}) = m(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath}) \right\}$$

$$\{\ \}\cdot\hat{\imath} \Rightarrow \boxed{mg - \frac{c(x^2\dot{x} + xy\dot{y})}{\ell^2} = m\ddot{x}} \quad (a)$$

$$\{\ \}\cdot\hat{\jmath} \Rightarrow \boxed{-\frac{c\,xy\dot{x} + y^2\dot{y}}{\ell^2} = m\ddot{y}}$$

define $V_x = \dot{x}$, $V_y = \dot{y}$

$$\Rightarrow \quad \begin{aligned} \dot{z}_1 &= \dot{x} = V_x \\ \dot{z}_2 &= \dot{y} = V_y \\ \dot{z}_3 &= \dot{V}_x = g - c(x^2 V_x + xy V_y)/(\ell^2 m) \\ \dot{z}_4 &= \dot{V}_y = -c(xy V_x + y^2 V_y)/(\ell^2 m) \end{aligned} \quad *$$

(a) in first order form

(b) at $t = 0^+$ $\quad V_x = 0,\ V_y = 0 \Rightarrow \dot{V}_x = g,\ \dot{V}_y = 0$

eqn *

$$\Rightarrow \boxed{\underline{a}(0^+) = g\hat{\imath}} \text{ (b)} \qquad (= 10\,\text{m/s}^2\ \hat{\imath})$$

```
tspan=[0 16];
z0 = [0 5 0 0]';
[t z]=ode23('sponge', tspan, z0);
xe = z(end,1)
```

useful Matlab command (c)

run this after saving sponge.m

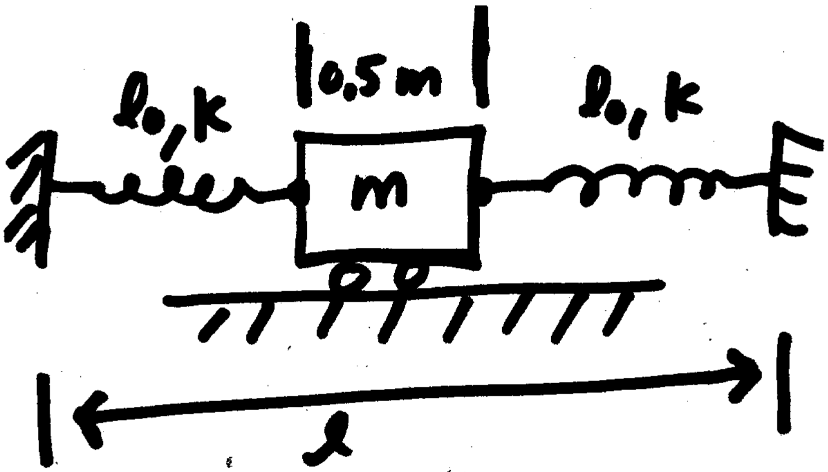
sponge.m

```
function zdot= sponge(t,z)
m=7; g=10; c=13
x= z(1); y= z(2); Vx= z(3); Vy= z(4);
xdot= Vx ;
ydot= Vy;
l2=      x^2 + y^2 ;
Vxdot = g - c*(x^2 * Vx + x*y* Vy)/(l2*m);
Vydot = -c*(x*y*Vx + y^2*Vy)/( l2*m);
zdot= [xdot ydot Vxdot Vydot]';
```

3) **(25 pt)** A mass $m = 7\,\text{kg}$ is held in place by two equal springs with $k = 5\,\text{N/m}$ and $\ell_0 = 3\,\text{m}$.

a) How long does one oscillation take if $\ell = 6.5\,\text{m}$?

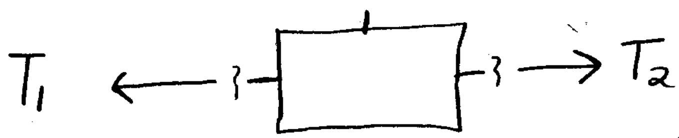
b) How long does one oscillation take if $\ell = 12.5\,\text{m}$?



$\rightarrow x =$ displacement from middle

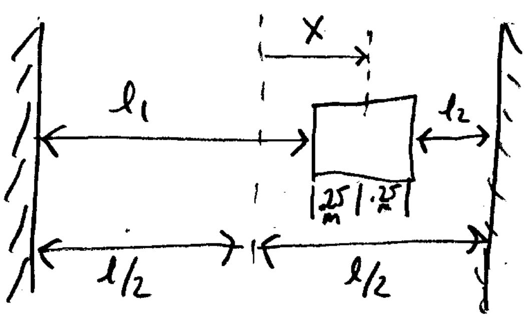
FBD

$\rightarrow \hat{\imath}$



$$T_2 = (\ell_2 - \ell_0)\,k$$

$$T_1 = (\ell_1 - \ell_0)\,k$$



$$\Rightarrow \quad \ell_1 = \frac{\ell}{2} + x - 0.25\,\text{m}$$

$$\ell_2 = \frac{\ell}{2} - x - 0.25\,\text{m}$$

LMB $\quad \{\sum \underline{F}_i = m\underline{a}\}\cdot\hat{\imath}$

$$T_2 - T_1 = m\ddot{x}$$

$$(\ell_2 - \ell_0)\,k - (\ell_1 - \ell_0)\,k = m\ddot{x}$$

(cont'd)

$$\left[\underbrace{\left(\frac{l}{2} - x - .25m\right)}_{l_2} - l_0\right]k - \left[\underbrace{\left(\frac{l}{2} + x - 0.25m\right)}_{l_1} - l_0\right]k = m\ddot{x}$$

*

$(l, l_0, .25m$ all drop out!$)$

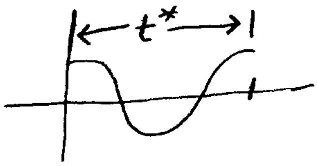
$$\Rightarrow \quad -2kx = m\ddot{x}$$

$$\boxed{\ddot{x} + \left(\frac{2k}{m}\right)x = 0}$$

Classic Harmonic Oscillator eqn.

ODE Soln: $x = A\cos\left(\underbrace{\sqrt{2k/m}\,t}\right) + B\sin\left(\underbrace{\sqrt{\frac{2k}{m}}\,t}\right)$

at one oscillation this whole $() = 2\pi$



$$\sqrt{2k/m}\,t^* = 2\pi \Rightarrow t^* = \frac{2\pi}{\sqrt{2k/m}}$$

$$t^* = \frac{2\pi}{\sqrt{2 \cdot (5 N/m)/(7 kg)}}$$

$1N = 1\, kg \cdot m/s^2$

$$t^* = \frac{2\pi}{\sqrt{(10/7)/s^2}}$$

$$\boxed{\boxed{t^* = \frac{2\pi}{\sqrt{10/7}} \text{ seconds}}}$$

(a) &
(b)

*Note: stretching a linear spring increases the force but not the stiffness.

"Solutions"

Your Name: Andy Ruina

Section day and time: ______

T&AM 203 Prelim II

Tuesday Oct 26, 2004

Draft September 27, 2004

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

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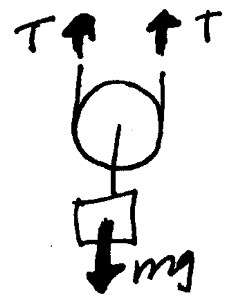
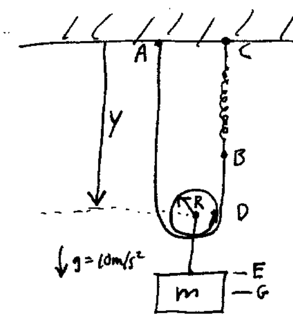
Problem	1:	/25
Problem	2:	/25
Problem	3:	/25

1) The system below is released from rest at $y = 1.25$ m. In terms of the quantities given, what is y at $t = 5$ s?

FBD





$L_{AB} = 2$ m = length of string
$R = \cancel{0.25}$ m $(0.5/\pi)$ m
L_s = rest length of spring $= 1$ m
$m = 3$ kg
$k = 70$ N/m
$y(0) = 1.25$ m
$y(5s) = ?$

Spring Law: $T = k\Delta$ ← spring stretch.

total spring length

Kinematics:
$$2y + \pi R = L_{AB} + \underbrace{(L_s + \Delta)}_{\text{total spring length}}$$
$$\Delta = 2y + \pi R - L_{AB} - L_s$$

LMB
$$\sum F_y = m\ddot{y} \Rightarrow -2T + mg = \ddot{y}m$$
$$T = k\Delta = k(2y + \pi R - L_{AB} - L_s)$$

$$\Rightarrow \quad \ddot{y} + \frac{4k}{m} y = \frac{2k}{m}(L_{AB} + L_s - \pi R) + g$$

$$y_h = A\cos\sqrt{4k/m}\,t + \underbrace{B}_{0 \text{ because } \dot{y}(0)=0}\sin\sqrt{\frac{4k}{m}}\,t$$

$$y_p = \frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4k}$$

$$y(t) = A\cos\sqrt{\frac{4k}{m}}\,t + \frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4k}$$

$$y(0) = y_0 \Rightarrow \quad y(t) = \left[y_0 - \left(\frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4k}\right)\right]\cos\sqrt{4k/m}\,t$$
$$+ \frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4k}$$

1) cont'd

$$y(t) = \left[\left[1.25 - \left(\frac{2+1-1/2}{2}\right) - \frac{30}{280}\right]\cos\sqrt{\frac{280}{30}}\,t/s + \left(1.25 + \frac{30}{280}\right)\right] m$$

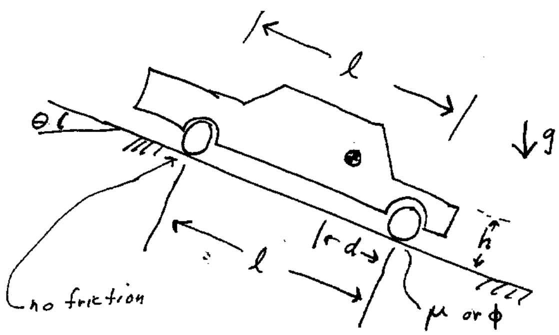
$$= \left[\frac{-3}{28}\cos\left(\sqrt{\frac{280}{3}}\,5\right) + \frac{5}{4} + \frac{3}{28}\right] m$$

$t = 5\,s$

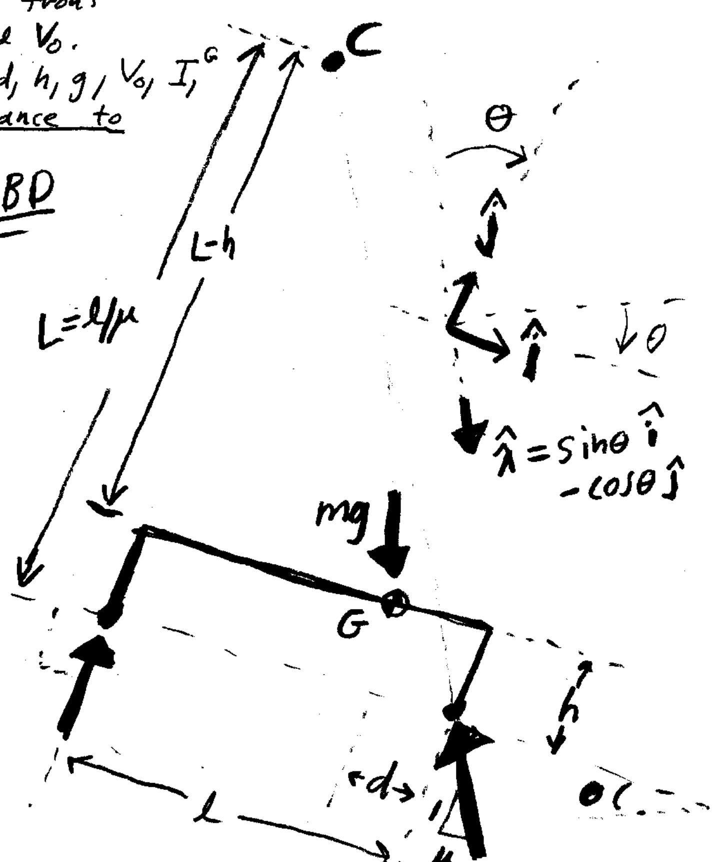
$$\boxed{y(5s) = \left[\frac{3}{28}\left(1 - \cos\left(5\sqrt{\frac{280}{3}}\right)\right) + \frac{5}{4}\right] m}$$

Prob 1

2) The car shown slams on its front brakes when it is going a speed V_0. In terms of some or all of ℓ, d, h, g, V_0, I^G, μ (or ϕ) and θ <u>what is the distance to stop?</u>



FBD



AMB/C: $\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$

$$\underline{r}_{G/C} \times (mg\hat{\lambda}) = \underline{r}_{G/C} \times (m\underline{a}_G)$$

$$\underline{a}_G = a\hat{i}\ ;\ \underline{r}_{G/C} = (\ell-d)\hat{i} - (L-h)\hat{j}$$

$$\Rightarrow \left[(\ell-d)\hat{i} - (L-h)\hat{j}\right] \times \left[mg(\sin\theta\,\hat{i} - \cos\theta\,\hat{j})\right] = \left[(\ell-d)\hat{i} - (L-h)\hat{j}\right] \times (ma\hat{i})$$

$$\left\{ \cancel{m} g\hat{k}\left[-(\ell-d)\cos\theta + (L-h)\sin\theta\right] = \cancel{m} a (L-h)\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad a = \left[\frac{-(\ell-d)}{(L-h)}\cos\theta + \sin\theta\right] g < 0 \quad \text{(slowing down)}$$

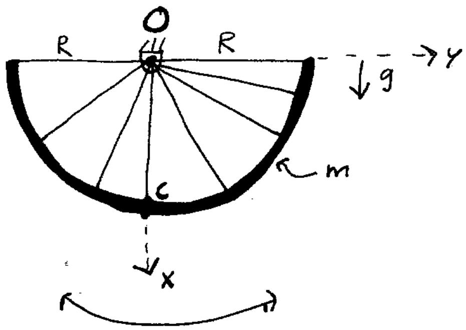
$$a = \frac{dv}{dt} \Rightarrow \quad av = v\frac{dv}{dt} \Rightarrow \quad a\,v\,dt = v\,dv \Rightarrow a\,dx = d(v^2/2) \Rightarrow a\Delta x = \Delta v^2/2$$

$$\left[\Delta(v^2/2) = -V_0^2/2\right]$$

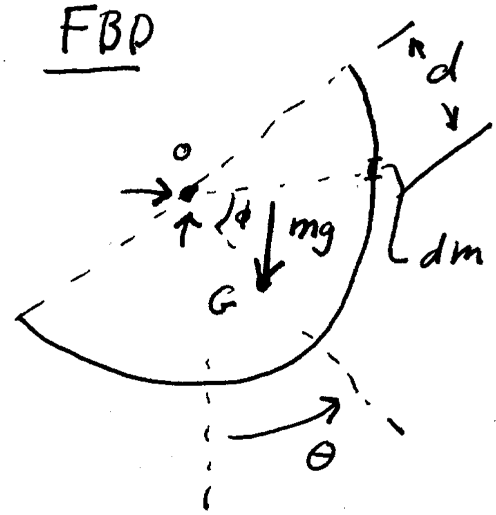
$$\Rightarrow \Delta x = x = \text{stopping distance} = -2V_0^2/2a$$

$$x = \frac{V_0^2}{2\left[\frac{\ell-d}{(L-h)}\cos\theta - \sin\theta\right]g} = \boxed{\frac{V_0^2\left[\ell/\mu - h\right]}{2g\left[(\ell-d)\cos\theta - \left(\frac{\ell}{\mu} - h\right)\sin\theta\right]}}$$

3) A bicycle wheel is cut in half. <u>What is the period of oscillation?</u> Answer in terms of some or all of R, m, g & the release angle θ_0. Neglect the mass & weight of the spokes & hub. Assume small oscillations.



FBD



$\underline{AMB_{/O}}$:
$$\Sigma \underline{M}_{/O} = \dot{\underline{H}}_{/O}$$

$$-mdg\sin\theta\,\hat{k} = \int \underbrace{\underline{r}}_{\text{radial from } O} \times \underbrace{\underline{a}}_{(\text{radial part}) + \overbrace{(\text{circumferential part})}^{R\ddot{\theta}}} dm$$

$$\Big\{ \qquad = \ddot{\theta} R^2 m \hat{k} \Big\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \quad \ddot{\theta} + \frac{dg}{R^2}\sin\theta = 0$$

small angles $\Rightarrow \quad \ddot{\theta} + \frac{dg}{R^2}\theta = 0 \Rightarrow \theta = A\cos\sqrt{\frac{dg}{R^2}}\,t + B\sin\left(\sqrt{\frac{dg}{R^2}}\,t\right)$

$t^* =$ period: $\sqrt{\frac{dg}{R^2}}\,t^* = 2\pi$

$\boxed{d = 2R/\pi \text{ from next page}}$

$$t^* = 2\pi\sqrt{R^2/dg}$$
$$= 2\pi\sqrt{R\pi/2g}$$

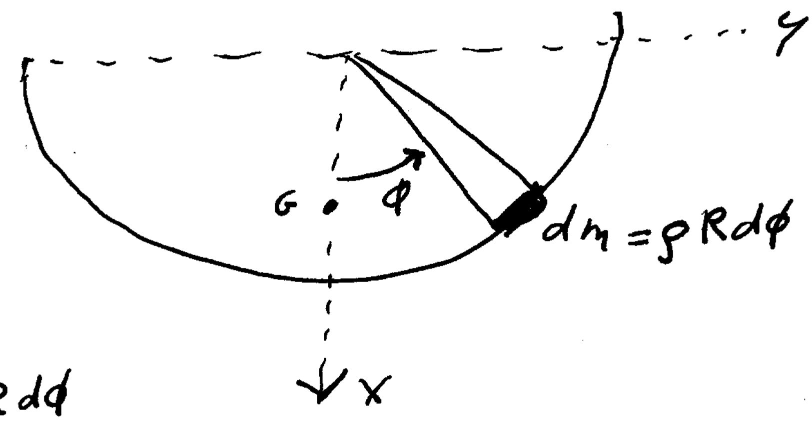
$$\Rightarrow \boxed{t^* = \sqrt{2}\,\pi^{3/2}\sqrt{\frac{R}{g}}}$$

prob 3

3 cont'd

Find d

$\rho = \frac{m}{\pi R}$



$$m X_G = \int x\, dm$$

$$= \int_{-\pi/2}^{\pi/2} (R\cos\phi)\, \rho R\, d\phi$$

$$= R^2\left(\frac{m}{\pi R}\right) \int_{-\pi/2}^{\pi/2} \cos\phi\, d\phi$$

$$= \frac{Rm}{\pi} \sin\phi \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{2Rm}{\pi}$$

$$\Rightarrow \boxed{d = X_G = \frac{2R}{\pi}}$$ ← used on previous page

"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: Tu,Th 9:05-9:55

T&AM 203 Prelim 3

Tuesday Nov 23, 2004

Draft November 23, 2004

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

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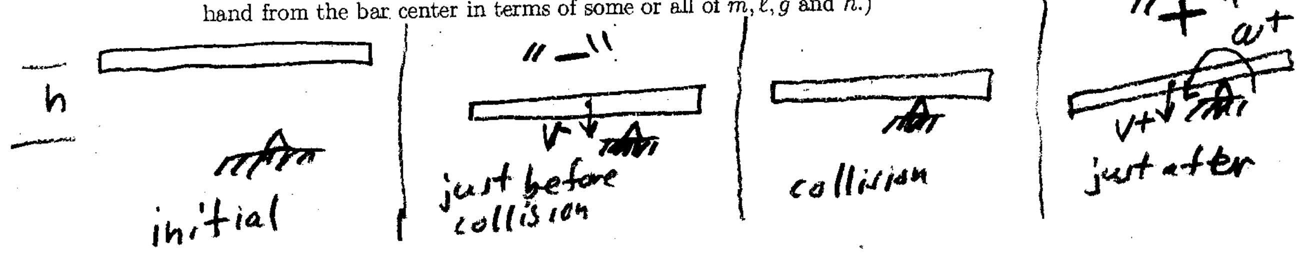
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

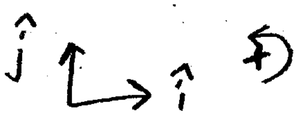
Problem 1:	/25
Problem 2:	/25
Problem 3:	/25

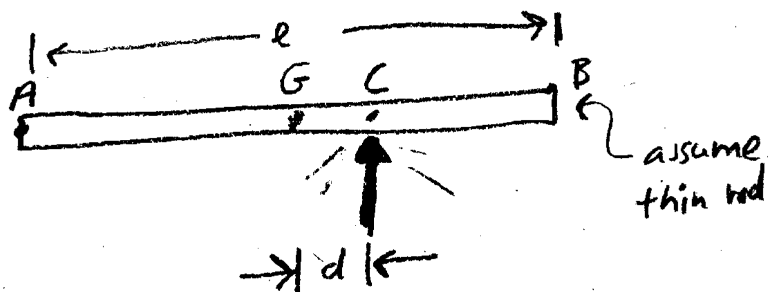
1) (25 pt) A uniform horizontal bar with mass m and length ℓ falls a height h without rotation in gravitational field g. The bar is then caught by a stationary hand. The hand does not move, but imparts an impulse but no torque (it freely allows rotation). *Clearly indicate where on the bar the hand should catch the bar so that one end of the bar (indicate which end) does not have any change of velocity at the impact.* (Indicate the distance d of the hand from the bar center in terms of some or all of m, ℓ, g and h.)

$\downarrow g$



Collision FBD





AMB /C $\qquad \underline{H}^-_{/C} = \underline{H}^+_{/C}$

$$\{ v^- m d \hat{k} = v^+ m d \hat{k} + \overset{m\ell^2/12}{I^G} \omega^+ \hat{k} \}$$

$$v^+ = \omega^+ d$$

$$\{\}\cdot\hat{k} \Rightarrow v^- \cancel{m} d = \omega^+ \cancel{m}\left[d^2 + \tfrac{1}{12}\ell^2\right]$$

$$\Rightarrow \omega^+ = \frac{v^- d}{d^2 + \frac{1}{12}\ell^2} \Rightarrow v_A^+ = \omega^+\left(d + \tfrac{\ell}{2}\right) = \frac{v^- d(d + \ell/2)}{d^2 + \ell^2/12}$$

Problem statement: $v_A^+ = v_A^-$

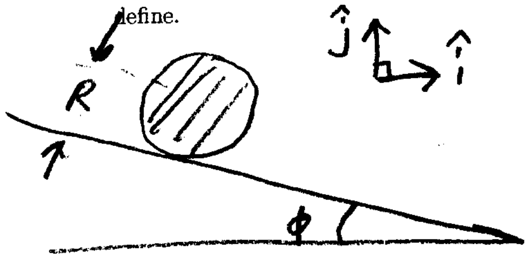
$$\Rightarrow \frac{v^- d(d + \ell/2)}{d^2 + \ell^2/12} = v^-$$

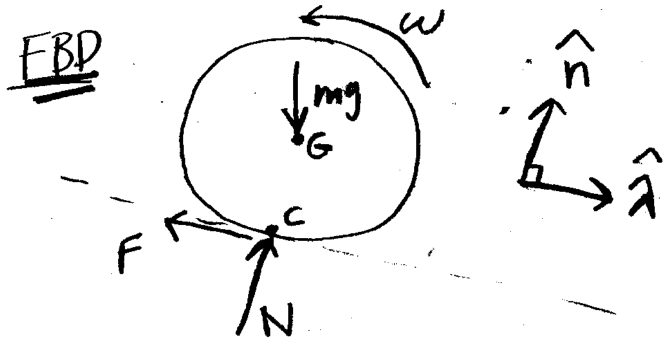
$$\Rightarrow \frac{d\ell}{2} = \frac{\ell^2}{12}$$

$$\Rightarrow \boxed{d = \ell/6}$$

Pt. C is called the center of percussion. Thinking of AB as a baseball bat its where to hit the ball so the hand end at A has no jump in velocity. An approximation of "the sweet spot".

2) **(25 pt)** A uniform disk of mass m and radius R is released from rest at $t = 0$ to roll-without-slip down a slope ϕ (measured relative to the horizontal) as accelerated by gravity g. *At time t what is the acceleration of the point on the disk that is then touching the ground?* Answer in terms of some or all of m, R, g, ϕ, t and any base vectors that you choose that you clearly define.





FBD shown is valid at all times.

Kinematics: $\underline{V}_G = V_G \hat{\lambda} = \underbrace{-\omega R}_{\text{rolling}} \hat{\lambda} \Rightarrow a_G = -\dot{\omega} R$

AMB$_{/C}$: $\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$

$$\underline{r}_{G/C} \times (-mg\hat{j}) = \underline{r}_{G/C} \times m\underline{a}_G + I^G \dot{\omega} \hat{k}$$

$\underline{r}_{G/C} = R\hat{n}$ $\qquad$ $\underline{a}_G = -\dot{\omega} R \hat{\lambda}$ $\qquad$ $I^G = \frac{1}{2} m R^2$ (see next page)

$\hat{n} \times \hat{j} = \sin\phi \hat{k} \Rightarrow \{ -Rmg\sin\phi \hat{k} = R^2 m \dot{\omega} \hat{k} + \frac{1}{2} m R^2 \dot{\omega} \hat{k} \}$

$\{ \} \cdot \hat{k} \Rightarrow -Rg\sin\phi = (R^2 + \frac{1}{2} R^2)\dot{\omega}$ (right hand side = constant)

$$\Rightarrow \boxed{\dot{\omega} = \frac{-2g\sin\phi}{3R}}$$

$$\omega_0 = 0 \Rightarrow \boxed{\omega = \frac{-2g\sin\phi}{3R} t}$$

What is accel. of pt. C?

$$\underline{a}_C = \underline{a}_G + \underline{a}_{C/G} = -\dot{\omega} R \hat{\lambda} + \dot{\underline{\omega}} \times \underline{r}_{C/G} + (-\omega^2 \underline{r}_{C/G})$$

$$\underline{a}_c = -\dot{\omega}R\hat{\lambda} + \dot{\omega}\hat{k} \times (-R\hat{n}) + \omega^2 R\hat{n}$$

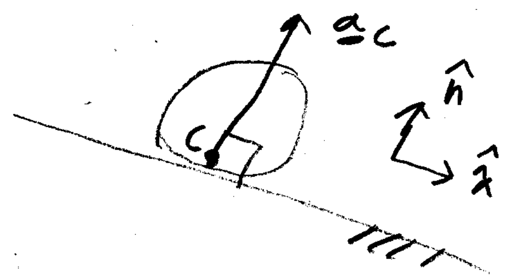
$\hat{k} \times \hat{n} = -\hat{\lambda}$

$$= -\dot{\omega}R\hat{\lambda} + \dot{\omega}R\hat{\lambda} + \omega^2 R\hat{n}$$

$$= \omega^2 R\hat{n}$$

$$= \left(\frac{-3g\sin\phi}{R}t\right)^2 R\hat{n}$$

$$\boxed{\underline{a}_c = \frac{4g^2\sin^2\phi\, t^2}{9R}\hat{n}}$$



Calculate I^G for uniform disk $\quad \rho = \frac{m}{A} = \frac{m}{\pi R^2}$

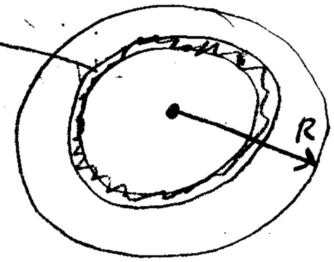
$$I^G = \int r^2\, dm$$

$$= \int_0^R r^2 (\rho 2\pi r\, dr)$$

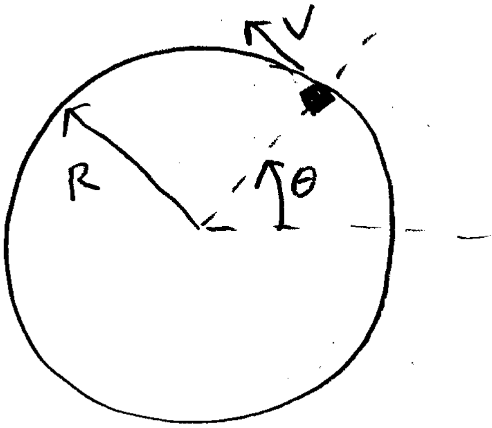
$$= 2\pi\rho \int_0^R r^3\, dr = 2\pi \frac{m}{\pi R^2}\left(\frac{r^4}{4}\right)\Big|_0^R$$

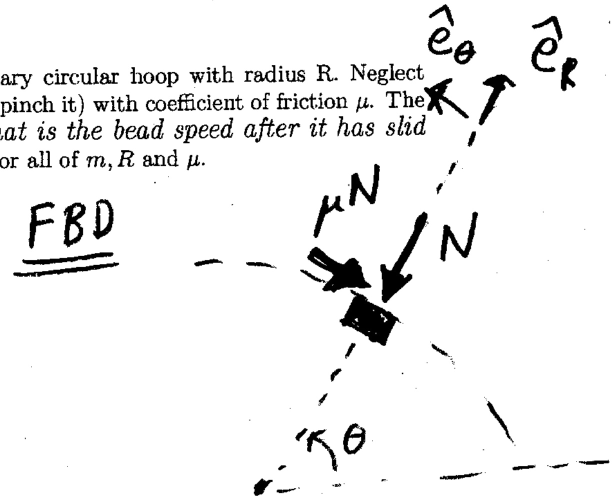
$$= 2\cancel{\pi}\frac{m}{\cancel{\pi}R^2}\frac{R^4}{4}$$

$$\boxed{I^G = mR^2/2}$$



3) (25 pt) A small bead with mass m slides on a rigid stationary circular hoop with radius R. Neglect gravity. The bead slides loosely on the wire (does not pinch it) with coefficient of friction μ. The initial speed of the bead is v_0 (along the circle). *What is the bead speed after it has slid once around the hoop?* Answer in terms of some or all of m, R and μ.





LMB

$$\underline{F} = m\underline{a} \Rightarrow -N\hat{e}_R - \mu N\hat{e}_\theta = m\left[(\cancel{\ddot{R}}^{\,0} - R\dot{\theta}^2)\hat{e}_R + (R\ddot{\theta} + 2\cancel{\dot{R}}^{\,0}\dot{\theta})\hat{e}_\theta\right]$$

$$\left\{-N\hat{e}_R - \mu N\hat{e}_\theta = -mR\dot{\theta}^2\hat{e}_R + mR\ddot{\theta}\hat{e}_\theta\right\}$$

$$\{\}\cdot\hat{e}_R \Rightarrow \quad N = mR\dot{\theta}^2$$

$$\{\}\cdot\hat{e}_\theta = \quad -\mu N = mR\ddot{\theta} \quad \Rightarrow mR\ddot{\theta} = -\mu mR\dot{\theta}^2$$

$\boxed{\text{Define } \omega = \dot{\theta}}$

$$\Rightarrow \ddot{\theta} = -\mu\dot{\theta}^2 \Rightarrow \boxed{\dot{\omega} = -\mu\omega^2} *$$

Why? Because we are given final θ not final t.

Now think of ω as $\omega(\theta)$. $\boxed{\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt}}$

$$\Rightarrow \frac{d\omega}{d\theta}\omega = -\mu\omega^2$$

$$\Rightarrow \frac{d\omega}{d\theta} = -\mu\omega$$

$$\Rightarrow \omega = \omega_0 e^{-\mu\theta}$$

[Note $V = \omega R$]

$$\Rightarrow V = \underbrace{\omega_0 R}_{V_0} e^{-\mu\theta}$$

$$\Rightarrow V = V_0 e^{-\mu\theta}$$

$$\underbrace{\theta = 2\pi}, \Rightarrow \boxed{V = V_0 e^{-2\pi\mu}}$$

↑ one revolution

Alternative soln. to ODE *1 (the long way around)

$$\frac{d\omega}{dt} = -\mu\omega^2 \Rightarrow \frac{d\omega}{\omega^2} = -\mu\, dt$$ (seperable 1st order ODE)

$$\Rightarrow -\omega^{-1} - (-\omega_0^{-1}) = -\mu t \Rightarrow \frac{1}{\omega_0} - \frac{1}{\omega} = -\mu t$$

$$\Rightarrow \frac{1}{\omega} = \frac{1}{\omega_0} + \mu t \Rightarrow \boxed{\omega = \frac{1}{\mu t + 1/\omega_0} = \frac{\omega_0}{1+\omega_0\mu t}} \quad (1)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\omega_0}{1+\omega_0\mu t} \Rightarrow d\theta = \frac{\omega_0\, dt}{1+\omega_0\mu t}$$ (seperable again)

$$\Rightarrow \int_0^\theta d\theta' = \int_0^t \frac{\omega_0\, dt'}{1+\omega_0\mu t'}$$

substitution: let $u = 1+\omega_0\mu t$, $du = \omega_0\mu\, dt$

$$\theta = \int_1^u \frac{1}{\mu}\frac{du'}{u'} = \frac{1}{\mu}\left[\ln(u) - \ln(1)\right] = \frac{1}{\mu}\ln u$$

$$\theta = \frac{1}{\mu}\ln(1+\omega_0\mu t)$$

$\underline{\underline{\theta = 2\pi}}$ $\Rightarrow$ $2\pi\mu = \ln(1+\omega_0\mu t)$

$$\Rightarrow e^{2\pi\mu} = 1+\omega_0\mu t$$

$$t = \frac{e^{2\pi\mu}-1}{\omega_0\mu} \quad (2)$$

Apply (2) to (1) $\Rightarrow$ $$\omega\Big|_{\theta=2\pi} = \frac{\omega_0}{1+\omega_0\mu\left[\frac{e^{2\pi\mu}-1}{\omega_0\mu}\right]} = \frac{\omega_0}{e^{2\pi\mu}}$$

$$\Rightarrow \omega = \omega_0 e^{-2\pi\mu} \Rightarrow \omega R = \omega_0 R e^{-2\pi\mu}$$

$$\Rightarrow \boxed{V = V_0 e^{-2\pi\mu}}$$ (again)

at $\theta = 2\pi$

Solutions

①

Your Name: Manish Agarwal

Section time: ______________

T&AM 203 Prelim 1

Tuesday September 26, 2006

Draft September 25, 2006

3 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25

1) (25 pt) Pulleys. In the problems below you are asked "What is the relation" between this and that. This means you should write the simplest possible equation in which this and that are the only unknowns.)

a) (1 point) Please read all the rules and hints at the front of the exam. Write here: "I read the cover page.":

____

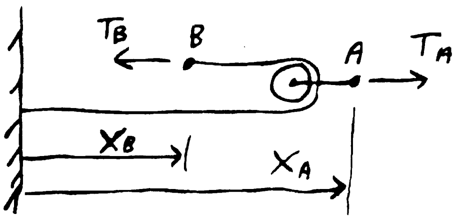
b) (3 points) The ideal pulley system (make the usual assumptions) in (b,c) below shown is part of a larger mechanism. What is the relation between T_A and T_B? Clearly justify your work from first principles.

c) (10 points) For the same pulley system what is the relation between a_A and a_B? Clearly justify your work from first principles.

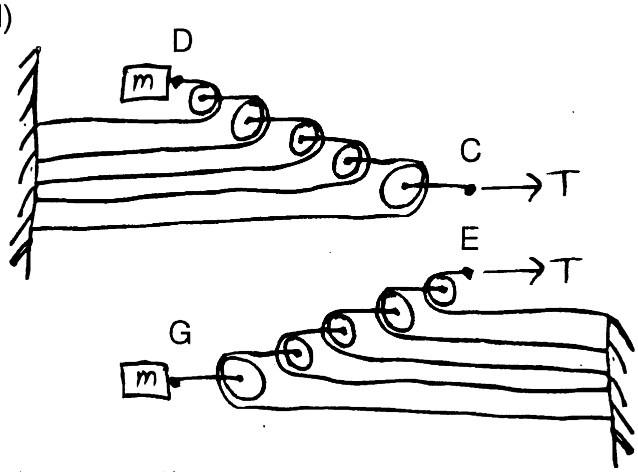
[Part (d) will only be graded if (b) and (c) above are correct.]

d) (11 points) The two pulley systems below (d) are treated as having all ideal components. What is the relation between a_C and a_E? You may use the results from parts (b) and (c) above without re-deriving them again and again. When comparing the systems use $m = m$ and $T = T$.

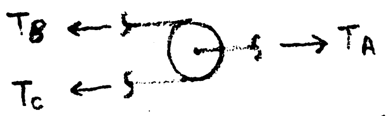
b,c)



d)



b)



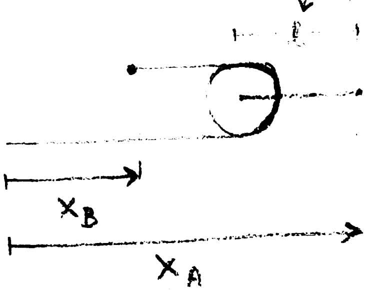
assuming massless pulley

$T_A - T_B - T_c = 0$

also $T_B = T_c$ assuming massless, inextensible string.

=> $\boxed{T_A = 2T_B}$

c)



length of the string

$= (x_A - \ell) + (x_A - \ell - x_B)$

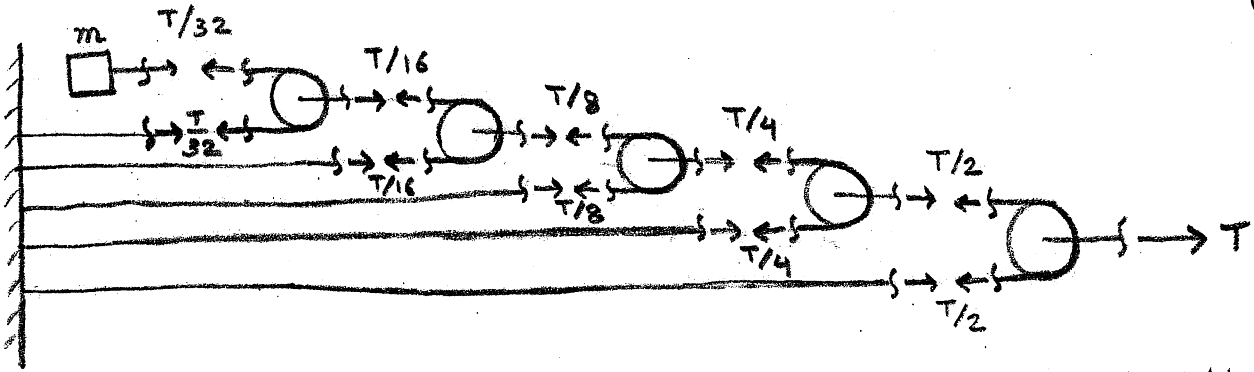
$=$ constant

differentiating wrt time

$\dot{x}_A + \dot{x}_A - \dot{x}_B = 0$ $(\because \dot{\ell} = 0)$

differentiating again $2\ddot{x}_A = \ddot{x}_B$ => $\boxed{2a_A = a_B}$

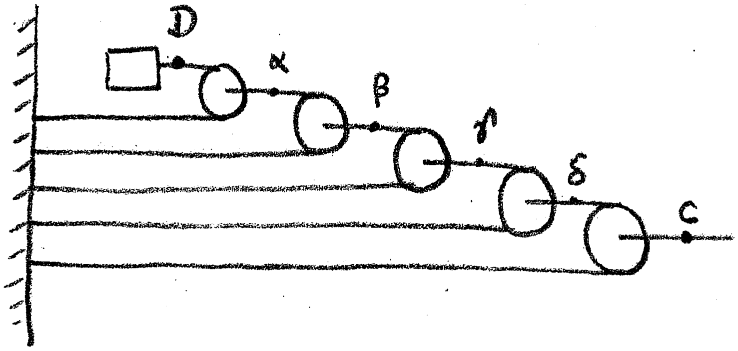
d)



using part b) the above FBDs can be established

N2L on m gives

$$\frac{T}{32} = m a_D \Rightarrow a_D = \frac{T}{32m}$$

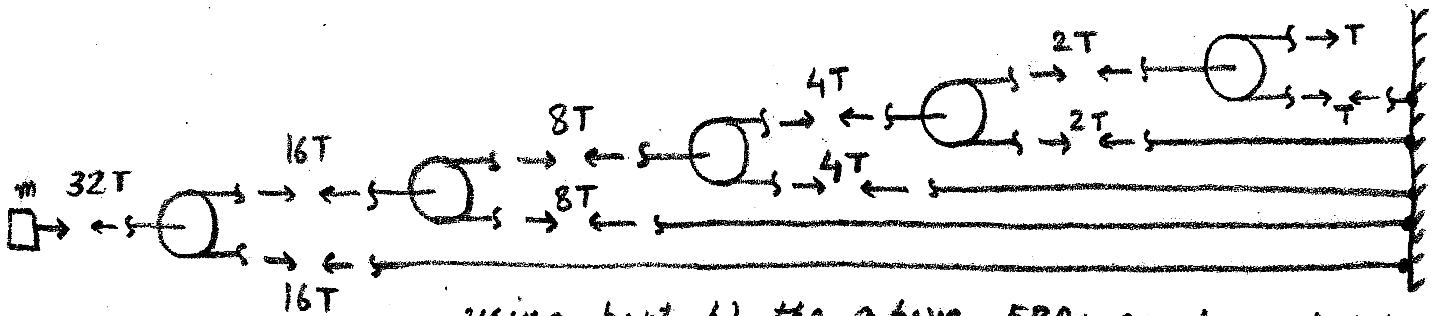


by part c)

$$2a_\alpha = a_D$$
$$2a_\beta = a_\alpha$$
$$2a_\gamma = a_\beta$$
$$2a_\delta = a_\gamma$$
$$2a_c = a_\delta$$

$$\Rightarrow 2a_c = \frac{a_\gamma}{2} = \frac{a_\beta}{4} = \frac{a_\alpha}{8} = \frac{a_D}{16} = \frac{T}{32m} \cdot \frac{1}{16}$$

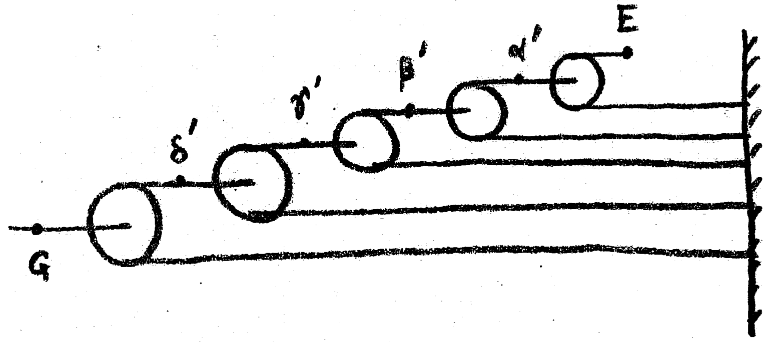
$$\Rightarrow a_c = \frac{T}{m}\left(\frac{1}{32}\right)^2 \quad \text{— (1)}$$



using part b) the above FBDs can be established

N2L on m gives $32T = m a_G$ $\quad a_G = 32\frac{T}{m}$

by part c)



$$2a_{\alpha'} = a_E$$
$$2a_{\beta'} = a_{\alpha'}$$
$$2a_{\gamma'} = a_{\beta'}$$
$$2a_{\delta'} = a_{\gamma'}$$
$$2a_G = a_{\delta'}$$

$$\Rightarrow \quad a_E = 2a_{\alpha'} = 4a_{\beta'} = 8a_{\gamma'} = 16a_{\delta'} = 32a_G$$

$$\Rightarrow \quad a_E = 32 \cdot \underbrace{32\frac{T}{m}}_{\text{substituting the value of } a_G} = (32)^2 \frac{T}{m} \qquad — \text{②}$$

dividing ① by ②

$$\frac{a_c}{a_E} = \left(\frac{1}{32}\right)^4$$

$$\boxed{a_E = (32)^4 a_c}$$

$$a_E \approx 1{,}000{,}000 \; a_c \quad !$$

5

2) (25 pt) **MATLAB etc.** The block of code shown calculates motions from a dynamics problem relevant to this course. It runs without error. (This code is no good outside a test, obviously, because the variable names are not suggestive, intermediate variables aren't used, and there is no commenting.)

a) (10 points) Write a mechanics question (with values, units, basic assumptions etc.) that the output of this code answers. Your question should make no reference to matlab or computers but should be in the language of mechanics.

[Grading of parts (b) and (c) below depend on the answer to (a) above being correct. So be confident before moving on.]

b) (10 points) Assume that the command `plot(t,z(:,2))` is added just below the command `t(end)`. Draw, as accurately as you can, the resulting plot. Label (give numerical values) key points and asymptotes which you find using your own pencil-and-paper analysis. Label the axes (even though the code does not do this).

c) (5 points) Get as far as you can towards finding a numerical value for `t(end)` without using the computer. Ultimately you will be stuck without a calculator. But get to a point where the job of the calculator is clear.

this is what the code prints. So the 'mechanics question' should demand time.

initial conditions

```
function prelim1q2
options=odeset('events',@fSally);
[t,z,tev, zev, i] = ode45(@fgeorge,[0 1000],[0 10],options);
t(end)
end

function zdot = fgeorge(t,z);
zdot = [z(2) -2*z(2)-10]';
end

function [value,done,dir] = fSally(t,z)
value = z(1) ; done = 1; dir= -1;
end
```

Remember g!

This suggests a diffrential equation of the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_2 - 10$$

requires to detect an event corresponding to $x_1 = 0$ when x_1 crosses from +ive to -ive

a) realizing $x_1 = x$ (position)

$x_2 = v$ (velocity)

we get the diff equ^n as

$$\dot{x} = v$$
$$\dot{v} = -2v - 10$$

or $\ddot{x} = -2\dot{x} - 10$

noting that its a one-dimensional motion and comparing it with the standard equ^n in mechanics

$$m\ddot{x} + c\dot{x} + kx = F$$

or $$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}$$

or $$\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x + \frac{F}{m}$$

we see $\frac{c}{m} = 2$ $k = 0$ $\frac{F}{m} = -10 = -g$!

linear damping   no stiffness   force opposing the motion in all probability should be gravity!

the mechanics question can, hence, be

" A projectile of mass 1 kg is projected vertically up with velocity 10 m/s. The air applies a linear drag proportional to the velocity (drag constant 2 N/(m/s)). Find the time projectile take to hit the ground again."

b) plot (t, z(:,2)) wants a plot of v vs t

now $\quad \dot{v} = -2v - 10$

$$\int_{10}^{v} \frac{dv}{2v+10} = -\int_0^t dt \quad \Rightarrow \quad \frac{1}{2} \ln(2v+10) \Big]_{10}^{v} = -t$$

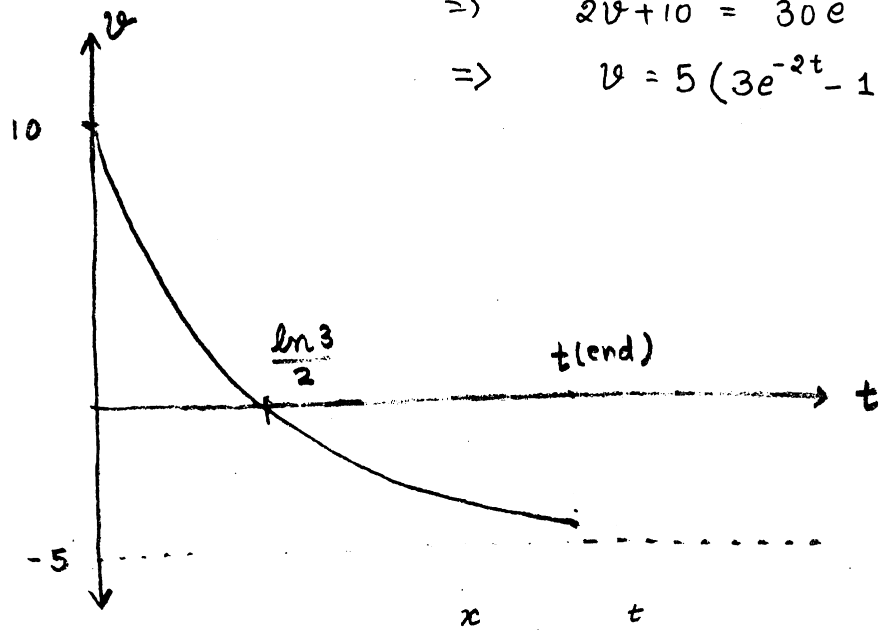
initial condition $\longrightarrow$ 10

$$\Rightarrow \quad \ln\left(\frac{2v+10}{30}\right) = -2t$$

$$\Rightarrow \quad 2v + 10 = 30 e^{-2t}$$

$$\Rightarrow \quad v = 5(3e^{-2t} - 1)$$

$$\boxed{\begin{aligned} &v = 0 \\ \Rightarrow\ &e^{-2t} = 1/3 \\ &2t = \ln 3 \\ &t = \tfrac{1}{2}\ln 3 \end{aligned}}$$



c) $\displaystyle \frac{dx}{dt} = 5(3e^{-2t} - 1) \quad \Rightarrow \quad \int_0^x dx = 5\int_0^t (3e^{-2t} - 1)\, dt$

$$\Rightarrow \quad x = 5\left\{ \frac{3e^{-2t}}{-2} - t \right\}_0^t \quad \Rightarrow \quad x = 5\left[-\frac{3}{2}e^{-2t} - t + \frac{3}{2} \right]$$

$$x = 5\left[\frac{3}{2}(1 - e^{-2t}) - t \right]$$

now t(end) occurs at $x = 0$

$$\Rightarrow \quad \boxed{\frac{3}{2}(1 - e^{-2\,t(end)}) - t(end) = 0}$$

is to be solved for t(end)

now use the calculator!

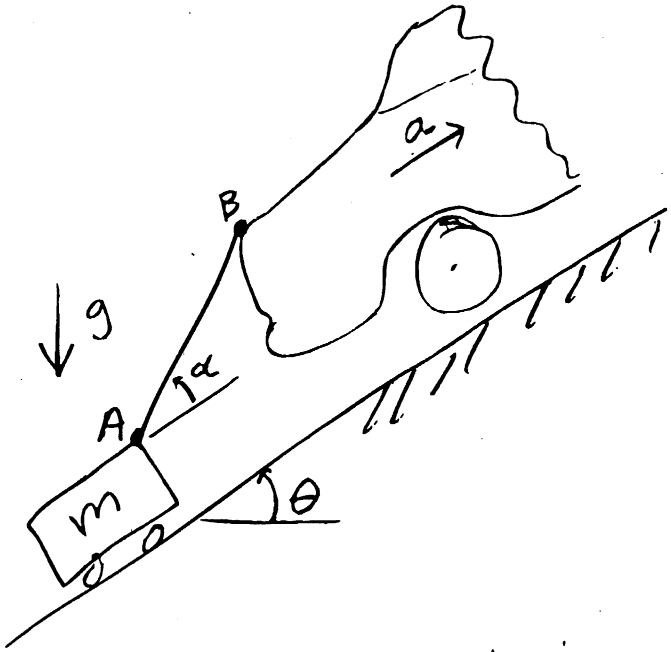
7

3) **(25 pt)** A car drags a cart. A car with known acceleration a accelerates up a hill dragging a cart.

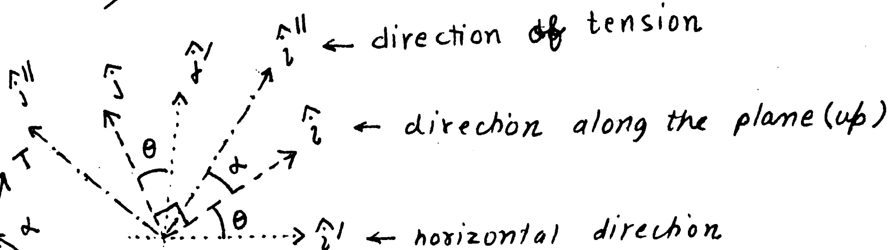
a) (15 points) Assume no friction. Find the force of the ground on the cart. Answer in terms of some or all of m, g, a, θ, α and L_{AB}.

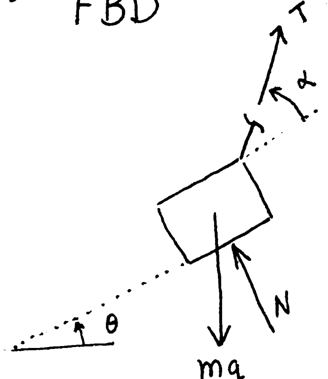
[Your score is the better of the two scores from part (a) and part (b).]

b) (25 points) Assume there is friction between the cart and the ground. Find the tension in the cable AB. Answer in terms of some or all of m, g, a, θ, α, L_{AB} and the friction coefficient μ (or the friction angle ϕ, defined as $\tan\phi = \mu$).



a) FBD





applying N2L on mass m gives

$$\vec{F}_{ext} = m\vec{a}$$

$$T\hat{i}'' - mg\hat{j}' + N\hat{j} = m(a_t\hat{i} + a_n\hat{j})$$

(tangential acceleration) ← normal acceleration $= 0$

$$\{\}\cdot\hat{i} \Rightarrow T\underbrace{(\hat{i}''\cdot\hat{i})}_{\cos\alpha} - mg\underbrace{(\hat{j}'\cdot\hat{i})}_{\cos(\frac{\pi}{2}-\theta)} + N\underbrace{(\hat{j}\cdot\hat{i})}_{0} = m a_t$$

$$\Rightarrow \quad T\cos\alpha - mg\sin\theta = m a_t = m a \quad — (1)$$

($a_t = a$ assuming the string remains tight.)

$$\{\ \}\cdot\hat{j} \Rightarrow \quad T\underbrace{(\hat{i}''\cdot\hat{j})}_{\cos(\frac{\pi}{2}-\alpha)} - mg\underbrace{(\hat{j}'\cdot\hat{j})}_{\cos\theta} + N\underbrace{(\hat{j}\cdot\hat{j})}_{1} = 0$$ ⑧

$$T\sin\alpha - mg\cos\theta + N = 0 \quad — ②$$

from ②
$$N = mg\cos\theta - T\sin\alpha \quad — ③$$

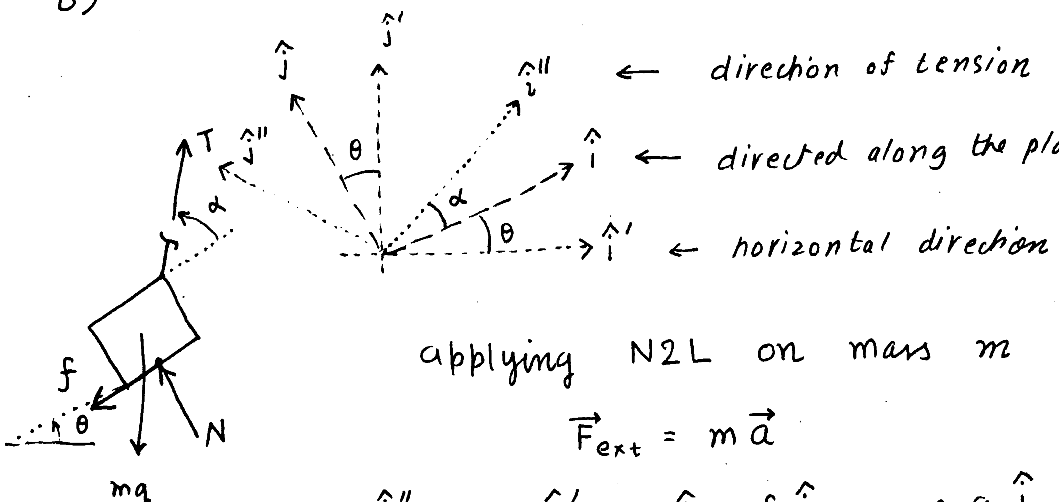
from ①
$$T = \frac{mg\sin\theta + ma}{\cos\alpha}$$

Substituting in ③

$$N = mg\cos\theta - \frac{\sin\alpha}{\cos\alpha}(mg\sin\theta + ma)$$

$$\boxed{N = m\left[g(\cos\theta - \tan\alpha\sin\theta) - a\tan\alpha\right]}$$

b) AN extra frictional force appear in this part !



applying N2L on mass m gives

$$\vec{F}_{ext} = m\vec{a}$$

$$T\hat{i}'' - mg\hat{j}' + N\hat{j} - f\hat{i} = ma\hat{i} \qquad \text{noting } f = \mu N$$

$$\{\ \}\cdot\hat{i} \quad T\underbrace{(\hat{i}''\cdot\hat{i})}_{\cos\alpha} - mg\underbrace{(\hat{j}'\cdot\hat{i})}_{\cos(\frac{\pi}{2}-\theta)} + N\underbrace{(\hat{j}\cdot\hat{i})}_{0} - \mu N\underbrace{(\hat{i}\cdot\hat{i})}_{1} = ma\underbrace{(\hat{i}\cdot\hat{i})}_{1}$$

$$\Rightarrow \quad T\cos\alpha - mg\sin\theta - \mu N = ma \quad — ①$$

$$\{\ \}\cdot\hat{j} \quad T\underbrace{(\hat{i}''\cdot\hat{j})}_{\cos(\frac{\pi}{2}-\alpha)} - mg\underbrace{(\hat{j}'\cdot\hat{j})}_{\cos\theta} + N\underbrace{(\hat{j}\cdot\hat{j})}_{1} - \mu N\underbrace{(\hat{i}\cdot\hat{j})}_{0} = ma\underbrace{(\hat{i}\cdot\hat{j})}_{0}$$

$$\Rightarrow \quad T\sin\alpha - mg\cos\theta + N = 0 \quad — ②$$

from ②
$$N = mg\cos\theta - T\sin\alpha$$

9

substituting the value of N in (1)

$$T\cos\alpha - mg\sin\theta - \mu(mg\cos\theta - T\sin\alpha) = ma$$

$$\Rightarrow \quad T(\cos\alpha + \mu\sin\alpha) = ma + mg\sin\theta + \mu mg\cos\theta$$

$$\boxed{T = \frac{m(a + g\{\sin\theta + \mu\cos\theta\})}{\cos\alpha + \mu\sin\alpha}}$$

Note :- from (2) and answer above

$$N = mg\cos\theta - T\sin\alpha$$

$$= mg\cos\theta - \frac{m\sin\alpha(a + g\{\sin\theta + \mu\cos\theta\})}{\cos\alpha + \mu\sin\alpha}$$

put $\mu = 0$ to get the answer to part a)

$$N(\mu=0) = mg\cos\theta - m\tan\alpha(a + g\sin\theta)$$

$$= m[g(\cos\theta - \tan\alpha\sin\theta) - a\tan\alpha]$$

which is what we found in a) !

Your Name: MANISH AGARWAL

Section time: ______________________

T&AM 203 Prelim 2

Tuesday October 24, 2006

Draft October 24, 2006

3 problems, 25^{+} points each, and 90^{+} minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	4:	/25
Problem	5:	/25
Problem	6:	/25

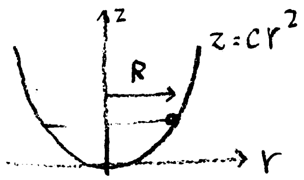
4) (25 pt) **Particle sliding in circles in a parabolic bowl.** As if in a James Bond adventure (in a big slippery Cornell-managed radio-telescope bowl in Puerto Rico), a particle-like human with mass m is sliding with negligible friction around in level circles at speed v. The equation describing the bowl is $z = CR^2 = C(x^2 + y^2)$

a) (20 points) Find v in terms of any or all of R, g, and C.

b) (5 points) Now say you are given ω, C and g. Find v and R if you can. Explain any oddities.

Sol

(a)



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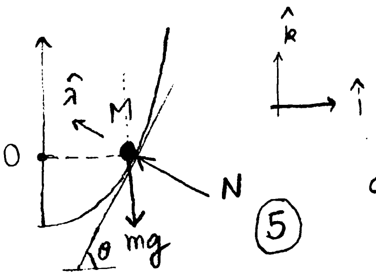


5

mass M executes circular motion about center O.

3

note $\quad \tan\theta = \left.\dfrac{dz}{dr}\right|_{r=R} = 2CR \quad$ — 1

also $\quad \hat{\lambda} = \cos\theta\,\hat{k} - \sin\theta\,\hat{i} \quad$ — 2

applying N2L

$$\vec{F}_{ext} = m\vec{a}$$

$$-mg\hat{k} + N\hat{\lambda} = m(\underbrace{a_z}_{0}\hat{k} + \overset{-\frac{v^2}{R}}{\overset{\parallel}{a_r}}\hat{i})$$

$\because$ motion in Level circle

$$\Rightarrow \quad -mg\hat{k} + N(\cos\theta\hat{k} - \sin\theta\hat{i}) = -\frac{mv^2}{R}\hat{i} \qquad [\text{using } 2]$$

$\{\ \}\cdot\hat{k} \Rightarrow \quad -mg + N\cos\theta = 0 \Rightarrow N\cos\theta = mg$ — 3 5

$\{\ \}\cdot\hat{i} \Rightarrow \quad -N\sin\theta = -\dfrac{mv^2}{R} \Rightarrow N\sin\theta = \dfrac{mv^2}{R}$ — 4 5

dividing 4 by 3 and using 1

$$\tan\theta = 2CR = \frac{v^2}{Rg}$$

$$\Rightarrow \boxed{v = R\sqrt{2Cg}} \quad 2$$

(b) ω, C, g given, v and R to be found

(2) $\omega = \frac{v}{R} = \sqrt{2Cg}$ from answer of part (a)

$\Rightarrow$ v and R can have any values untill

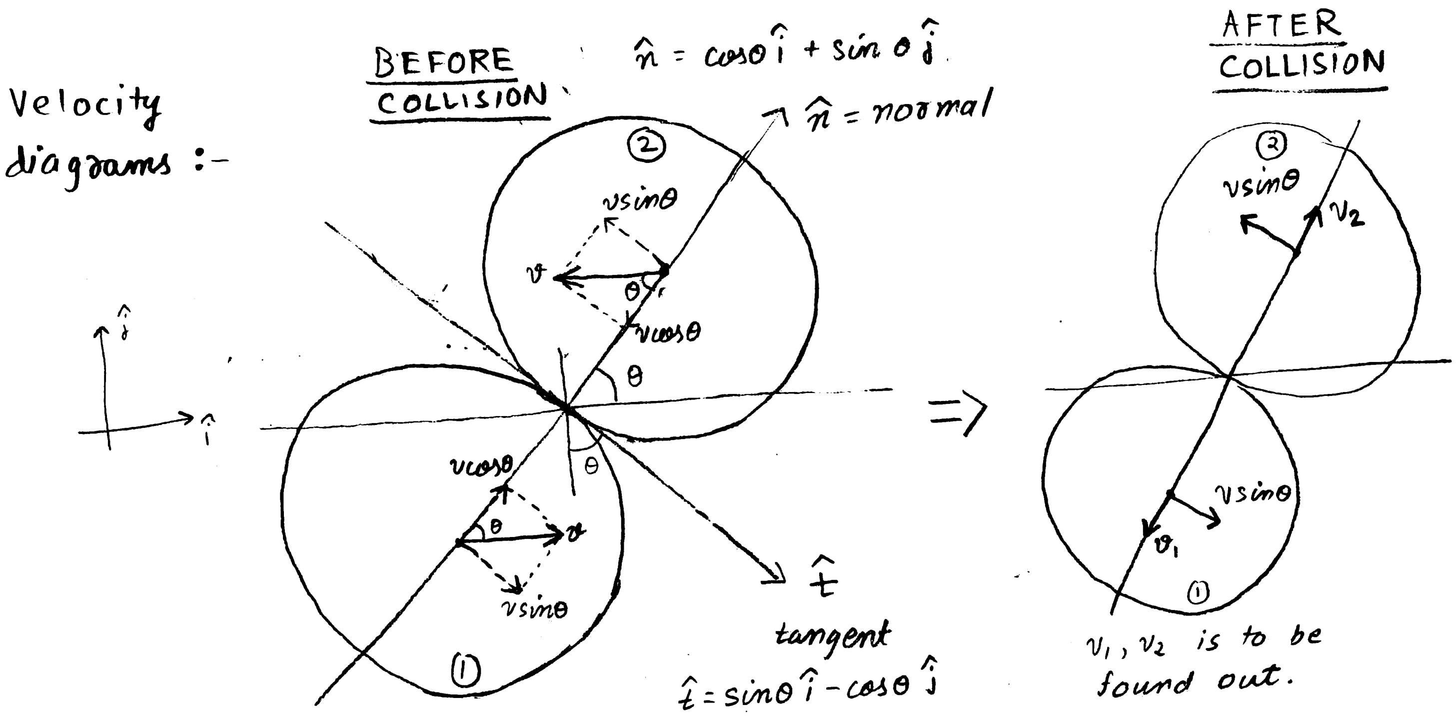
(3) $\frac{v}{R} = \sqrt{2Cg} = \omega$

$\uparrow$ $\uparrow$

this has to match in the given data.

5) (25 pt) Collision. Two equal mass m spherical particles have a frictionless collision with coefficient of restitution e. Before the collision their two velocities are $\underline{\mathbf{v}}_1^- = v\,\hat{\mathbf{i}}$ and $\underline{\mathbf{v}}_2^- = -v\,\hat{\mathbf{i}}$. The normal to their common tangent plane at contact is $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$. In terms of some or all of $v, m, e, \theta, \hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, find the velocity of particle 2 after the collision.

Velocity diagrams :-



Since impact is only in $\hat{n}$ direction, only normal component of velocities change.

by conservation of momentum for system in $\hat{n}$ direction

$$mv\cos\theta - mv\cos\theta = mV_2 - mV_1$$

(10) $\Rightarrow \boxed{V_1 = V_2}$ — ①

by definition of coefficient of restitution

$$e = \frac{\text{velocity of seperation}}{\text{velocity of approach}}$$

$$e = \frac{v_1 + v_2}{V\cos\theta + V\cos\theta} = \frac{v_2}{V\cos\theta} \qquad [\ v_1 = v_2 \text{ from ① }]$$

(10)

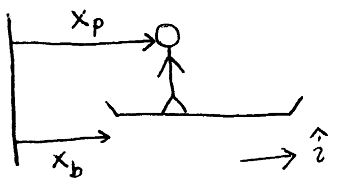
$$\Rightarrow \quad v_2 = eV\cos\theta$$

$\Rightarrow$ velocity of particle ② after impact $= -v\sin\theta\,\hat{t} + eV\cos\theta\,\hat{n}$

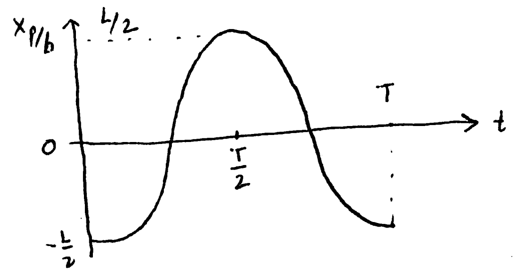
$= -v\sin\theta\,(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) + ev\cos\theta\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

(5) $\boxed{v\left\{(-\sin^2\theta + e\cos^2\theta)\,\hat{i} + \sin\theta\cos\theta\,(e+1)\,\hat{j}\right\}}$

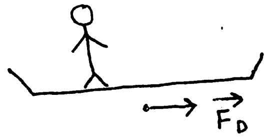
6) (25 pt) A person mass m_p walks up and back, all the way to the bow and to the stern, in a boat mass m_b. The person walks continuously and repeatedly, over and over and over again, moving relative to the boat sinusoidally in time, with period T. The length of the boat is L and the drag force on the boat from the water is $F = cv_b$. After a while the boat just moves back and forth also. How far does the boat go back and forth? (That is, the bow of the boat goes back and forth between two points, what is the distance between those two points?) Answer in terms of some or all of m_p, m_b, L and T.



motion of person wrt boat



FBD



3

$$x_{P/b} = -\frac{L}{2}\cos\left(\frac{2\pi}{T}t\right)$$

$$\Rightarrow v_{P/b} = \frac{2\pi}{T}\frac{L}{2}\sin\left(\frac{2\pi}{T}t\right) \qquad \text{let } \frac{2\pi}{T} = \omega$$

Material property

$$\vec{F}_D = -c\vec{V}_b \qquad 3$$

$$\Rightarrow a_{P/b} = \frac{L}{2}\omega^2\cos(\omega t) \quad \text{— } \circledast$$

5

LMB $\quad \Sigma\vec{F} = \dot{\vec{L}}$

$$\vec{F}_D = m_p\vec{a}_p + m_b\vec{a}_b$$

$$-c\vec{V}_b = m_p\vec{a}_p + m_b\vec{a}_b \qquad 3$$

by, <u>KINEMATICS</u>

$$\vec{a}_p = \vec{a}_{P/b} + \vec{a}_b$$

given by $\circledast$

$$\Rightarrow \left\{ -c\vec{V}_b = (m_p + m_b)\vec{a}_b + m_p\vec{a}_{P/b} \right\}$$

$$\{\ \}\cdot\hat{\imath} \Rightarrow \quad -cV_b = (m_p+m_b)\dot{v}_b + m_p\frac{L}{2}\omega^2\cos\omega t \qquad [a_b = \dot{V}_b]$$

$$\dot{v}_b + \underbrace{\frac{c}{m_p+m_b}}_{C_1} V_b = -\underbrace{\frac{m_p\frac{L}{2}\omega^2}{m_p+m_b}}_{A}\cos\omega t$$

5

$$\Rightarrow \boxed{\dot{v}_b + C_1 V_b = A\cos\omega t} \text{ — } (1)$$

$\omega = 2\pi/T$

$C_1 = c/(m_p + m_b)$

$$A = -\frac{m_p\frac{L}{2}\omega^2}{m_p+m_b}$$

we need to solve for V_b and then x_b

The exponential part of solution $\to 0$ as $t \to \infty$. For the oscillating part guess the solution of the form

$$V_b = B\cos\omega t + D\sin\omega t$$

Plug it in ① to get

$$-B\omega\sin\omega t + D\omega\cos\omega t + BC_1\cos\omega t + DC_1\sin\omega t = A\cos\omega t$$

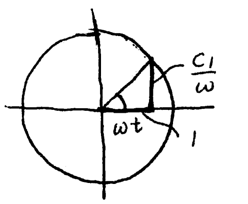
equating $\sin\omega t$ and $\cos\omega t$ terms

$$-B\omega + DC_1 = 0$$
$$D\omega + BC_1 = A$$

Solving it we get $\quad D = \dfrac{A\omega}{\omega^2 + C_1^2} \quad \& \quad B = \dfrac{AC_1}{\omega^2 + C_1^2}$

$$\Rightarrow \quad V_b = \frac{A}{\omega^2 + C_1^2}\left(C_1\cos\omega t + \omega\sin\omega t\right)$$

$$\Rightarrow \quad X_b = \frac{A}{\omega^2 + C_1^2}\left(\frac{C_1}{\omega}\sin\omega t - \cos\omega t\right) + \text{constant of integration}$$



$$\Rightarrow \quad \text{amplitude} = \frac{A}{\omega^2 + C_1^2}\sqrt{\left(\frac{C_1}{\omega}\right)^2 + 1} = \frac{A}{\omega\sqrt{C_1^2 + \omega^2}}$$

so, distance boat goes back and forth is $2 \times |\text{Amplitude}|$

$$\left|\frac{2A}{\omega\sqrt{C_1^2 + \omega^2}}\right| \qquad ⑥$$

which is

$$\boxed{\frac{\left(\dfrac{m_p}{m_p + m_b}\right)\left(\dfrac{2\pi}{T}\right)L}{\sqrt{\left(\dfrac{c}{m_p + m_b}\right)^2 + \left(\dfrac{2\pi}{T}\right)^2}}} \quad \text{Ans.}$$

in dimensionless form dividing by $\frac{2\pi}{T}$ througout

$$\boxed{\frac{\left(\dfrac{m_p}{m_p + m_b}\right)L}{\sqrt{\left[\dfrac{CT}{2\pi(m_p + m_b)}\right]^2 + 1}}} \quad \text{Ans}$$

Solutions to Homework exam

Your Name: Manish Agarwal
ma327

Section time: ____________

T&AM 203 Homework Exam

Tuesday Dec 12, 2006

Draft December 12, 2006

4 problems, 25^+ points each, 4 hours.

If you can do all the homework you are guaranteed a grade of C. These 4 problems are based on homework problems (see next page), or parts of homework problems, with slight changes so that memorizing answers won't help. If you can do 3 of them *fully correctly* (good work, correct answer) in 4 hours you are guaranteed a grade of at least C.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, or books allowed. You can bring a one-sided formula sheet, but not any worked out HW etc.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ `= 18`" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 5/91:	/25
Problem 6/15:	/25
Problem 6/79:	/25
Problem 6/168:	/25

Replace the text from the book (copied with the figures on the next page) with the problem statements below. For all problems all quantities are described with letters and not numbers. Answers should be expressed with letters. Don't use the numbers given with the book problem statement. Ignore the numbers in the figures.

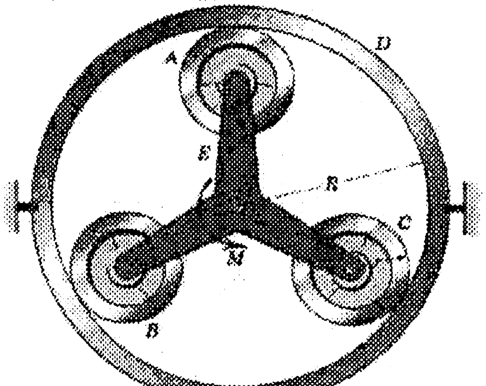
5/91 Given $\dot{a}, L_{CD}, L_{AB}, a$ and b find $\dot{b}$. (Note $\underline{\mathbf{v}}_A = -\dot{a}\,\hat{\mathbf{i}}$.)

6/15 The car has mass m and gravity points down with constant g. The coefficient of friction between wheels and road is μ. For rear wheel drive what is the maximum possible car acceleration? (Answer in terms of some or all of a, b, h, m, g and μ.)

6/79 A uniform disk with radius R and mass m rolls down a slope θ. The friction coefficient μ is large enough so the disk rolls without slipping. Gravity g points down. Find the component of the force that acts on the disk from the ramp that is tangent to the ramp. Answer in terms of some or all of R, m, g, θ and μ.

6/168 The angular velocity and acceleration of the spider are given as ω_s and α_s. Find the acceleration of a point on one of the planet gears that is, at the instant in question, in contact with the ring gear. Answer in terms of some or all of R, r, ω_s, α_s and unit vectors you clearly define.

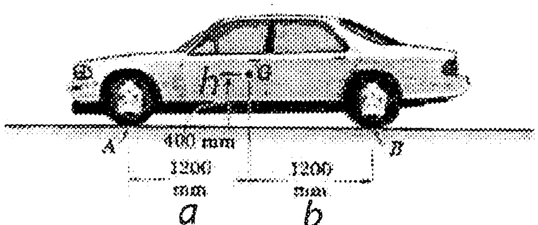
6/168 A planetary gear system is shown, where the gear teeth are omitted from the figure. Each of the three identical planet gears A, B, and C has a mass of 0.8 kg, a radius $r = 50$ mm, and a radius of gyration of 30 mm about its center. The spider E has a mass of 1.2 kg and a radius of gyration about O of 60 mm. The ring gear D has a radius $R = 150$ mm and is fixed. If a torque $M = 5$ N·m is applied to the shaft of the spider at O, determine the initial angular acceleration α of the spider.



Problem 6/168

6/15 The 1650-kg car has its mass center at G. Calculate the normal forces N_A and N_B between the road and the front and rear pairs of wheels under conditions of maximum acceleration. The mass of the wheels is small compared with the total mass of the car. The coefficient of static friction between the road and the rear driving wheels is 0.8.

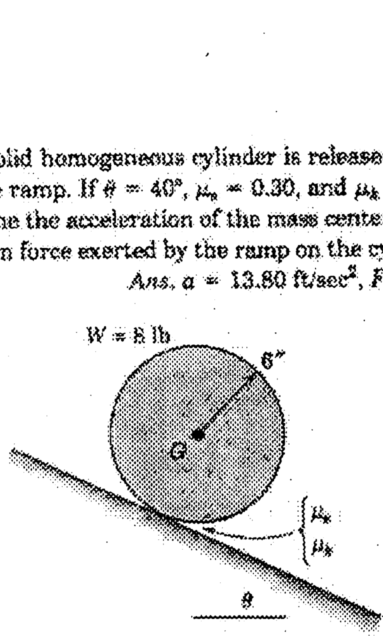
Ans. $N_A = 6.85$ kN, $N_B = 9.34$ kN



Problem 6/15

6/79 The solid homogeneous cylinder is released from rest on the ramp. If $\theta = 40°$, $\mu_s = 0.30$, and $\mu_k = 0.20$, determine the acceleration of the mass center G and the friction force exerted by the ramp on the cylinder.

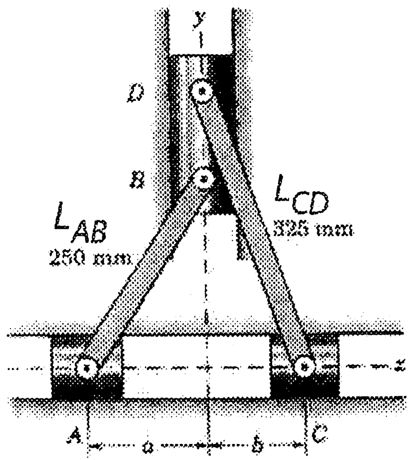
Ans. $a = 13.80$ ft/sec^2, $F = 1.714$ lb



Problem 6/79

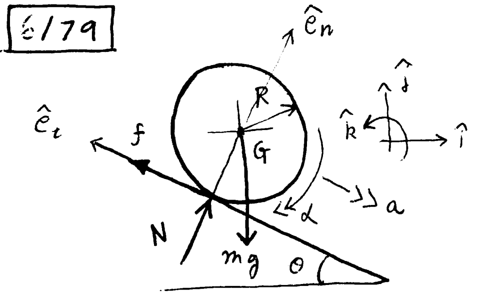
►5/91 At the instant represented, $a = 150$ mm and $b = 125$ mm, and the distance $a + b$ between A and C is decreasing at the rate of 0.2 m/s. Determine the common velocity v of points B and D for this instant.

Ans. $v = 0.0536$ m/s



Problem 5/91

6/179



LMB: $-mg\hat{j} + N\hat{e}_n + f\hat{e}_t = -ma\hat{e}_t$

$\{\ \}\cdot\hat{e}_t \Rightarrow -mg\sin\theta + f = -ma$

$-f + mg\sin\theta = ma$ — (1)

$\{\ \}\cdot\hat{e}_n \Rightarrow \quad N = mg\cos\theta$ — (2)

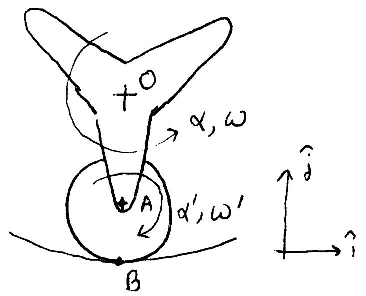
AMB: about G

$-fR = -I\alpha \qquad fR = \frac{1}{2}mR^2\alpha$ — (3)

No slip condition: $\alpha R = a$ — (4)

(1), (3) and (4) $\Rightarrow$ $\boxed{f = \dfrac{mg\sin\theta}{3}}$ Ans

6/168



$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$= \cancel{\vec{a}_O}_{\,0} + \vec{a}_{A/O} + \vec{a}_{B/A}$$

$$= \vec{\alpha}\times\vec{OA} + \vec{\omega}\times(\vec{\omega}\times\vec{OA}) + \vec{\alpha}'\times\vec{AB} + \vec{\omega}'\times(\vec{\omega}'\times\vec{AB})$$

$$= \alpha(R-r)\hat{i} + \omega^2(R-r)\hat{j} - \alpha' r\hat{i} + \omega'^2 r\hat{j}$$

$$= [\alpha(R-r) - \alpha' r]\hat{i} + [\omega^2(R-r) + \omega'^2 r]\hat{j} \qquad 10$$

by no slip condition, tangential acceleration is zero.

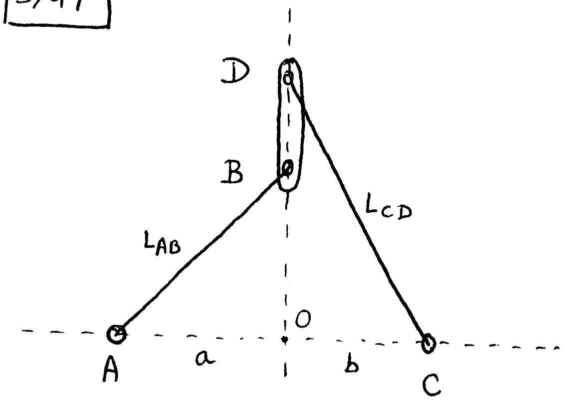
$\vec{a}_B\cdot\hat{i} = 0 \Rightarrow \alpha(R-r) - \alpha' r = 0$

$\Rightarrow \omega(R-r) - \omega' r = 0 \Rightarrow \omega' = \dfrac{\omega(R-r)}{r}$ 10

$$\Rightarrow \vec{a}_B = \left\{\omega^2(R-r) + \left[\frac{\omega(R-r)}{r}\right]^2\right\}\hat{j} \qquad 5$$

$$= \boxed{\frac{\omega^2(R-r)R}{r}\hat{j}} \text{ Ans}$$

5/91



$\mathring{a}$ given

to find $\mathring{b}$

$$OB = \sqrt{L_{AB}^2 - a^2}$$

$$OD = \sqrt{L_{CD}^2 - b^2}$$

$$DB = OD - OB = \sqrt{L_{CD}^2 - b^2} - \sqrt{L_{AB}^2 - a^2}$$

$$\frac{d(DB)}{dt} = 0 \quad \text{since } DB \text{ is of fixed length}$$

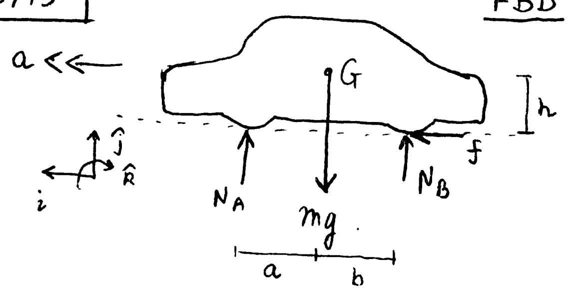
$$\Rightarrow \quad \frac{1}{\cancel{2}} \frac{-\cancel{2} b \mathring{b}}{\sqrt{L_{CD}^2 - b^2}} - \frac{1}{\cancel{2}} \frac{-\cancel{2} a \mathring{a}}{\sqrt{L_{AB}^2 - a^2}} = 0$$

$$\Rightarrow \quad \boxed{\mathring{b} = \frac{a}{b} \sqrt{\frac{L_{CD}^2 - b^2}{L_{AB}^2 - a^2}} \, \mathring{a}}$$

Ans

6/15

FBD



LMB: $f\hat{i} + N_A\hat{j} + N_B\hat{j} - mg\hat{j} = ma\,\hat{i}$

$\{\} \cdot \hat{i}$ $\quad f = ma \quad$ — ①

$\{\} \cdot \hat{j}$ $\quad N_A + N_B = mg \quad$ — ②

AMB: about point G

$$N_A a + f h = N_B b \quad — ③$$

acceleration is max for max f

$\Rightarrow \quad f = \mu N_B \quad$ — ④

②, ③ and ④ $\Rightarrow \quad N_B = \dfrac{mga}{a+b-\mu h}$

$$\Rightarrow \quad a = \frac{f}{m} = \frac{\mu N_B}{m} = \frac{\mu g a}{a+b-\mu h} = \boxed{\frac{\mu g}{1 + \frac{b}{a} - \mu \frac{h}{a}}}$$

Ans

Your Name: Joe Burns

Your TA: Andy Ruina

T&AM 203 FINAL EXAM

Wednesday May 17, 2000

Draft May 9, 2000

4 problems, 100 points, and 150 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

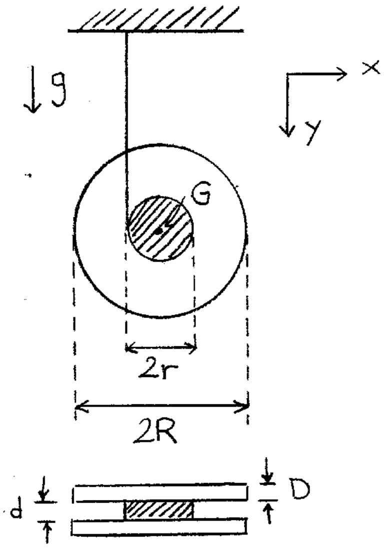
b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	3:	/25
TOTAL:		enough/100 to pass

1)(25 pts) Yo-yo. A yo-yo of mass $2m$ is made of two identical disks (mass m, radius R, thickness D) glued on either side of a massless spindle (radius r, thickness d). A string is wrapped around the spindle and unwinds without friction. The string has total length L and is infinitesimally thin and massless. G is at the yo-yo's center of mass.



a) (3 pts) Does G move in the x-direction as the yo-yo falls and unwinds? Why or why not?

b) (6 pts) Find G's vertical acceleration. Comment on the two cases:
i) $R << r$ and ii) $R >> r$,

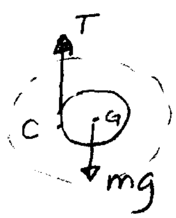
c) (4 pts) Find the tension in the string.

d) (5 pts) Write an expression for the total kinetic energy of the yo-yo when G's speed is v.

e) (5 pts) If $r << L$ and the yo-yo starts from rest, find v when the string is fully unwound.

f) (2 pts) Under what circumstances will the yo-yo rewind completely?

FBD



a.) Since $\Sigma \underline{F}$ is only in the $\hat{j}$ direction, $\boxed{\underline{a}_G \text{ is only in } \hat{j}}$, $\underline{v}_0 = 0$ ∴ no x motion.

b.) Method I

$\Sigma \underline{M}_C = \dot{\underline{H}}_C$

$r\hat{i} \times mg\hat{j} = \dot{\underline{H}}_G + \underline{r} \times m\underline{a}$

$\cancel{m} rg\,\hat{k} = \frac{1}{2}\cancel{m}R^2\dot{\omega}\,\hat{k} + r\hat{i} \times \cancel{m}a\hat{j}$

$rg = \frac{1}{2}R^2\frac{a}{r} + ra = \left(\frac{1}{2}\frac{R^2}{r} + \frac{r^2}{r}\right)a$

∴ $\boxed{a = \dfrac{g}{1 + \frac{1}{2}\frac{R^2}{r^2}}}$ $\approx g$ if $R << r$; $a << g$ if $r << R$

Method II

Kinematics: $\dot{\omega} r = a$ —(1)

$(LMB)\cdot\hat{j} \Rightarrow -T + mg = ma$ —(2)

$AMB_G \Rightarrow \frac{1}{2}mR^2\dot{\omega} = Tr$ —(3)

3 eqns for the 3 unknowns $\dot{\omega}, a, T$

c.) Plug a back into (2) $T = m(g-a) = mg\left(\dfrac{1+\frac{1}{2}\frac{R^2}{r^2} - 1}{1+\frac{1}{2}\frac{R^2}{r^2}}\right) = \boxed{\dfrac{mg}{1+\frac{2r^2}{R^2}} = T}$

d.) For planar motion

$KE = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$ [or $= \frac{1}{2}I_C\omega^2$ since in pure rotation about C]

$= \frac{1}{2}mv^2 + \frac{1}{2}\left[\frac{1}{2}mR^2\frac{v^2}{r^2}\right] = \boxed{\frac{mv^2}{2}\left[1 + \frac{1}{2}\frac{R^2}{r^2}\right] = KE}$

e.) Use conservation of energy with PE measured from top

We ignore r & R compared to L

$PE + KE = -mgR + 0 = -\cancel{m}gL + \frac{1}{2}\cancel{m}v^2\left(1 + \frac{1}{2}\frac{R^2}{r^2}\right)$

E@ TOP E@ BOTTOM

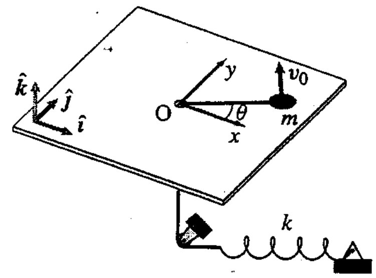
$\Rightarrow \boxed{v = \sqrt{\dfrac{2g(L-R)}{1+\frac{1}{2}\frac{R^2}{r^2}}}}$

could also get by $v = \sqrt{2aL}$

f.) As long as energy is conserved (which it never will be in a real system), the yo-yo will rewind perfectly.

loss in string, slip & "bounce" at bottom.

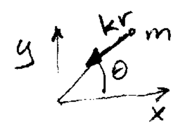
2)(25 pts) Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.



a) (2 pts) The force of the plane on the particle is $mg\hat{k}$.
b) (2 pts) $\ddot{x} + \frac{k}{m}x = 0$
c) (2 pts) $\ddot{y} + \frac{k}{m}y = 0$
d) (3 pts) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{r}|$
e) (2 pts) $r =$ constant
f) (3 pts) $\dot{\theta} =$ constant
g) (3 pts) $r^2\dot{\theta} =$ constant
h) (2 pts) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant
i) (3 pts) The trajectory is a straight line segment.
j) (3 pts) The trajectory is a circle.

a.) Since the particle has no vertical ($\hat{k}$) motion, $\Sigma F_z = 0$ $\downarrow mg$, $\uparrow N$ $\Rightarrow N = mg$ Always true

b.), c.) Writing LMB in horizontal plane and using Cartesian coords,



$$\Sigma \underline{F} = m\underline{a} \Rightarrow -k\underline{r} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) = -k(x\hat{\imath} + y\hat{\jmath})$$

$$\therefore \boxed{\begin{array}{l}\ddot{x} + \frac{k}{m}x = 0 \\ \ddot{y} + \frac{k}{m}x = 0\end{array}}$$ always true as long as a frictionless

d.) If we express $\underline{a}$ in polar coords

$$-kr\hat{e}_r = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right] \quad *$$

$$\hat{e}_r \cdot (\text{LMB}) \Rightarrow -kr = m(\ddot{r} - r\dot{\theta}^2) \Rightarrow \boxed{\ddot{r} + \frac{k}{m}r = 0 \text{ if } \dot{\theta} = 0}$$ motion is purely radial

e.) $r =$ constant when $m(-r\dot{\theta}^2) = -kr$ (part of radial $*$)

$$\therefore \dot{\theta} = \sqrt{\frac{k}{m}} \text{ and is const}$$

f.) This is a variant of e.), $\dot{\theta}$ will be constant if it is $= \sqrt{\frac{k}{m}}$ and r is constant

g.) $r^2\dot{\theta}$ is angular momentum per unit mass (i.e., for planar motion $H = mr(r\dot{\theta})$; or $r^2\dot{\theta} = \frac{H}{m}$)

From $*$, we see the $\hat{e}_\theta$ component is $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{2\cancel{r}\dot{r}\dot{\theta} + r^{\cancel{2}}\ddot{\theta}}{\cancel{r}} = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

Since there is no $\hat{e}_\theta$ force component, this is always true $\Rightarrow r^2\dot{\theta} = \text{const}$

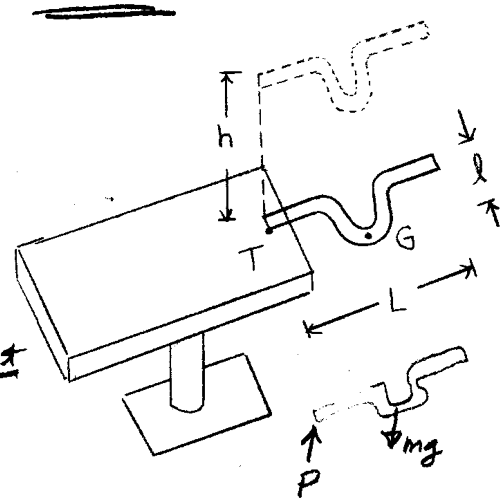
h.) This is twice total energy $2\left(\frac{1}{2}mv^2 + \frac{1}{2}kr^2\right)$. Thus always true for this conservative system.

i.) As part d) showed, get straight line motion if purely radial motion

j.) As parts e) and f) mentioned, circular motion is possible when $\dot{\theta} = \sqrt{\frac{k}{m}}$

3)(25 pts) The problems below (a-d) are independent.

a) (5 pts) A falling wire. A U-shaped wire (of the given dimensions) falls from a height h without rotation and strikes a tabletop at T completely inelastically. Briefly defend your answers to the following questions.



During impact:

i) (1 pt) Is the wire's linear momentum conserved?
ii) (1 pt) Is the wire's angular momentum about G constant?
iii) (1 pt) Is the wire's angular momentum about T constant?
iv) (1 pt) Is the wire's angular velocity conserved?
v) (1 pt) Is the wire's total mechanical energy (kinetic + potential) constant?

i.) LMB $\Rightarrow$ $\underline{F} = \frac{d}{dt} m\underline{v}$, External forces act ∴ linear momentum not const

ii.) AMB $\Rightarrow$ $\underline{M}_{/G} = \dot{\underline{H}}_{/G}$ $\underline{P}$ has moment about G ∴ $\underline{H}_{/G} \neq$ constant

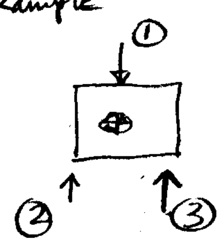
iii) The impulsive force $\underline{P}$ has no moment about T; we can ignore mg in comparison to P and thus $\underline{H}_T$ is approximately constant

iv.) It wasn't rotating beforehand and will afterward (due to $\underline{M}_G$). $\underline{\omega}$ not conserved.

v.) Energy is lost in collisions generally unless they are elastic. This is an inelastic collision ∴ not constant E
(heating, sound)

b) (5 pts) Mars Polar Lander. Last December when the Mars Polar Lander was lost, some blamed its simple thrusters (devices that eject gas and thus are capable of providing an impulse in a single direction). Argue using dynamical principles, the minimum number of thrusters required to have complete three-dimensional control.

The full dynamical description of a body is given by the 3-D position of the CM (LMB) plus three angles to give its orientation (AMB). For each of these, you need to be able to go + and −. We can control the angular orientations with the same thrusters. For example

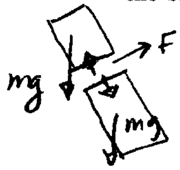


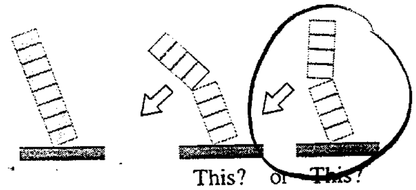
① for downward motion
②+③ for pure upward
①+③ for + rotation
②+① for − rotation

$$\boxed{3 \times 3 = 9 \text{ thrusters}}$$

c) **(5 pts) A falling tower.** Frequently parents will build a tower of blocks for their children. Just as frequently, kids knock them down. In falling (even when they start to topple aligned), these towers invariably break in two (or more) pieces at some point along their length. Why does this occur? What condition is satisfied at the point of the break? Will the stack bend towards or away from the floor after the break?

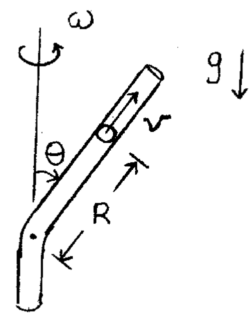
$\dot{\theta}, \ddot{\theta}$





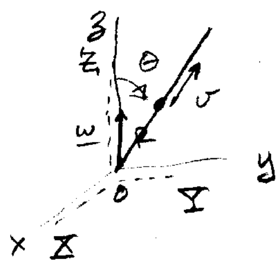
The tower will rotate as a single piece only as long as the forces between the various blocks are compressive and friction sufficient to prevent sliding. Once they become tensile, it will separate since unglued blocks cannot support tension. Top block has (largest r) largest acceleration & least compressive force. Thus the top breaks free first. And then the others follow. The angular motion of the solid stack is given by $\underline{M} = I\dot{\omega}\hat{k}$ where $I \sim h^2$ by $M \sim h$. Thus taller stacks fall more slowly. Hence the break-aways will lag behind & will appear to bend away from the floor.

d) **(10 pts) A pea shooter.** A pea of mass m is being blown out of a tube at constant speed v. The tube itself is at a constant angle θ to the vertical and spins at constant angular velocity $\omega_o\mathbf{\hat{k}}$ (i.e., it sweeps out a cone). At the instant shown, the tube is in the yz-plane and the pea is at a distance R along the tube.

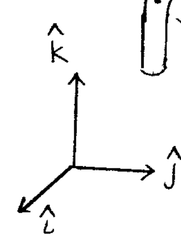


i) (7 pts) What is the pea's acceleration?

ii) (3 pts) What force acts on it?



Choose XYZ (inertial) as given. Fix moving system xyz such that it spins with the tube and is instantaneously parallel to XYZ (as shown).



$$\underline{a}_{pea} = \underline{a}_{rel} + \underline{a}_o + \underline{\omega} \times (\underline{\omega} \times \underline{R}) + \dot{\underline{\omega}} \times \underline{R} + 2\underline{\omega} \times \underline{v}$$

$\underline{a}_{rel} = 0$ pea moves with constant speed along tube

$\underline{a}_o = 0$ origins are together always

$\dot{\underline{\omega}} = 0$ moving system spins at constant rate

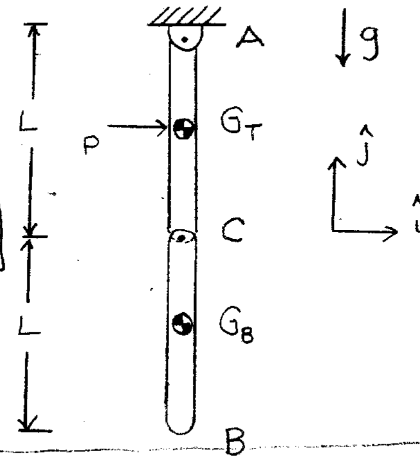
$$\therefore \underline{a}_{pea} = \omega\hat{k} \times \left(\omega\hat{k} \times R[\sin\theta\,\hat{j} + \cos\theta\,\hat{k}]\right) + 2\omega\hat{k} \times v(\sin\theta\,\hat{j} + \cos\theta\,\hat{k})$$

$$= \omega\hat{k} \times \omega R\sin\theta(-\hat{i}) - 2\omega v\sin\theta\,\hat{i}$$

$$\boxed{\underline{a}_{pea} = -\omega^2 R\sin\theta\,\hat{j} - 2\omega v\sin\theta\,\hat{i}}$$

LMB $\Rightarrow$ $\boxed{\underline{F}_{EXT} = m\,\underline{a}_{pea}}$ with $\underline{a}_{pea}$ given above. This force is caused by wall forces (contact) gravity and gas pressure.

4)(25 pts) **Double pendulum.** Two identical homogeneous slender bars (weight W, length L, frictionless hinges) hang vertically in a gravity field. They are initially at rest when a horizontal force $P\hat{\imath}$ is suddenly applied at the center of the top bar.



a) (10 pts) Write out expressions for the accelerations of the centers of mass G_T and G_B in terms of the angular motions of the bars.

b) (10 pts) Write down sufficient equations to solve for the reaction forces at A and C, and for the angular motions.

c) (5 pts) Describe how to use Matlab to solve these equations. You need not solve them.

Instantaneously,

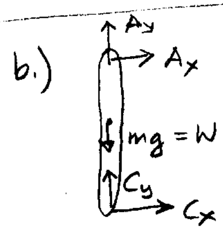
a.) $\underline{a}_{G_T} = \underline{\alpha}_T \times \frac{L}{2}(-\hat{j}) = \alpha_T \frac{L}{2}\hat{\imath}$

Does not include $\underline{\omega}\times(\underline{\omega}\times\frac{L}{2})$ because $\underline{\omega}=0$ initially

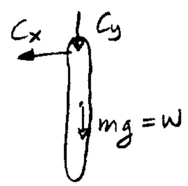
$\underline{a}_{B_T} = \underline{a}_C + \underline{a}_{B/C} = \alpha_T L\hat{\imath} + \alpha_B \hat{k}\times\frac{L}{2}(-\hat{j})$

$= \left(\alpha_T L + \alpha_B \frac{L}{2}\right)\hat{\imath}$

b.)



We have 6 unknowns : $\underline{A}$ & $\underline{C}$ (2 components each) and 2 αs and 6 eqns (LMB for top and bottom bars - 2 components each) and AMB for each bar.



Top : LMB $\sum \underline{F}_{EXT} = -W\hat{j} + \underline{A} + \underline{C} = m\underline{a}_{G_T}$

$$\boxed{\frac{W}{g}\alpha_{G_T}\frac{L}{2}\hat{\imath} = (A_x + C_x)\hat{\imath} + (A_y + C_y - W)\hat{j}}$$

AMB $\sum \underline{M}_{G_T} = \underline{\dot{H}}_{G_T} = \frac{1}{12}\frac{W}{g}L^2\dot{\omega}_{G_T}\hat{k}$

$$\boxed{(C_x - A_x)\frac{\cancel{L}}{2} = \frac{WL^{\cancel{2}}}{\cancel{12}_{6}g}\alpha_T}$$

Bottom

LMB $\sum \underline{F}_{EXT} = -W\hat{j} - \underline{C} = m\underline{a}_{G_B}$

$$\boxed{-C_x\hat{\imath} - (+W + C_y)\hat{j} = \frac{W}{g}L\left(\alpha_G + \frac{\alpha_B}{2}\right)\hat{\imath}}$$

AMB $\sum \underline{M}_{G_B} = \underline{\dot{H}}_{G_T} = \frac{W}{12g}L^2\alpha_B\hat{k}$

$$\boxed{C_x\frac{L}{2} = \frac{W}{12g}L^2\alpha_B}$$

ⓒ The boxed equations are 6 algebraic eqns for the unknowns $A_x, A_y, C_x, C_y, \alpha_T$ & α_B. We write them in matrix form $Ax = b$, where x is a column vector of these. and solve $x = A^{-1}B$. In Matlab we would write

$\underline{\underline{A}} = [6 ; 6 ; 6 ; 6 ; 6 ; 6]$; entries

$b = [; ; ; ; ;]$;

$x = A \backslash b$

"Solutions" *

Your Name: ANDY RUINA

Your TA: BURNS

T&AM 203 Prelim 1

Tuesday February 29, 2000 7:30 — 9:00+ PM

Draft February 26, 2000

3 problems, 100 points, and 90+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Two pages of formulas, from the front of the text, and a blank page for tentative scrap work are provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems poorly defined;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

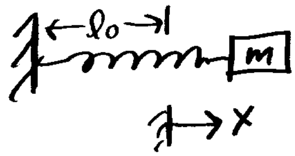
* The quotations are because often my "Solutions" have some remaining errors. (error in 1c corrected here)

I hope!

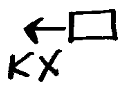
Problem	1:	30/30
Problem	2:	35/35
Problem	3:	35/35
TOTAL:		100 /100

1)(30 pts) The problems below are independent. MATLAB commands are not allowed except in part (c).

1a) (10 pts) A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? (Answer in terms of some or all of m, k, and d.) [Neglect gravity and friction.]



FBD:



LMB: $F = ma$

$\Rightarrow -kx = m\ddot{x}$

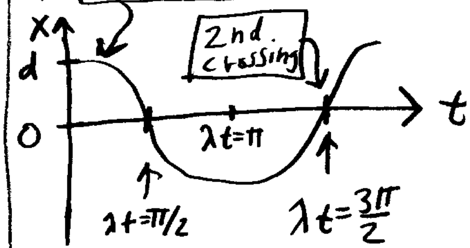
$\Rightarrow \ddot{x} + (k/m)x = 0$

$\Rightarrow x = A\cos(\lambda t) + B\sin(\lambda t)$

$\lambda = \sqrt{k/m}$

$x(0) = d \Rightarrow A = d$, $\dot{x}(0) = 0 \Rightarrow B = 0$

$\Rightarrow x = d\cos(\lambda t)$,



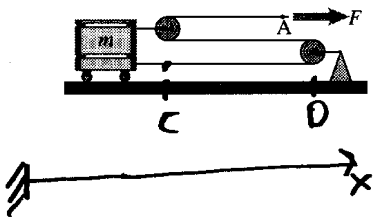
At second crossing

$\lambda t = 3\pi/2$

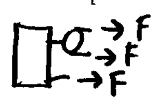
$\Rightarrow t = 3\pi/2\lambda$

$$\Rightarrow \boxed{t = \frac{3\pi}{2\sqrt{k/m}}}$$

1b) (10 pts) Assuming equal masses and equal forces in the two cases, what is the ratio of the acceleration of point A to that of point B? [assume massless ideal pulleys etc]



FBD



LMB

$3F = m\ddot{x}_c$

$\ddot{x}_c = 3F/m$ (1)

Kinematics

$const = \ell_{CA} + 2\ell_{CD}$

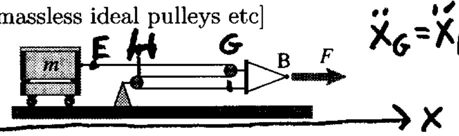
$= (x_A - x_c) + 2(x_D - x_c)$

$(\ddot{const}) = (\ddot{x}_A - \ddot{x}_c) + 2(\cancel{\ddot{x}_D} - \ddot{x}_c)$ $\to 0$

$0 = \ddot{x}_A - 3\ddot{x}_c$ (2)

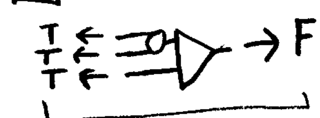
$\Rightarrow \ddot{x}_A = 3\ddot{x}_c$

(1)&(2) $\Rightarrow$ $\ddot{x}_A = 9F/m$ (3)



$\ddot{x}_G = \ddot{x}_B$

FBDs



$\Rightarrow T = F/3$

LMB $T = m\ddot{x}_E$

$F/3 = m\ddot{x}_E$ (4)

Kinematics

$const = \ell_{GE} + 2\ell_{GH}$

$= (x_G - x_E) + 2(x_G - x_H)$

$(\ddot{const}) = (\ddot{x}_G - \ddot{x}_E) + 2(\ddot{x}_G - \cancel{\ddot{x}_H})$ $\to 0$

$0 = 3\ddot{x}_G - \ddot{x}_E$

$\Rightarrow \ddot{x}_G = \ddot{x}_B = \ddot{x}_E/3$ (5)

(4)&(5) $\Rightarrow$ $\ddot{x}_B = F/9m$

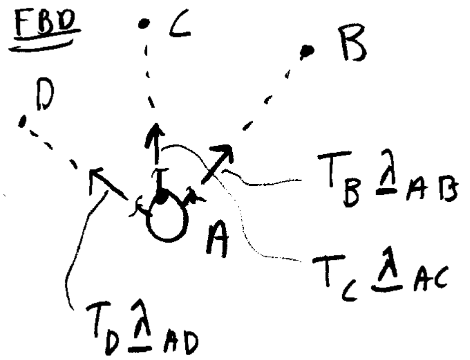
Comparison

$$\frac{\ddot{x}_A}{\ddot{x}_B} = \frac{9F/m}{F/9m}$$

$= \boxed{81}$ (6)

(Point B "feels" 81 times less massive than pt. A)

1c) (10 pts) In three-dimensional space with no gravity a particle with $m = 3\,\text{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\underline{\mathbf{a}} = (1\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}})\,\text{m/s}^2$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]



LMB $\quad T_B \underline{\lambda}_{AB} + T_C \underline{\lambda}_{AC} + T_D \underline{\lambda}_{AD} = m\underline{a}$

$$\Rightarrow \overbrace{\underbrace{\begin{bmatrix} [\underline{\lambda}_{AB}] & [\underline{\lambda}_{AC}] & [\underline{\lambda}_{AD}] \end{bmatrix}}_{\text{comps of unit vectors in columns}}}^{A} \begin{bmatrix} T_A \\ T_B \\ T_C \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underbrace{[A]\,[T] = m[a]}_{\text{need to solve for } [T]}$$

```
% a MATLAB script file to find 3 tensions
m   = 3;
a   = [ 1 2 3]';

rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';

uAB = rAB/norm(rAB); % norm gives vector magnitude
% You write the code below (4 to 5 lines).
% Don't copy any of the numbers above.
% Don't do any arithmetic on the side.
```

This code does the job.

```
uAC = rAC/norm(rAC);   % The other two
uAD = rAD/norm(rAD);   % unit vectors

A = [uAB  uAC  uAD];   % assemble A

T = A\(m*a)            % Solve with backslash
```

2)(35 pts) A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\underline{v}|$, where $\underline{v}$ is the velocity. Do not neglect gravity.

a) (10 ptss) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) (15 pts) Starting with a free body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (10 pts) (challenge, do last, long calculation) Assume the particle is thrown from $\underline{r} = \underline{0}$ with $\underline{v} = v_{x0}\,\hat{\mathbf{i}} + v_{y0}\,\hat{\mathbf{j}}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.)

[Hint: i) find $x(t)$ and $y(t)$ like in the homework, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.). If you use Matlab, start your code by assigning any non-trivial values to all constants.]

FBD: $\underline{F} = -cv\left(\frac{\underline{v}}{v}\right)$ $\quad -mg\underline{j}$

LMB: $\underline{F} = m\underline{a}$

$$\Rightarrow \{-c\underline{v} - mg\underline{j} = m(\dot{v}_x\underline{i} + \dot{v}_y\underline{j})\}$$

$$\{\}\cdot\underline{i} \Rightarrow -cv_x = m\dot{v}_x \Rightarrow \boxed{m\ddot{x} + c\dot{x} = 0}$$

$$\{\}\cdot\underline{j} \Rightarrow -cv_y - mg = m\dot{v}_y \Rightarrow \boxed{m\ddot{y} + c\dot{y} = -mg}$$

(b)

Steady State $\Rightarrow \dot{v}_x = 0,\ \dot{v}_y = 0 \Rightarrow v_x = 0,\ v_y = \frac{-mg}{c}$

$\Rightarrow$ steady state falling speed $= \boxed{mg/c}$ (a)

Trajectory eqs

(1) $\dot{v}_x + \frac{c}{m}v_x = 0$

(3) $\dot{x} = v_x$

(5) $\dot{v}_y + \frac{c}{m}v_y = -g$

(7) $\dot{y} = v_y$

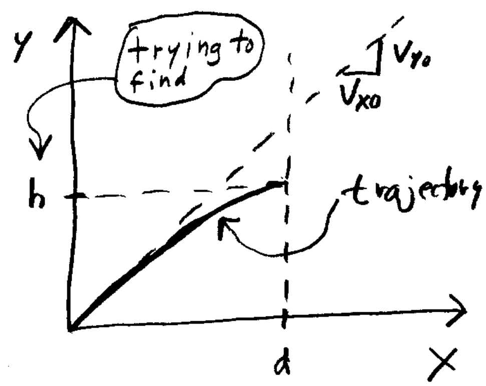
ICs

(2) $v_x(0) = v_{x0}$

(4) $x(0) = 0$

(6) $v_y(0) = v_{y0}$

(8) $y(0) = 0$



Find $x(t)$

$$(1) \Rightarrow v_x = C_1 e^{-(c/m)t},\quad (2) \Rightarrow C_1 = v_{x0} \Rightarrow v_x = v_{x0}e^{-(c/m)t}$$

$$(3) \Rightarrow x(t) = -\frac{mv_{x0}}{c}e^{-(c/m)t} + C_2,\ (4) \Rightarrow C_2 = \frac{mv_{x0}}{c} \Rightarrow \boxed{x = \frac{mv_{x0}}{c}\left(1 - e^{-(c/m)t}\right)}$$

(like homework)

Find t from x

$$\frac{cx}{mv_{x0}} = 1 - e^{-(c/m)t} \Rightarrow e^{-(c/m)t} = 1 - \frac{cx}{mv_{x0}} \Rightarrow -(c/m)t = \ln\left(1 - \frac{cx}{mv_{x0}}\right)$$

$$\Rightarrow \boxed{t = -\frac{m}{c}\ln\left(1 - \frac{cx}{mv_{x0}}\right)} \ (1)$$

{note, this inversion is possible because particle always moves to right.}

Problem 2 continued.

Solve for $y(t)$

$$(5) \Rightarrow \quad V_y = c_3 e^{-(c/m)t} - \frac{mg}{c}, \quad (6) \Rightarrow c_3 = V_{y0} + \frac{mg}{c}$$

$$\Rightarrow \quad V_y(t) = \left(V_{y0} + \frac{mg}{c}\right) e^{-(c/m)t} - \frac{mg}{c}$$

$$(7) \Rightarrow \quad y(t) = -\frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right) e^{-(c/m)t} - \frac{mg}{c} t + c_4$$

$$(8) \Rightarrow \quad 0 = -\frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right) + c_4 \Rightarrow c_4 = \frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right)$$

$$\Rightarrow \quad \boxed{y(t) = \frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{mg}{c} t} \quad (2)$$

(like homework)

[Define: char. time, $t_c = m/c$ | V steady state, $V_s = \frac{mg}{c}$
{to simplify algebra} max. x, $x_m = mV_{x0}/c$]

$$(1) \Rightarrow \quad t = -t_c \ln(1 - x/x_m) \qquad (3)$$

$$(2) \Rightarrow \quad y(t) = t_c(V_{y0} + V_s)\left(1 - e^{-t/t_c}\right) - V_s t \qquad (4)$$

Substitute (4) into (3)

$$y = t_c\left(V_{y0} + V\right)\left(1 - (1 - x/x_m)\right) + V_s t_c \ln(1 - x/x_m)$$

$$y = t_c\left(V_{y0} + V_s\right)\left(\frac{x}{x_m}\right) + V_s t_c \ln(1 - x/x_m)$$

$$\Rightarrow h = t_c(V_{y0} + V_s)\left(\frac{d}{x_m}\right) + V_s t_c \ln(1 - d/x_m)$$

$$\boxed{h = V_s t_c \left[\left(\frac{V_{y0}}{V_s} + 1\right)\left(\frac{d}{x_m}\right) + \ln(1 - d/x_m)\right]} \quad (c)$$

$\boxed{x_m = mV_{x0}/c}$ $\boxed{V_s = mg/c}$ $\boxed{t_c = m/c}$

Checks:

$d \to 0 \Rightarrow$
$\ln(1 - d/x_m) \approx -d/x_m$

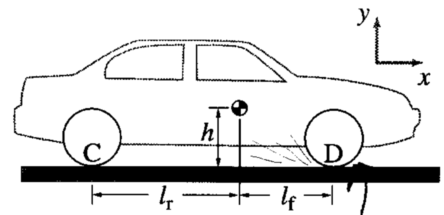
$$\Rightarrow h \sim \cancel{V_s} t_c \left(\frac{V_{y0}}{\cancel{V_s}}\right) \frac{d}{x_m} = \frac{\cancel{m}}{\cancel{c}} V_{y0} \frac{d}{\cancel{m} V_{x0}/\cancel{c}} = \frac{V_{y0}}{V_{x0}} d \quad \text{(which makes sense, see sketch)}$$

$$\underline{c \to 0} \Rightarrow \ln(1 - d/x_m) \approx -d/x_m - \tfrac{1}{2}(d/x_m)^2 \Rightarrow h \sim \frac{V_{y0}}{V_{x0}} d - \frac{1}{2}\left(\frac{m}{c}\right)\left(\frac{mg}{c}\right)\left(\frac{d}{mV_{x0}/c}\right)^2$$

$\boxed{\text{Taylor series}}$

$$\sim \frac{V_{y0}}{V_{x0}} - \frac{1}{2} g \left(\frac{d}{V_{x0}}\right)^2 \quad \text{(parabolic trajectory)}$$

3) 35 pts) Car accelerating. A car (mass $=m$) with a big motor, front-wheel drive, and a stiff suspension accelerates to the right with the front wheels over-powered and skidding (friction coefficient $= \mu$) and back wheels turning freely.



skids w/ μ

a) (5 pts) Assuming the car starts from rest and has constant acceleration a, how far has it travelled in time t? (Answer in terms of a and t.) [Not a trick, just easy.].

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1$$

$$v(0) = 0 \Rightarrow \quad v(t) = at$$

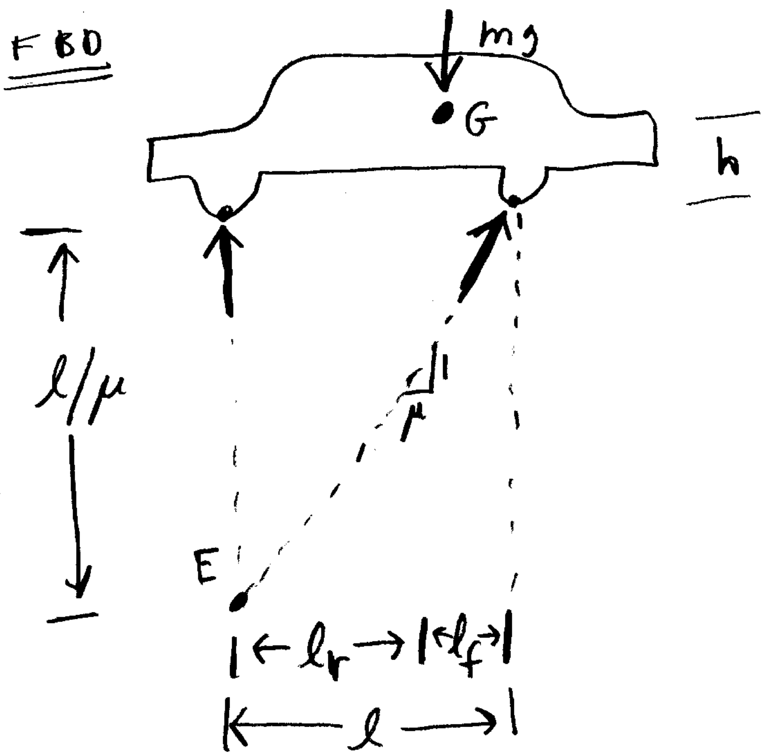
$$x(t) = \int v(t)\,dt = \int at\,dt = \frac{at^2}{2} + C_2$$

$$x(0) = 0 \Rightarrow \quad C_2 = 0$$

$$\Rightarrow \quad \boxed{x(t) = \frac{1}{2}at^2} \quad (a)$$

b) (30 pts) Find a in terms of any or all of ℓ_r, ℓ_f, h, m, g and μ. [Hint: all the directions on the cover page apply. Your answer should reduce to $a = \ell_r g/h$ in the limit $\mu \to \infty$.]

FBD



$\ell = \ell_f + \ell_r$

Assume: massless tires, rigid body,

$\underline{a}_G = \underline{a} = a\,\underline{i} = a_G\,\underline{i}$

AMB

$$\sum \underline{M}_{/E} = \dot{\underline{H}}_{/E} \quad (1)$$

Evaluate left hand side of (1)

$$\sum \underline{M}_{/E} = \underline{r}_{EG} \times (-mg\,\underline{j})$$

$$= \left(\ell_r \underline{i} + \left(\frac{\ell}{\mu} + h\right)\underline{j}\right) \times (-mg\,\underline{j})$$

$$= -\ell_r mg\,\underline{k} \quad (2)$$

Ev. right hand side of (1)

$$\dot{\underline{H}}_{/E} = \underline{r}_{EG} \times m\underline{a}_G$$

$$= \left(\ell_r \underline{i} + \left(\frac{\ell}{\mu} + h\right)\underline{j}\right) \times (ma\,\underline{i})$$

$$= -ma\left(\frac{\ell}{\mu} + h\right)\underline{k} \quad (3)$$

Plug (2) & (3) back into (1)

$$\Rightarrow \quad \sum \underline{M}_{/E} = \dot{\underline{H}}_{/E}$$

$$\left\{ -\ell_r m g \underline{K} = -ma\left(\frac{\ell}{\mu} + h\right)\underline{K} \right\}$$

$$\{ \}\cdot\underline{K} \Rightarrow \ell_r \cancel{m} g = \cancel{m} a\left(\frac{\ell}{\mu} + h\right)$$

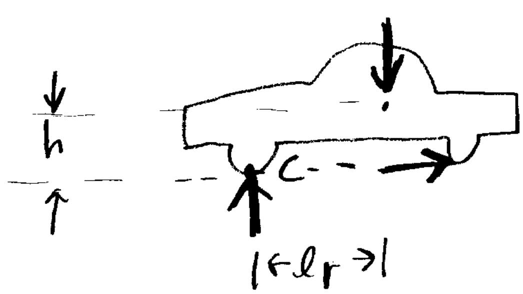
$$a = g\frac{\ell_r}{\left(\frac{\ell}{\mu} + h\right)}$$

$$\boxed{a = g\left[\frac{\ell_r}{\frac{\ell_r + \ell_f}{\mu} + h}\right]} \quad \text{(b)}$$

Check: $a \underset{\mu\to\infty}{=} g\frac{\ell_r}{h}$

[No matter how big is μ, a front wheel drive car can't have more accel. than this.]

Why? when $\mu \to 0$ FBD looks like this



If we look at

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_C$$

$$\Rightarrow mg\ell_r = ahm$$

$$\Rightarrow a = g\ell_r/h$$

(gravity "balances" acceleration.)

"SOLUTIONS"

Your Name: STAFF

Your TA: "

T&AM 203 Prelim 2

Tuesday March 28, 2000 7:30 — 9:00$^+$ PM

Draft March 28, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- * you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

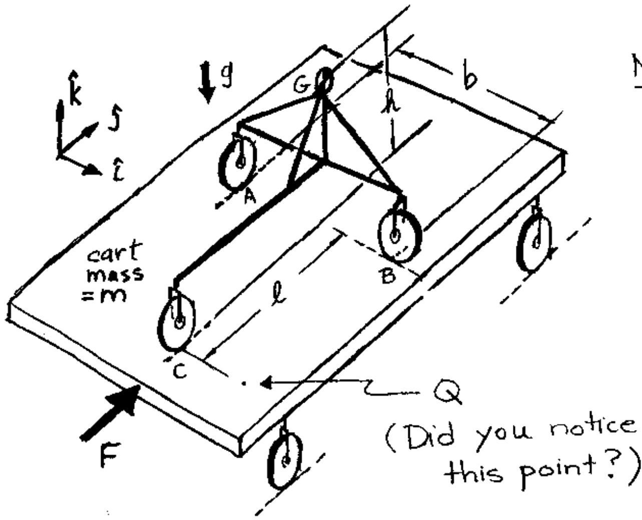
Hopefully,

Problem	1:	30 /30
Problem	2:	35 /35
Problem	3:	35 /35
TOTAL:		100 /100

1)(30 pts) **3-wheeled robot.** A 3-wheeled robot with mass m is being transported on a level flatbed trailer also with mass m. The trailer is being pushed with a force $F\hat{\mathbf{j}}$. The ideal massless trailer wheels roll without slip. The ideal massless robot wheels also roll without slip. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\mathbf{j}}$ direction and the wheel at B is free to roll in the $\hat{\mathbf{i}}$ direction. The center of mass of the robot at G is h above the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance b apart. The length of the robot is ℓ.

Find the force vector $\underline{\mathbf{F}}_A$ of the trailer on the robot at A in terms of some or all of m, g, ℓ, F, b, h, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

[Hints: Use a free body diagram of the cart with robot to find their acceleration. With reference to a free body diagram of the robot, use angular momentum balance about axis BC to find F_{Az}.]



Note: From the announcement,

$$\boxed{\underline{a}_G = \underline{a}_{CART} = \underline{a}}$$

The robot does not move with respect to the cart.

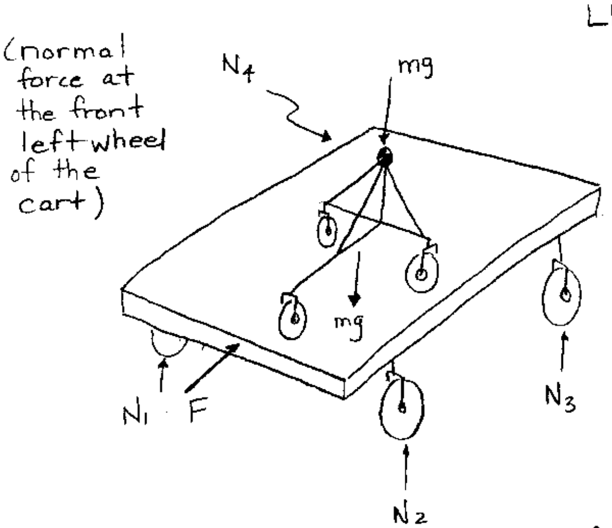
FBD (cart w/robot)

$m_{ROBOT} + m_{CART} = 2m$

LMB : $\Sigma \underline{F} = m_{TOTAL}\, \underline{a}$

$LMB \cdot \hat{j} \Rightarrow F = 2m a_y$

$$\therefore \boxed{a_y = \frac{F}{2m}}$$



$LMB \cdot \hat{i} \Rightarrow 0 = 2m a_x$

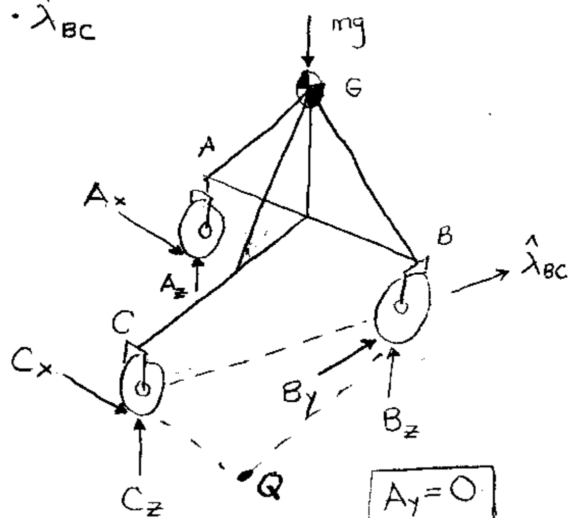
$$\therefore \boxed{a_x = 0}$$

$a_z = 0$ by the assumption that the cart doesn't leave the ground

$$\therefore \boxed{\underline{a} = \frac{F}{2m}\hat{j}}$$

(Continue work for problem 1 here)

FBD (robot)



$\boxed{A_y = 0}$

↑ wheel rolls freely in $\hat{j}$-dir.

$\underline{AMB}$/axis BC : $\{\Sigma \underline{M}_{/C} = \dot{\underline{H}}_{/C}\} \cdot \hat{\lambda}_{BC}$

where $\hat{\lambda}_{BC} = \dfrac{\underline{r}_{BC}}{|\underline{r}_{BC}|}$

$$\Rightarrow \hat{\lambda}_{BC} = \frac{\frac{b}{2}\hat{i} + l\hat{j}}{\sqrt{(\frac{b}{2})^2 + l^2}}$$

The only forces creating moments about axis BC are A_z, mg :

$$\{\Sigma \underline{M}_{/C}\} \cdot \hat{\lambda}_{BC} = \{\underline{r}_{A/C} \times A_z\hat{k} + \underline{r}_{G/C} \times -mg\hat{k}\} \cdot \hat{\lambda}_{BC}$$

$$= \{(-\tfrac{b}{2}\hat{i} + l\hat{j}) \times A_z\hat{k} + (l\hat{j} + h\hat{k}) \times -mg\hat{k}\} \cdot \hat{\lambda}_{BC}$$

$$= \{+A_z\tfrac{b}{2}\hat{j} + A_z l\hat{i} - mgl\hat{i}\} \cdot \frac{\frac{b}{2}\hat{i} + l\hat{j}}{\sqrt{(\frac{b}{2})^2 + l^2}}$$

$$= \left(\frac{b}{2}l(A_z - mg) + \frac{bl}{2}A_z\right)\frac{1}{\sqrt{(\frac{b}{2})^2 + l^2}}$$

$$\{\dot{\underline{H}}_{/C}\} \cdot \hat{\lambda}_{BC} = \{\underline{r}_{G/C} \times m\underline{a}\} \cdot \hat{\lambda}_{BC}$$

$$= \{(l\hat{j} + h\hat{k}) \times m(\tfrac{F}{2m}\hat{j})\} \cdot \hat{\lambda}_{BC}$$

$$= \{-F\tfrac{h}{2}\hat{i}\} \cdot \frac{\frac{b}{2}\hat{i} + l\hat{j}}{\sqrt{(\frac{b}{2})^2 + l^2}}$$

$$= -\frac{Fbh}{4} \cdot \frac{1}{\sqrt{(\frac{b}{2})^2 + l^2}}$$

$$\Rightarrow \{\Sigma \underline{M}_{/C} = \dot{\underline{H}}_{/C}\} \cdot \hat{\lambda}_{BC}$$

$$\frac{bl}{2}(A_z - mg) + \frac{bl}{2}A_z = -\frac{Fbh}{4}$$

$$bl A_z = \frac{mgbl}{2} - \frac{Fhb}{4}$$

$$\therefore \boxed{A_z = \frac{mg}{2} - \frac{Fh}{4l}}$$

• Now get A_x by taking AMB/Q • $\hat{k}$

$\underline{AMB}_{/Q}$: $\{\Sigma \underline{M}_{/Q} = \dot{\underline{H}}_{/Q}\} \cdot \hat{k}$

The only force creating a moment about Q in the $\hat{k}$-direction is A_x !

$$\Rightarrow \Sigma \underline{M}_{/Q} \cdot \hat{k} = \{\underline{r}_{A/Q} \times A_x\hat{i}\} \cdot \hat{k}$$

$$= \{(-b\hat{i} + l\hat{j}) \times A_x\hat{i}\} \cdot \hat{k}$$

$$= \{-A_x l\hat{k}\} \cdot \hat{k} = -A_x l$$

$$\dot{\underline{H}}_{/Q} \cdot \hat{k} = \{\underline{r}_{G/Q} \times m\underline{a}\} \cdot \hat{k}$$

$$= \{(-\tfrac{b}{2}\hat{i} + l\hat{j} + h\hat{k}) \times m(\tfrac{F}{2m}\hat{j})\} \cdot \hat{k}$$

$$= \{\tfrac{F}{2}(-\tfrac{b}{2}\hat{k} - h\hat{i})\} \cdot \hat{k}$$

$$= -\frac{Fb}{4}$$

$$\Rightarrow \{\Sigma \underline{M}_{/Q} = \dot{\underline{H}}_{/Q}\} \cdot \hat{k}$$

$$-A_x l = \frac{-Fb}{4} \quad \Rightarrow \boxed{A_x = \frac{Fb}{4l}}$$

2)(35 pts) **Slippery money.** A round uniform flat horizontal platform with radius R and mass m is mounted on frictionless bearings with a vertical axis at 0. At the moment of interest it is rotating counter clockwise (looking down) with angular velocity $\underline{\omega} = \omega\hat{\mathbf{k}}$. A force in the xy plane with magnitude F is applied at the perimeter at an angle of 30° from the radial direction. The force is applied at a location that is ϕ from the fixed positive x axis. At the moment of interest a small coin sits on a radial line that is an angle θ from the fixed positive x axis (with $\boxed{\text{mass much much smaller than } m).}$ Gravity presses it down, the platform holds it up, and friction (coefficient=μ) keeps it from sliding.

Find the biggest value of d for which the coin does not slide in terms of some or all of $F, m, g, R, \omega, \theta, \phi$, and μ.

Let m_1 = coin mass

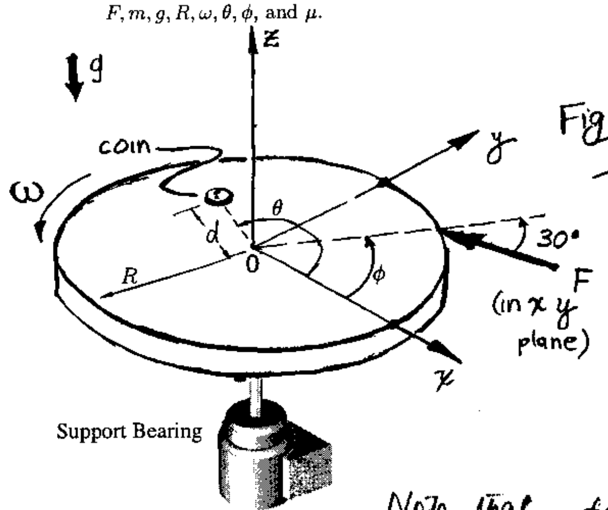
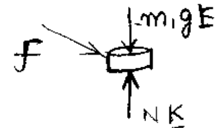


Fig I

FBD of the coin



LMB in z dir:

$N = m_1 g$ — (1)

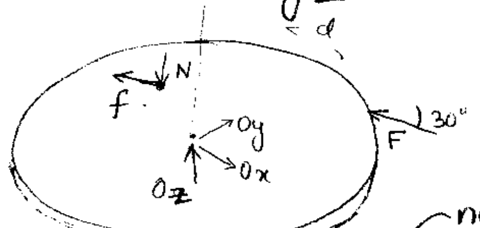
f is the frictional force acting on the coin (the direction is yet unknown, but its in XY plane)

Note that frictional force will act in a direction which provides the acceleration

So let us first calculate the acceleration of the coin

Fig II

FBD of the disc



Let us do AMB about O

$$\sum \vec{M}_0 = \dot{\vec{H}}_0 .$$

neglect

$$\sum \vec{M}_0 = F\sin 30^\circ R\,\hat{k} + (\vec{r}\times\vec{f}) + \underbrace{dN(-\hat{j})}_{\substack{\| \\ dm_1 g \text{ (very small)}}}$$

$\vec{r}$ is the vector from 0 to the coin

Now the maximum magnitude of $\vec{r}\times\vec{f} = \mu m_1 g d$

and since $m_1 << m$ & $\mu < 1$

Therefore we can neglect the contribution of f.

So $\sum \vec{M}_0 = RF\sin 30^\circ \hat{k} = \frac{FR}{2}\hat{k}$

Since 0 is a fixed point

from table IV

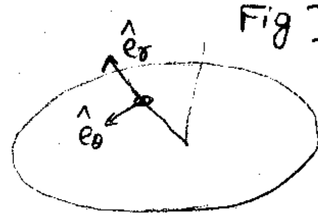
$\dot{\vec{H}}_0 = I_{0_{zz}}\dot{\vec{\omega}} = \frac{mR^2}{2}\dot{\omega}\hat{k}$

$\therefore \quad \frac{F\cancel{R}}{\cancel{2}} = \frac{mR^{\cancel{2}}}{\cancel{2}}\dot{\omega} \Rightarrow \boxed{\dot{\omega} = \frac{F}{mR}}$ — ② $\dot{\vec{\omega}} = \frac{F}{mR}\hat{k}$

So the acceleration of the coin is $\vec{a}_{coin} = \dot{\vec{\omega}} \times \vec{r}_{coin} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{coin})$

Let $\hat{e}_r$, $\hat{e}_\theta$ be as shown

Fig III



then $\vec{r}_{coin} = d\,\hat{e}_r$

$\vec{a}_{coin} = \dot{\omega}\hat{k} \times d\,\hat{e}_r - \omega^2 d\,\hat{e}_r$

$= d\dot{\omega}\,\hat{e}_\theta - d\omega^2\,\hat{e}_r$

Now from the LMB of the coin's FBD (Fig I)

$\vec{f} = m_1\vec{a}$

We want the limiting case when the coin is about to slip off so in that case f should be at its maximum & its magnitude $= \mu N = \mu m_1 g$

$\therefore \quad |\vec{f}| = |m_1\vec{a}|$

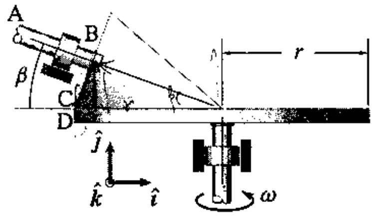
$\Rightarrow \quad \mu \cancel{m_1} g = \cancel{m_1}|\vec{a}| = \cancel{m_1}\sqrt{d^2\dot{\omega}^2 + d^2\omega^4}$

$\Rightarrow \quad d_{max} = \frac{\mu g}{\sqrt{\dot{\omega}^2 + \omega^4}}$ — ③

Substituting ② in ③ $\boxed{d_{max} = \frac{\mu g}{\sqrt{(F/mR)^2 + \omega^4}}}$

3) 35 pts) **Cone on Disk.** A disk rotates with constant rate ω about an fixed axis in the $\hat{\mathbf{j}}$ direction. A right cone held in a fixed bearing at B rolls at constant rate so that the point on the corner of the edge of the cone has the same velocity as the point it touches on the disk, $\underline{v}_C = \underline{v}_D$. Axis AB is in the xy plane.

Find the velocity and acceleration of point C on the cone in terms of some or all of $\omega, r, \beta, \hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.



$\underline{v} = \underline{\omega} \times \underline{r}$

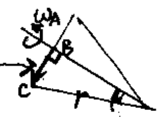
$\therefore \underline{v}_D = \omega\hat{j} \times (-r)\hat{i}$

$= r\omega\,\hat{k}$

$\underline{v}_C = \omega_A(-\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) \times r\sin\beta(-\sin\beta\,\hat{i} - \cos\beta\,\hat{j})$

$= r\omega_A \sin\beta(\cos^2\beta\,\hat{k} + \sin^2\beta\,\hat{k})$

$= r\omega_A \sin\beta\,\hat{k}$



$\underline{r}_{C/B} = \underline{r}_{BC} =$

$= r\sin\beta(-\sin\beta\,\underline{i} - \cos\beta\,\underline{j})$

Since $\underline{v}_C = \underline{v}_D$

$\therefore \{\ \}\cdot\hat{k}: \quad r\omega = r\omega_A\sin\beta$

$\therefore \omega_A = \frac{\omega}{\sin\beta} = \text{const}.$

$\underline{a} = \cancel{\dot{\underline{\omega}}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$

$= \underline{\omega} \times (\underline{\omega} \times \underline{r})$

$\therefore \underline{a}_C = \underline{\omega}_A \times (\underline{\omega}_A \times \underline{r}_{BC})$

$= \omega_A(-\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) \times [\omega_A(-\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) \times r\sin\beta(-\sin\beta\,\hat{i} - \cos\beta\,\hat{j})]$

$= \omega_A(-\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) \times r\omega_A\sin\beta\,\hat{k}$

$= r\omega_A^2\sin\beta\,(\cos\beta\,\hat{j} + \sin\beta\,\hat{i})$

$= r\cdot\frac{\omega^2}{\sin^2\beta}\sin\beta(\cos\beta\,\hat{j} + \sin\beta\,\hat{i})$

$= \frac{r\omega^2}{\sin\beta}(\sin\beta\,\hat{i} + \cos\beta\,\hat{j})$

$$\therefore \boxed{\begin{aligned} \underline{v}_C &= r\omega\hat{k} \\ \underline{a}_C &= \frac{r\omega^2}{\sin\beta}(\sin\beta\,\hat{i} + \cos\beta\,\hat{j}) \end{aligned}}$$

" SOLUTIONS "

Your Name: STAFF

Your TA: "

T&AM 203 Prelim 3

Tuesday April 25, 2000 7:30 — 9:00$^+$ PM

Draft April 25, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

b) Full credit if

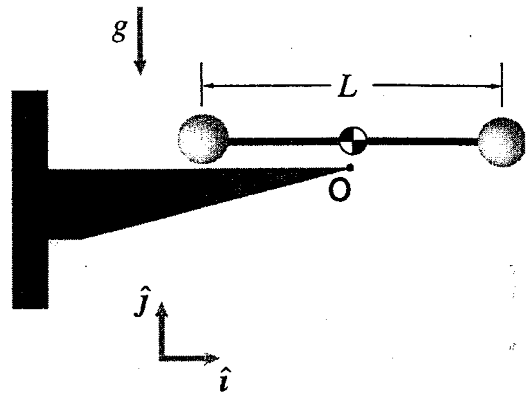
- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/35
Problem	2:	/30
Problem	3:	/35
TOTAL:		/100

1) 35 pts) Bouncing baton. Two equal point masses are connected by a massless rigid rod of length L. While horizontal, the baton falls without rotation until it reaches the speed v and the left ball strikes the rigid surface of a table. At this instant the center of the rod is just over the right edge of the table. The collision is elastic (conserves energy).



a) Immediately after impact, what are the velocity of the rod's center and the angular velocity of the rod? Answer in terms of some or all of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, v, L, g$, and m.

b) Assuming no other interaction with the table, accurately describe —using equations if appropriate— the subsequent motion and rotation of the baton. Answer in terms of some or all of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, v, L, g$, and m.

c) What is the minimum value of v for which the left mass will miss the table in its subsequent motion. Assume no subsequent collision of the massless rod with the surface. Answer in terms of some or all of v, L, g, and m.

a) Can use AMB/A :

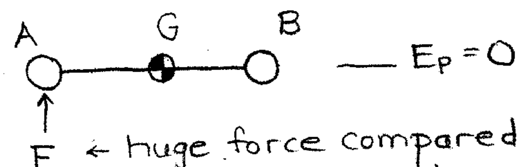
$$\underbrace{\sum \underline{M}_{/A}}_{\underline{0}} = \dot{\underline{H}}_{/A}$$

$\Rightarrow$ $\underline{H}_{/A}$ conserved

$$\underline{H}_{/A}^{①} = \sum \underline{r}_{i/A} \times m_i \underline{v}_i$$
$$= \cancel{\underline{r}_{A/A}} \times m \underline{v}_A + \underline{r}_{B/A} \times m \underline{v}_B$$
$$= L\hat{\imath} \times m(-v\hat{\jmath}) = -mLv\hat{k}$$

$$\underline{H}_{/A}^{②} = \sum \underline{r}_{i/A} \times m_i \underline{v}_i$$
$$= \underline{0} + \underline{r}_{B/A} \times m \underline{v}_B$$
$$= L\hat{\imath} \times m(-v_B\hat{\jmath})$$
$$= -mLv_B\hat{k}$$

FBD during collision



← huge force compared to mg ⇒ neglect mg

① just before collision

② " after "

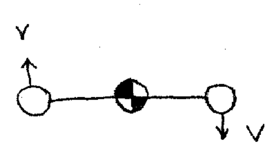
$$\{\underline{H}_{/A}^{①} = \underline{H}_{/A}^{②}\} \cdot \hat{k}$$

$\Rightarrow$ $v_B = v$

$\Rightarrow$ $\boxed{\underline{v}_B = -v\hat{\jmath}}$

Energy conservation $\Rightarrow$ $E_{K_1} + \underbrace{\cancel{E_{P_1}}}_{0} = E_{K_2} + \underbrace{\cancel{E_{P_2}}}_{0}$

$$\Rightarrow \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_B^2 + \tfrac{1}{2}mv_A^2$$
$$mv^2 = \tfrac{1}{2}m(-v)^2 + \tfrac{1}{2}mv_A^2$$
$$\Rightarrow v_A = v$$
$$\Rightarrow \boxed{\underline{v}_A = v\hat{\jmath}}$$



(Continue work for problem 1 here)

Use relative motion to get $\underline{v}_{CM}$, $\underline{\omega}$:

$$\underline{v}_{CM} = \underline{v}_A + \underline{v}_{CM/A} = v\hat{j} + -\omega\hat{k} \times \tfrac{L}{2}\hat{i} = (v - \tfrac{1}{2}\omega L)\hat{j} \quad (1)$$

$$\underline{v}_{CM} = \underline{v}_B + \underline{v}_{CM/B} = -v\hat{j} + -\omega\hat{k} \times -\tfrac{L}{2}\hat{i} = (-v - \tfrac{1}{2}\omega L)\hat{j} \quad (2)$$

$\{(1) = (2)\}\cdot\hat{j} \Rightarrow v - \frac{1}{2}\omega L = -v - \frac{1}{2}\omega L$

$\Rightarrow \omega = \frac{2v}{L}$, $\boxed{\underline{\omega} = -\frac{2v}{L}\hat{k}}$

$\Rightarrow \boxed{\underline{v}_{CM} = 0}$

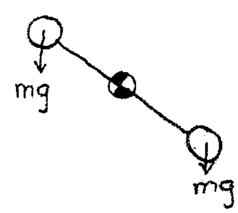
b) LMB: $\Sigma \underline{F} = (m+m)\underline{a}_{CM}$

$-2mg\hat{j} = 2m\underline{a}_{CM}$

$\Rightarrow \underline{a}_{CM} = -g\hat{j}$

$\Rightarrow$ The center of mass falls straight down with acceleration of magnitude g.

FBD after collision



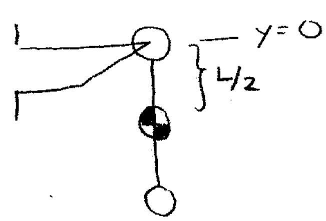
AMB/CM: $\underbrace{\Sigma \underline{M}_{/CM}}_{\underline{0}} = \dot{\underline{H}}_{/CM}$

gravity forces cancel each other

$\Rightarrow \underline{H}_{/CM}$ conserved $\Rightarrow \underline{\omega}$ constant

$\Rightarrow$ The baton rotates at rate $\underline{\omega} = -\frac{2v}{L}\hat{k}$.

c) For the minimum v, the baton should rotate by 90° when the center of mass has fallen a distance $L/2$.



$\omega = \frac{2v}{L}$ = constant

$\Rightarrow \omega = \frac{\Delta\theta}{\Delta t} \leftarrow$ find this

$$\left.\begin{aligned} \underline{a}_{CM} &= -g\hat{j} \\ \Rightarrow \underline{v}_{CM} &= -gt\hat{j} \\ \underline{r}_{CM} &= -\tfrac{1}{2}gt^2\hat{j} \end{aligned}\right\}$$

When $\underline{r}_{CM} = -\frac{L}{2}\hat{j}$

$\frac{L}{2} = \frac{1}{2}g(\Delta t)^2$

$\Rightarrow \Delta t = \sqrt{L/g}$

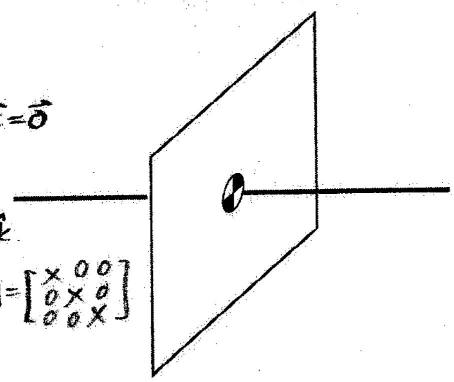
$\therefore \omega = \frac{2v}{L} = \frac{\pi/2}{\sqrt{L/g}}$

$\therefore \boxed{v = \frac{\pi}{4}\sqrt{gL}}$

2)(30 pts) **Static and Dynamic Balance** A series of bodies, each of uniform density and each with total mass m, rotate at a constant angular speed ω about a fixed horizontal axis. Ignore gravity. For each body state whether the body is (i) *statically* balanced and whether it is (ii) *dynamically* balanced. Give clear arguments using words or equations to support your claims.

(iii, iv) For each body you must *either* (a) add one point mass m or (b) add two point masses each of mass $m/2$ (your choice) that *maintain* static and dynamic balance if they are balanced, or that *make* the bodies statically and dynamically balanced. Justify your placement with words and/or equations. The masses need not be added to the bodies, but could be attached off the bodies by structures with negligible mass. [Hint: none of the placements are unique. You may draw a side view if that helps clarify your placement.]

a) A rectangular plate (height h, length ℓ) mounted with the axle perpendicular to the plate and through its center.



i) Statically balanced? (yes/no) Why?

yes. center of the mass is on the axis. $\Sigma\vec{F}=\vec{0}$

ii) Dynamically balanced? (yes/no) Why?

$\hat{i}$, $\hat{j}$, $\hat{k}$

yes. Look at $\vec{H}$, since $\vec{\omega}=\text{const}$ $\therefore \dot{\vec{H}}=\vec{\omega}\times\vec{H}$, $\vec{H}=[I]\cdot\vec{\omega}$ $\quad\vec{\omega}=\omega\hat{k}$ $\quad [I]=\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$

Since $\vec{\omega} \parallel [I]\cdot\vec{\omega}$ $\quad\therefore \dot{\vec{H}}=\vec{0}=\Sigma\vec{M}$

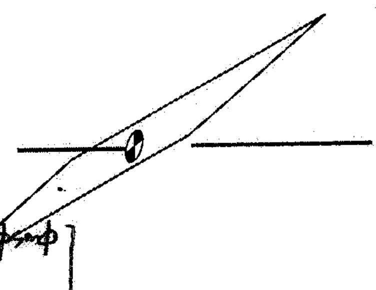
iii) Are you adding one mass or two?

One

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.

Center of the rectangle , it won't affect the previous answer since for this point mass, $[I]=[0]$.

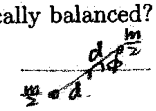
b) The same plate as in (a) above but mounted at an angle $\phi \neq \pi/2$ from the shaft.



i) Statically balanced? (yes/no) Why?

Yes. mass center is on the axis. $\Sigma\vec{F}=\vec{0}$

ii) Dynamically balanced? (yes/no) Why?

No, 

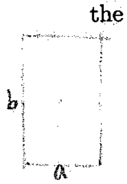
$\vec{\omega}=\omega\hat{k}$. $[I]=\begin{bmatrix} md^2\cos^2\phi & 0 & md^2\cos\phi\sin\phi \\ 0 & md^2 & 0 \\ md^2\cos\phi\sin\phi & 0 & md^2\sin^2\phi \end{bmatrix}$

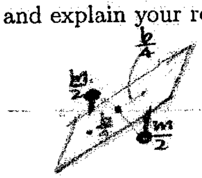
$[I]\cdot\vec{\omega}=md^2\sin\phi(\cos\phi\,\hat{i}+\sin\phi\,\hat{k}) \nparallel \vec{\omega}$. $\therefore \Sigma\vec{M}\neq\vec{0}$

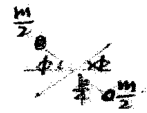
iii) Are you adding one mass or two?

Two.

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



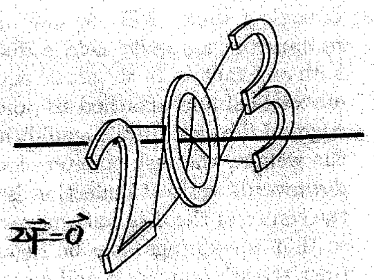




it will make the whole system symmetric. then. $I_{xy}=I_{yz}=I_{xz}=0$!

So $[I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \Sigma\vec{M}=\vec{0}$

c) The numerals '203' cut out of a plate and connecteed by massless rods. Each letter has mass $m/3$ and the three center-of-mass points of the individual letters are colinear and equally spaced. The shaft goes through the center of the '0' and is perpendicular to the plane of the letters.



i) Statically balanced? (yes/no) Why?

yes. center of the mass is on the axis. $\Sigma \vec{F} = \vec{0}$

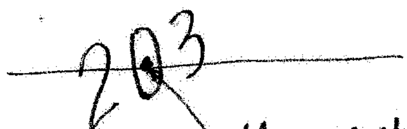
ii) Dynamically balanced? (yes/no) Why?

yes, since $\hat{k}$ is the eigenvector of the [I], and it's perpendicular to the other two eigenvectors which lie in the x-y plane. $\therefore [I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \vec{H} = \vec{\omega}\times([I]\cdot\vec{\omega}) = \vec{0} = \Sigma\vec{M}$

iii) Are you adding one mass or two?

One.

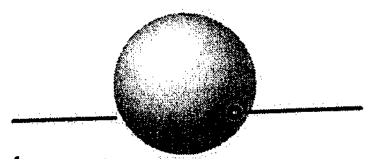
iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



add m at the center of '0'.

it won't affect the [I].

d) A sphere with radius R where the shaft passes a distance $d < R$ from the center.

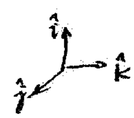


i) Statically balanced? (yes/no) Why?

No, center of the mass isn't on the axis.
$\therefore$ need extra force to provide centripetal acceleration.

ii) Dynamically balanced? (yes/no) Why?

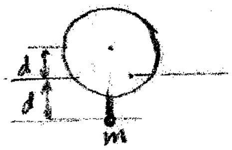
No, center of the mass is not on the axis.
$I_{xz} \neq 0$. $\therefore [I]\cdot\vec{\omega} \nparallel \vec{\omega}$. $\Sigma\vec{M} \neq \vec{0}$



iii) Are you adding one mass or two?

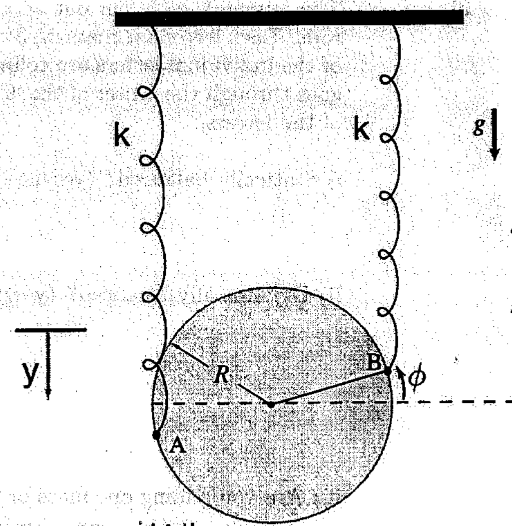
One.

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



After adding this point mass, $I_{xz} = 0$.
and $I_{yz} = I_{xy} = 0$
$\therefore [I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \Sigma\vec{M} = \vec{0}$

3)(35 pts) **Hanging disk, 2-D.** A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k. In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating thier stretch ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$).



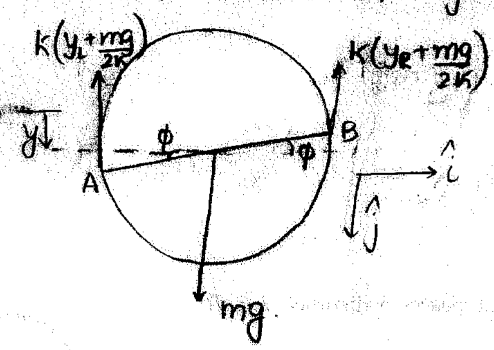
a) Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of $\phi, \dot{\phi}, y, \dot{y}, k, m, R,$ and g.

b) Find the natural frequencies of vibration in terms of some or all of $k, m, R,$ and g.

Ans :- Note that 'y' is measured from the equillibrium (and represents the displacement of the CM).

(a) So at equillibrium springs are stretched by $l_0 = \frac{mg}{2k}$

FBD



$y_R = y - R\phi$

$y_L = y + R\phi$

LMB in '$\hat{j}$' direction

$$m\ddot{y} = -k\left(y - R\phi + \frac{mg}{2k}\right) - k\left(y + R\phi + \frac{mg}{2k}\right) + mg$$

$$\Rightarrow m\ddot{y} = -ky + \cancel{kR\phi} \cancel{- mg} - ky - \cancel{kR\phi} + \cancel{mg}$$

$$\Rightarrow \boxed{\ddot{y} + \frac{2k}{m} y = 0} \quad (1) \quad \Rightarrow \ddot{y} = -\frac{2k}{m} y$$

AMB about CM :

$$k\left(y_R + \frac{mg}{2k}\right) \cdot R - k\left(y_L + \frac{mg}{2k}\right) \cdot R = I\ddot{\phi}$$

(Continue work for problem 3 here)

$$\Rightarrow K(y-R\phi)R - K(y+R\phi)R = I\ddot{\phi}$$

$$\Rightarrow \quad \ddot{\phi} + \frac{2KR^2}{I}\phi = 0$$

I for a disk $= \dfrac{mR^2}{2}$.

$$\Rightarrow \quad \ddot{\phi} + \frac{2KR^2 \cdot 2}{mR^2}\phi = 0$$

$$\Rightarrow \quad \boxed{\ddot{\phi} + \frac{4K}{m}\phi = 0} \quad ②$$

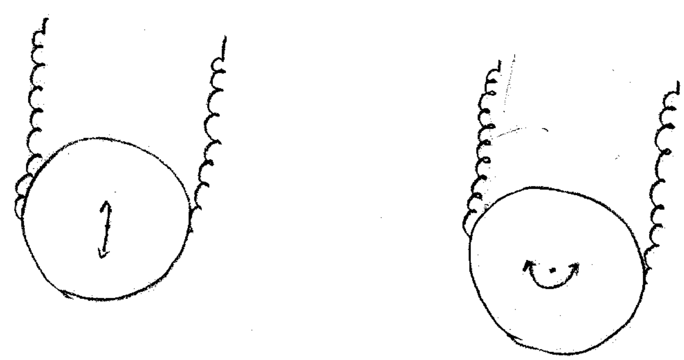
$$\ddot{\phi} = -\frac{4K}{m}\phi.$$

b) Since the system has 2 dof it has two natural frequencies.

which from equations ① & ② are

$$\boxed{\omega_1 = \sqrt{\frac{2k}{m}}, \quad \omega_2 = \sqrt{\frac{4k}{m}}}$$

The modes of vibration look like



"Solutions"

Your Name: ANDY RUINA

Section day and time: ______________

T&AM 203 Final exam

Tuesday May 14, 2002

Draft May 14, 2002

5 problems, 100 points, and 150 minutes (no extra).

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problem(s).

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

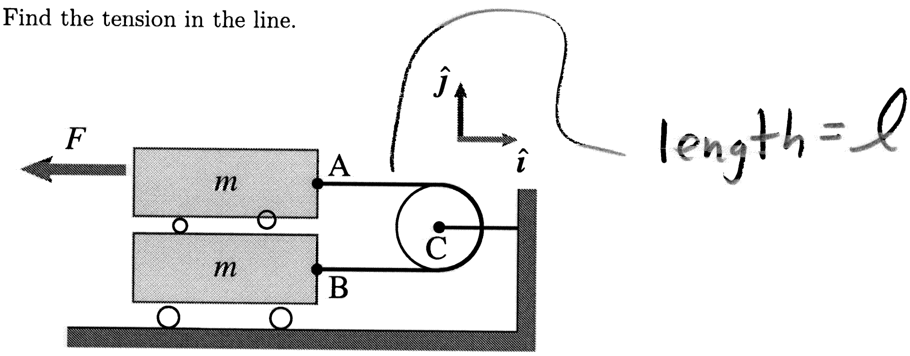
Problem 1:	/15
Problem 2:	/20
Problem 3:	/25
Problem 4:	/20
Problem 5:	/20
TOTAL:	**/100**

1) (15 pt) Two equal masses are stacked an tied together by the pully as shown. All bearings are frictionless. All rotating parts have negligible mass. The line is inextensible.

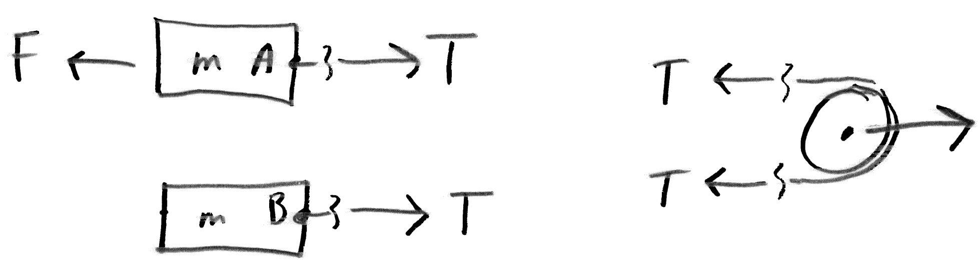
0) (~~10~~ 5 pt) for basic setup diagrams, assumptions, and equations needed to answer the questions below.

a) (5 pt) Find the acceleration of point A.

b) (5 pt) Find the tension in the line.



FBDs:



$$\{\text{LMB for } A\}\cdot\hat{\imath} \Rightarrow \quad -F + T = m\ddot{x}_A \qquad (1)$$

$$\{\text{LMB for } B\}\cdot\hat{\imath} \Rightarrow \quad T = m\ddot{x}_B \qquad (2)$$

Kimematics:

$$\ell = \text{const.} \Rightarrow \quad \ddot{x}_A = -\ddot{x}_B \qquad (3)$$

$$(1)-(2) \Rightarrow \quad -F = m\,\ddot{x}_A - \ddot{x}_B$$

$$(3) \Rightarrow \quad = 2m\ddot{x}_A$$

$$\Rightarrow \boxed{\underline{a}_A = -\frac{F}{2m}\hat{\imath}}\ (a)$$

$$(1)+(2) \Rightarrow \quad -F + 2T = m(\ddot{x}_A + \ddot{x}_B)$$

$$(3) \Rightarrow \quad = 0$$

$$\Rightarrow \boxed{T = F/2}\ (b)$$

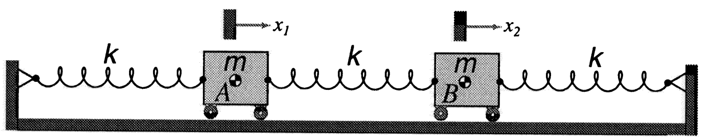
2) (20 pt) Two identical masses ($m = 2\,\text{kg}$) move in a straight line without friction. Three identical springs ($k = 7\,\text{N/kg}$) hold them in place (one between the left mass and a wall, one between the two masses, and one between the right mass and the wall). When the horizontal displacements x_1 and x_2 of the masses are zero all three springs are relaxed.

The system is released from rest at $t = 0$ with $x_1(0) = 0.3\,\text{m}$ and $x_2(0) = -.3\,\text{m}$.

a) (15 points) Write Matlab code where the final output will be the position of mass one at $t = 10\,\text{s}$. Your code should be general enough to handle arbitrary initial conditions. [Do not just use Matlab to evaluate the solution from (b) below.]

b) (5 points) Write a formula for the answer above. That is, evaluate an analytic solution of the resulting differential equations at $t = 10\,\text{s}$. [Hint: Using ideas from the lab makes this problem much easier than blindly grinding through the methods of Math 293, 294].

$\hat{j}$ $\hat{i}$



$X_1 = X_A$

$X_2 = X_B$

FBDs

$T_1 \leftarrow \square \rightarrow T_2 \qquad T_2 \leftarrow \square \rightarrow T_3$

$$T_1 = kX_1, \quad T_2 = k(X_2 - X_1), \quad T_3 = -kX_3$$

$\{LMB \text{ for } A\} \cdot \hat{i} \Rightarrow$
$$-T_1 + T_2 = m\ddot{X}_1$$
$$-kX_1 + k(X_2 - X_1) = m\ddot{X}_1$$
$$-2kX_1 + kX_2 = m\ddot{X}_1 \qquad (1)$$

$\{LMB \text{ for } B\} \cdot \hat{i} \Rightarrow$
$$T_3 - T_2 = m\ddot{X}_B$$
$$-kX_2 - (k(X_2 - X_1)) = m\ddot{X}_2$$
$$kX_1 - 2kX_2 = m\ddot{X}_2 \qquad (2)$$

(1) & (2) $\Rightarrow$
$$m\begin{bmatrix}\ddot{X}_1\\ \ddot{X}_2\end{bmatrix} = k\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}$$

Given ICs:

$X_1(0) = .3$
$X_2(0) = -.3$ } (using consistent mks units)

$V_1(0) = 0$
$V_2(0) = 0$ } defining $V_1 = \dot{X}_1$, $V_2 = \dot{X}_2$

(2 cont'd)

```
X10=.3; X20=-.3; V10=0; V20=0;
Z0=[X10 X20 V10 V20];
tspan= [0 10];
[t z]= ode23('twoblocks', tspan, Z0);
answer = z(end,1)          % note, no semicolon
```

driver file

```
function zdot = twoblocks(t, z)
K=7; m=2;
pos = [z(1) z(2)]';
vel = [z(3) z(4)]';
posdot= vel;
veldot= K*[-2 1;
            1 -2]*pos/m ;
zdot= [posdot' veldot'];
```

twoblocks.m

By inspection a normal mode of this system has blocks moving equally and oppositely. Appling $X_2=-X_1$ to ① $\Rightarrow$

$$m\ddot{x}_1 = -3k\,x_1 \Rightarrow x_1 = A\cos\sqrt{\tfrac{3k}{m}}\,t + B\sin\sqrt{\tfrac{3k}{m}}\,t$$

Init. cond. $\Rightarrow$ $A=.3$, $B=0$

So, in consistent units, at $t=10$

$$\boxed{x_1 = A\cos\sqrt{\tfrac{3k}{m}}\,t = .3\cos\left(10\sqrt{\tfrac{21}{2}}\right)}$$ (b)

3) (25 pt) A suitcase on level ground with wheels in front and skids in back is pulled by its handle with a forward force F. The handle is directly above the front wheels. In all cases you are given these quantities and can use them in your answer:

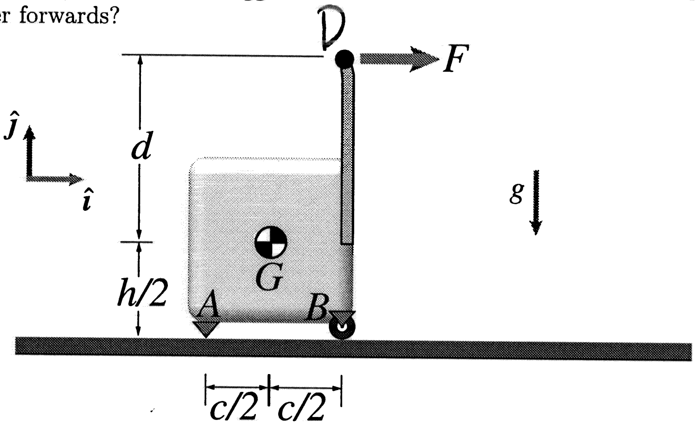
- $m =$ total mass of suitcase;
- $I^G =$ the polar moment of inertia about an axis through the center of mass G and in the ussual zz direction (perpendicular to the plane on which a side view of the suitcase is drawn);
- $h/2 =$ the height of G above the ground;
- $c/2 =$ the distance G is forward of the rear ground contact at A;
- $c/2 =$ the distance G is behind the front ground contact at B;
- $d =$ the height of the handle at D above the center of mass of the suitcase;
- $\mu =$ the coefficient of friction between the rear skids and ground;
- $g =$ the gravity constant.

0) (10 points) for the basic diagrams, stated assumptions, and mechanics equations needed to solve the parts below.

a) (5 points) Given F and that F is big enough for non-zero forward acceleration, and that F is not so big as to cause the suitcase to tip, what is the acceleration of the suitcase?

b) (5 points) What is the smallest F that can make the suitcase move at all?

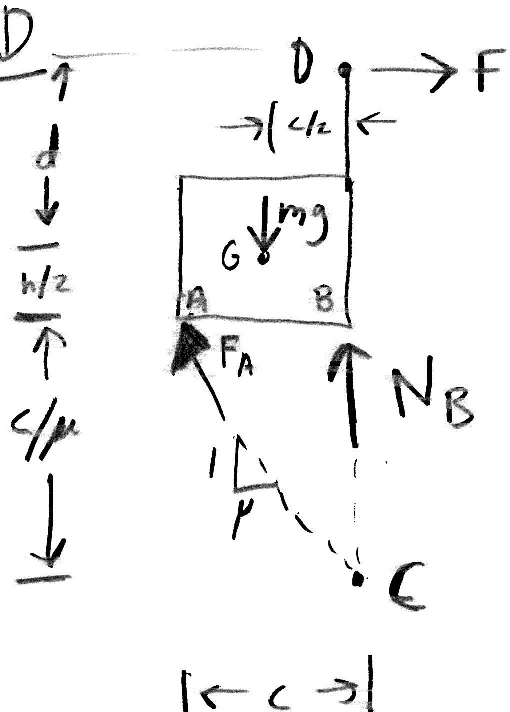
c) (5 points) What is the biggest acceleration that can be achieved without the suitcase tipping over forwards?



Note: in all cases $F = \mu N$

(slip is, or is about to, occur)

FBD



AMB/C

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$\left\{ \frac{mgc}{2}\hat{k} - F\left(\frac{c}{\mu} + \frac{h}{2} + d\right)\hat{k} = \underline{r}_{G/C} \times m\underline{a}_G + I^G \dot{\omega}\hat{k} \right\}$$

$\underline{a}_G = a\hat{\imath}$, $\dot{\omega} = 0$

$$\underline{r}_{G/C} = \left[\left(\frac{c}{\mu} + \frac{h}{2}\right)\hat{\jmath} - \frac{c}{2}\hat{\imath}\right]$$

$\{\}\cdot\hat{k} \Rightarrow$

$$\frac{mgc}{2} - F\left(\frac{c}{\mu} + \frac{h}{2} + d\right) = -ma\left(\frac{c}{\mu} + \frac{h}{2}\right)$$

(a)

$$\Rightarrow \boxed{a = \left[-\frac{mgc}{2} + F\left(\frac{c}{\mu} + \frac{h}{2} + d\right)\right] \Big/ m\left(\frac{c}{\mu} + \frac{h}{2}\right)}$$

$\underline{a} = a\hat{\imath}$

Sanity check: when $d = h = 0$ | 3, Cont'd

$$\Rightarrow \quad a = \frac{\left(-mg\,c/2 + F\frac{c}{\mu}\right)}{m\frac{c}{\mu}}$$

$$a = F/m - \mu m g/2$$

(as expected, good)

(b) One approach would be to do prob. again as a statics prob. Another is to set $\underline{a=0}$ in soln. above.

$$\Rightarrow \quad \frac{mgc}{2} = F_{min}\left(\frac{c}{\mu} + \frac{h}{2} + d\right)$$

$$\boxed{F_{min} = \frac{mgc}{2\left(\frac{c}{\mu} + \frac{h}{2} + d\right)}} \quad (b)$$

Sanity check: when $d=h=0$ $\Rightarrow F_{min} = \mu m g/2$ (as expected)

(c) In this case $F_A = 0$ (just barely)

$$\underline{AMB/D} \Rightarrow \quad \sum \underline{M}_{/D} = \dot{\underline{H}}_{/D}$$

$$\left\{ \frac{mgc}{2}\hat{k} = \underline{r}_{G/D} \times m\,\underline{a}_G \right\}$$

$$\underline{r}_{G/D} = -d\hat{j} - \frac{c}{2}\hat{i}, \qquad \underline{a}_G = a\hat{i}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \quad \frac{mgc}{2} = dma$$

$$a = \frac{gc}{2d}$$

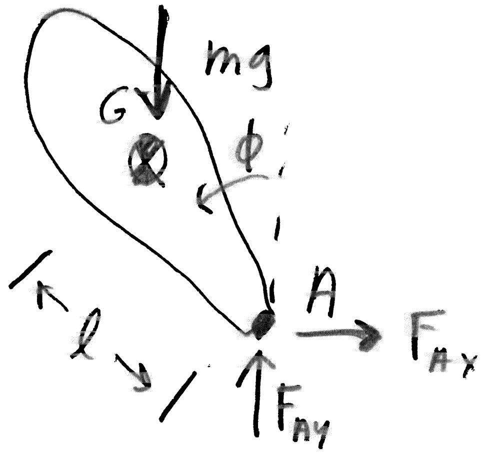
$$\boxed{\underline{a} = a\hat{i} = \frac{gc}{2d}\hat{i}} \quad (c)$$

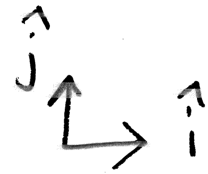
4) (20 pt) An inverted pendulum is supported at one end A by a hinge that moves up and down and, at the instant of interest, has an upwards acceleration a. The pendulum mass is m and its moment of inertia about the center of mass G is I^G. G is a distance ℓ from the end at A. At the instant in question the pendulum is tipped counter-clockwise from the vertical an angle ϕ and is tipping at the rate $\dot{\phi}$. Gravity g is pointing down.

Find $\ddot{\phi}$ in terms of some or all of a, ℓ, m, I^G, g, ϕ, and $\dot{\phi}$.

[Hint: you can check to see if your answer reduces to something you know well when $a = 0$ and $I^G = 0$. Another check is to set $a = -g$.]

FBD





given $\underline{a}_A = a\hat{j}$

$\underline{\omega} = \dot{\phi}\hat{k}$

AMB/A $\Rightarrow$ $\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$$\underline{r}_{G/A} \times (-mg\hat{j}) = \underline{r}_{G/A} \times (m\,\underline{a}_G) + I^G \ddot{\phi}\hat{k}$$

$\underline{r}_{G/A} = \ell(\cos\phi\hat{j} - \sin\phi\hat{i})$

$$\underline{a}_G = \underline{a}_A + \underline{a}_{G/A}$$
$$= a\hat{j} + \dot{\underline{\omega}} \times \underline{r}_{G/A} - \omega^2 \underline{r}_{G/A}$$

$\dot{\underline{\omega}} = \ddot{\phi}\hat{k}$

will drop out in the end ($\underline{r}_{G/A} \times \underline{r}_{G/A} = \underline{0}$)

$$\Rightarrow \Big\{ mg\ell\sin\phi\,\hat{k} = \ell(\cos\phi\hat{j} - \sin\phi\hat{i}) \times m\left[a\hat{j} + \ddot{\phi}\ell(-\cos\phi\hat{i} - \sin\phi\hat{j}) - \omega^2 \underline{r}_{G/A}\right] + I^G\ddot{\phi}\hat{k}$$

$[\cos^2 + \sin^2 = 1] \Rightarrow$

$$= -m\ell a\sin\phi\,\hat{k} + m\ell^2\ddot{\phi}\,\hat{k} + I^G\ddot{\phi}\hat{k} \Big\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \quad m(g+a)\ell\sin\phi = (I^G + m\ell^2)\ddot{\phi}$$

$$\Rightarrow \boxed{\ddot{\phi} = \frac{(g+a)m\ell\sin\phi}{I^G + m\ell^2}}$$

checks: 1) $a = 0$, $I^G = 0$

$\Rightarrow \ddot{\phi} = \frac{g}{\ell}\sin\phi$ (simple inverted pendulum)

2) $a = -g$

$\Rightarrow \ddot{\phi} = 0$ (pendulum in falling elevator. Like outer space)

5) (20 pt) A "centripital gun" consists of a rod hinged at one end at A and a frictionless collar that slides on the rod at the moving position C. The gun is powerred by the applied torque T. Neglect gravity. At the instant of interest you are given

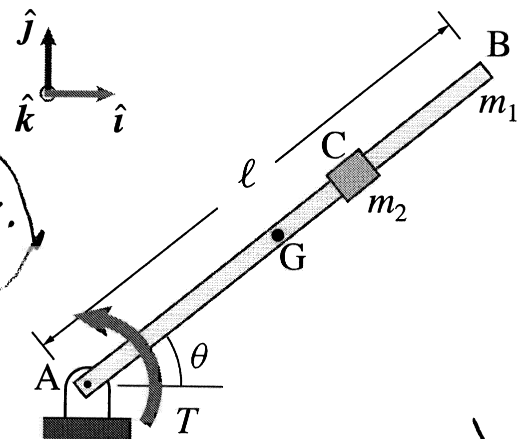
$T =$ the applied torque (counterclockwise is positive);
$m_1 =$ mass of the rod;
$m_2 =$ mass of the colar;
$I^G = m_1\ell^2/12 =$the polar moment of inertia of the rod about an axis through the center of mass G and in the ussual zz direction (perpendicular to the plane on which a side view of the suitcase is drawn);
$\ell/2 =$ the distance from A to G (from end of rod to COM);
$R = R(t) =$ the distance from A to C (the radius of the collar);
$\dot{R} = \dot{R}(t) =$ the rate of change of distance from O with time;
$\theta = \theta(t) =$ the counterclockwise angle of the rod relative to a fixed $+x$ axis.
$\dot{\theta} = \dot{\theta}(t) = \frac{d}{dt}\theta$;

Find $\ddot{R}$ in terms of some or all of $T, R, \dot{R}, \theta, \dot{\theta}, m_1, m_2, \ell$, and I^G .

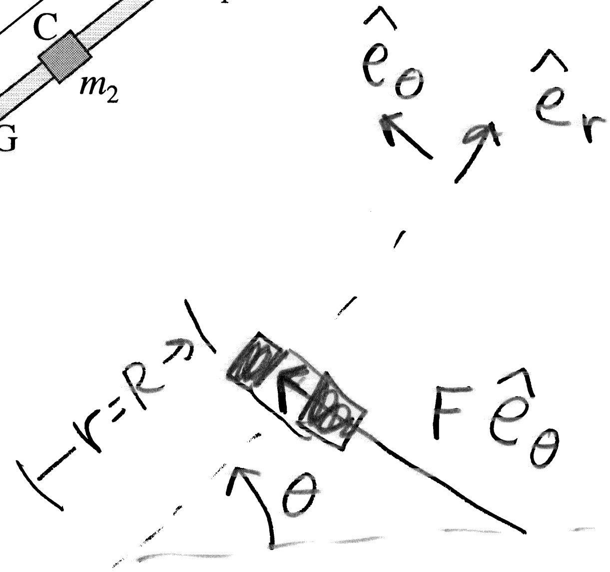
[Simplify your answer until it looks simple.]

Can also draw FBD of rod & look at its dynamics, but no need.

There are longer solutions. Here's a quick one.



FBD of collar!



LMB for collar

$$\sum \underline{F} = m\underline{a}$$

$$\left\{ F\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right]\right\}$$

$$\{\ \}\cdot\hat{e}_r \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = 0$$

$$\boxed{\ddot{r} = r\dot{\theta}^2}$$

That's it!

Note:
T has no effect! but to make $\dot{\theta}$ bigger for later on.

"SOLUTIONS"

[w/ comments on ODE solutions on pages 9-13]

Your Name: Andy Ruina

Section day and time: Wed. 10:10

T&AM 203 Prelim 1

Tuesday Feb 26, 2002

Draft February 26, 2002

3 problems, 100 points, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

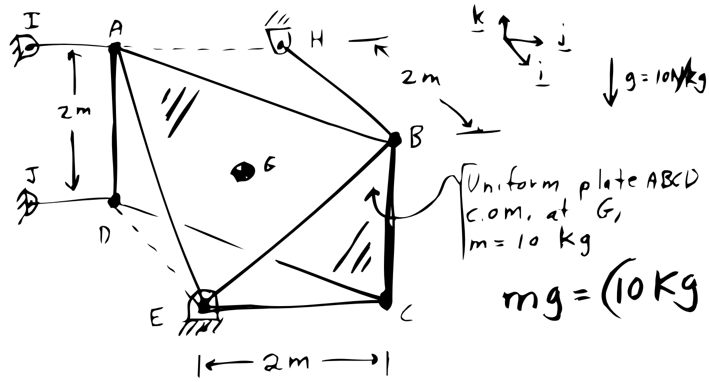
- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/50
TOTAL:	/100

1) **(25 pt)** Statics. The sign is held up by 6 bars. Find the tension in bar EB.

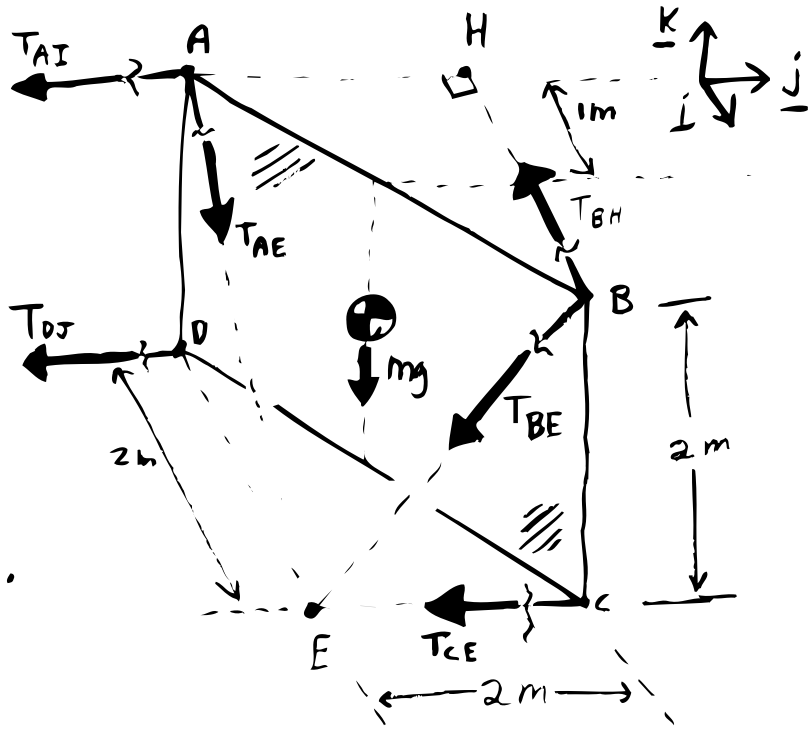


$mg = (10\,kg)(10\,N/kg) = 100\,N$

Consider axis AH:

* T_{AI}, T_{DJ}, T_{CE} are // to axis.
* T_{BH} & T_{AE} intersect axis,

$\Rightarrow$ Only T_{BE} and mg contribute to moment about axis AH. But mg is known.



$$\sum M_{\substack{axis\\AH}} = \left(\sum \underline{M}_{/H}\right)\cdot \underline{j} = 0$$

$$\Rightarrow \quad \underbrace{100\,Nm}_{\text{moment of mg about axis AH}} + \Big(\underline{r}_{B/H} \times \overbrace{T_{BE}\left(\frac{-\underline{j}-\underline{k}}{\sqrt{2}}\right)}^{\underline{T}_{BE}}\Big)\cdot \underline{j} = 0$$

$$\underline{r}_{B/H} = 2\,m\,\underline{i}$$

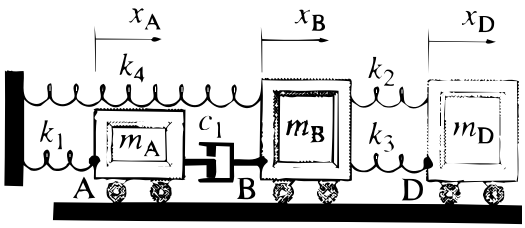
$$\Rightarrow \quad 100\,Nm + \left[2m\,T_{BE}(-\underline{k} + \underline{j})/\sqrt{2}\,\right]\cdot \underline{j} = 0$$

$$100\,N\cancel{m} + \sqrt{2}\,\cancel{m}\,T_{BE} = 0$$

$$\boxed{T_{BE} = \frac{-100}{\sqrt{2}}\,N \approx -70.7\,N}$$

(BE is in compression)

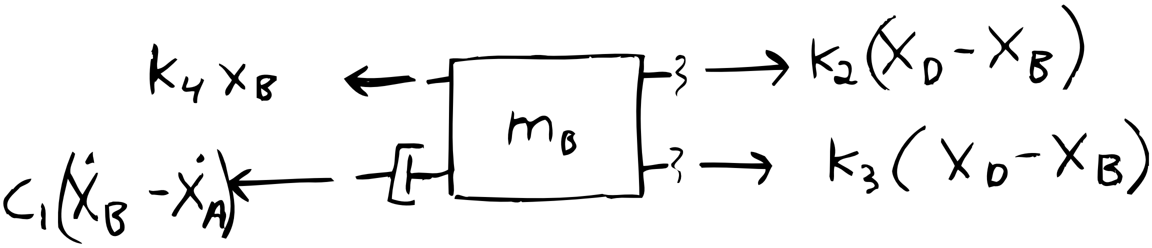
2) **(25 pt.)** In terms of some or all of $x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_D, k_1, k_2, k_3, k_4, m_A, m_B, m_D$ and c_1 find $\ddot{x}_B$. Assume the springs are relaxed when $x_A = x_B = x_D = 0$



FBD

$\rightarrow \underline{i}$

When m_B is at position x_B



LMB

$$\left\{ \sum \underline{F}_i = m_B \underline{a}_B \right\} \cdot \underline{i}$$

$$\Rightarrow -c_1(\dot{x}_B - \dot{x}_A) - k_4 x_B + k_2(x_D - x_B) + k_3(x_D - x_B) = m_B \ddot{x}_B$$

$$\boxed{\ddot{x}_B = \frac{1}{m_B}\left[-(k_2 + k_3 + k_4) x_B + (k_2 + k_3) x_D + c_1 \dot{x}_A - c_1 \dot{x}_B \right]}$$

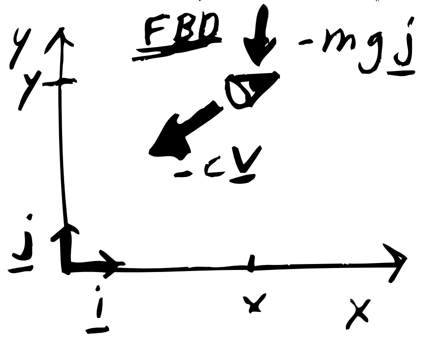
3) (50 pt) Trajectory. A 0.02 kg projectile (a badminton birdie, say) is launched from the origin at a 60° upwards angle at a speed of 50 m/s. The projectile stays near the earth so gravity $g = 10\text{m/s}^2$ is well approximated as constant (and all lines towards the center of the earth are effectively parallel). The air drag opposes motion and is proportional to speed with proportionality constant of $c = 0.1$ N/(m/s).

a) (20 pt) Write Matlab code to plot the trajectory, with the same vertical and horizontal scale, for 10 seconds [hints: FBD → LMB → first order ODEs → numerical solution → plotting].

b) (20 pt) Find, analytically, the position $\underline{\mathbf{r}}(t)$. [hints: same as above but use calculus instead of Matlab. The calculation has several steps (4 calculus problems, in one way of counting).]

c) (10 pt) More difficult. As accurately and neatly as you can, plot the trajectory. Label the units on the axis. The plot should go from when the projectile is launched until it hits the ground again. Key quantities to show are the peak height and the distance the projectile goes (which can be calculated very accurately). You can use the solution above or anything else you know or think. This will be graded on its correctness, not its agreement (or not) with the solution (b) above. But you should briefly rationalize your plot.

2D



given:
$m = .02$ Kg
$c = .1$ N/(m/s)
$g = 10$ N/Kg
$\underline{V}_0 = V_0 \underline{\lambda}_0$
$= 50$ m/s $(\cos 60° \underline{i} + \sin 60° \underline{j})$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\{-c\underline{v} - mg\underline{j} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j})\}$$
$$\underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$$

$$\{\}\cdot\underline{i} \Rightarrow -c\dot{x} = m\ddot{x} \Rightarrow \ddot{x} = \frac{-c}{m}\dot{x} \quad (1)$$
$$\{\}\cdot\underline{j} \Rightarrow -c\dot{y} - mg = m\ddot{y} \Rightarrow \ddot{y} = \frac{-c}{m}\dot{y} - g \quad (2)$$

Define $V_x = \dot{x}$, $V_y = \dot{y}$

(1), (2) $\Rightarrow$

$$\boxed{\begin{aligned} \dot{x} &= V_x & (3)\\ \dot{y} &= V_y & (4)\\ \dot{V}_x &= -(c/m)V_x & (5)\\ \dot{V}_y &= -(c/m)V_y - g & (6)\end{aligned}}$$

4 coupled 1st order ODEs

I.C.'s

$$\boxed{\begin{aligned} X_0 &= 0\\ Y_0 &= 0\\ V_{x0} &= 50\cos 60° \text{ m/s}\\ V_{y0} &= 50\sin 60° \text{ m/s}\end{aligned}}$$

Matlab solution (a)

```
% Ruina trajectory soln., assume consistent units
x0 = 0;  y0 = 0;
Vx0 = 50 * cos(60*pi/180); Vy0 = 50*sin(60*pi/180);
z0 = [x0 y0 Vx0 Vy0];
tspan = linspace [0, 10, 101];
[t z] = ode23('myrhs', tspan, z0);
x = z(:,1);  y = z(:,2);
plot (x, y); xlabel('x'); ylabel('y'); title('trajectory');
axis ('equal')
```

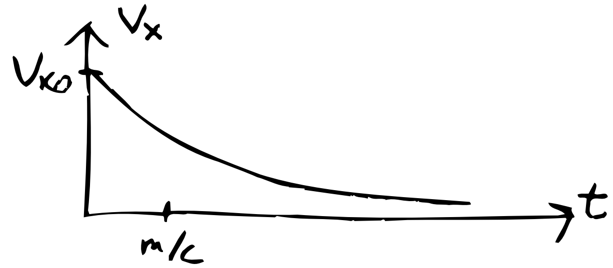
in file myrhs.m

```
function zdot = myrhs(t, z)
x = z(1); y = z(2); Vx = z(3); Vy = z(4);
c = .1; m = .02; g = 10;
xdot = Vx;
ydot = Vy;
Vxdot = -(c/m) * Vx;
Vydot = -(c/m) * Vy - g;
zdot = [xdot ydot Vxdot Vydot]';
```

Analytic Soln. (prob 3 cont'd)

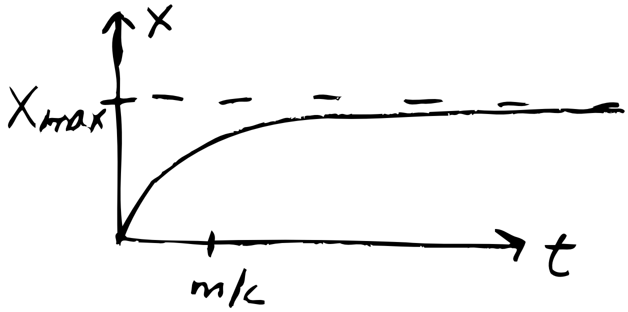
5 $\Rightarrow \quad V_x = V_{x0}\, e^{-(c/m)\,t}$

[See comments on pgs. 9-13 about ODE solutions.]



3 $\Rightarrow \quad x = x_0 + \int_0^t V_x(t')\,dt' = 0 + \int_0^t V_{x0}\, e^{-(c/m)\,t'}\,dt'$

$$= -\frac{mV_{x0}}{c}\, e^{-(c/m)\,t'}\Big|_0^t = \underbrace{\frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)}_{x(t)}$$



$$x_{max} \overset{7}{=} \frac{mV_{x0}}{c} = \frac{.02(kg)\;25(m/s)}{.1\,N/(m/s)}$$

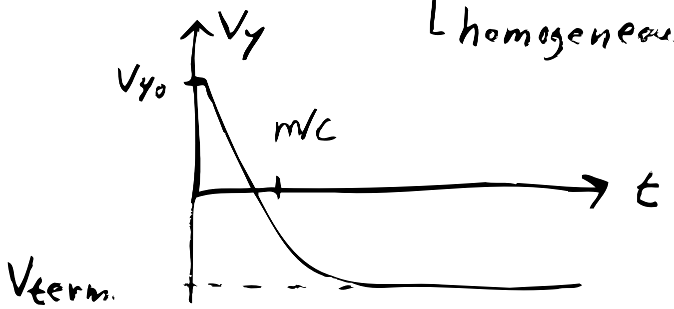
$$= \underbrace{5\,m}$$

[x expontially approaches 5 m w/ time

6 $\Rightarrow \quad V_y = \underbrace{\left(V_{y0} + \frac{gm}{c}\right) e^{-\frac{c}{m}t}} \; \underbrace{-\, gm/c}$

[partic. soln. from inspection (terminal vel.)

homogeneous soln., constant picked to sat. I.C.



V_{term} = terminal vel.

$= -gm/c$ (drag balances weight)

$= -10 \cdot .02/.1$ (m/s)

$= 2$ m/s

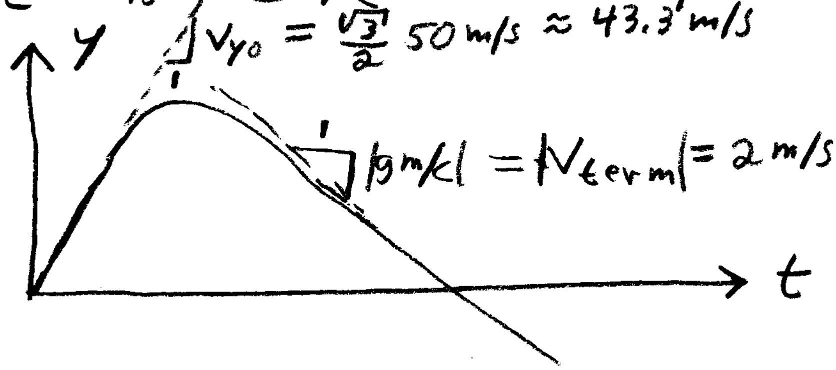
4 $\Rightarrow \quad y = y_0 + \int_0^t v_y(t')\,dt'$

$$= 0 + \int_0^t \left[\left(V_{y0} + \frac{gm}{c}\right) e^{-(c/m)\,t'} - gm/c\right] dt'$$

(prob (3) cont'd)

$$y = \left[-\frac{m}{c}\left(V_{y0} + \frac{gm}{c}\right) e^{-(c/m)t'} - \frac{gm}{c} t' \right]_0^{t'}$$

$$= \frac{m}{c}\left(V_{y0} + \frac{gm}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c} t$$



$$\underline{r}(t) = x\,\underline{i} + y\,\underline{j}$$

$$= \frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)\underline{i}$$

$$+ \left[\left(\frac{mV_{y0}}{c} + \frac{gm^2}{c^2}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c} t\right]\underline{j}$$

$c/m = (.1/.02)\,s^{-1} = 5/s$, $\frac{gm^2}{c^2} = 10 \cdot \frac{(.02)^2}{(.1)^2}\,m = .4\,m$

$\frac{mV_{x0}}{c} = 5\,m$ (see (7))

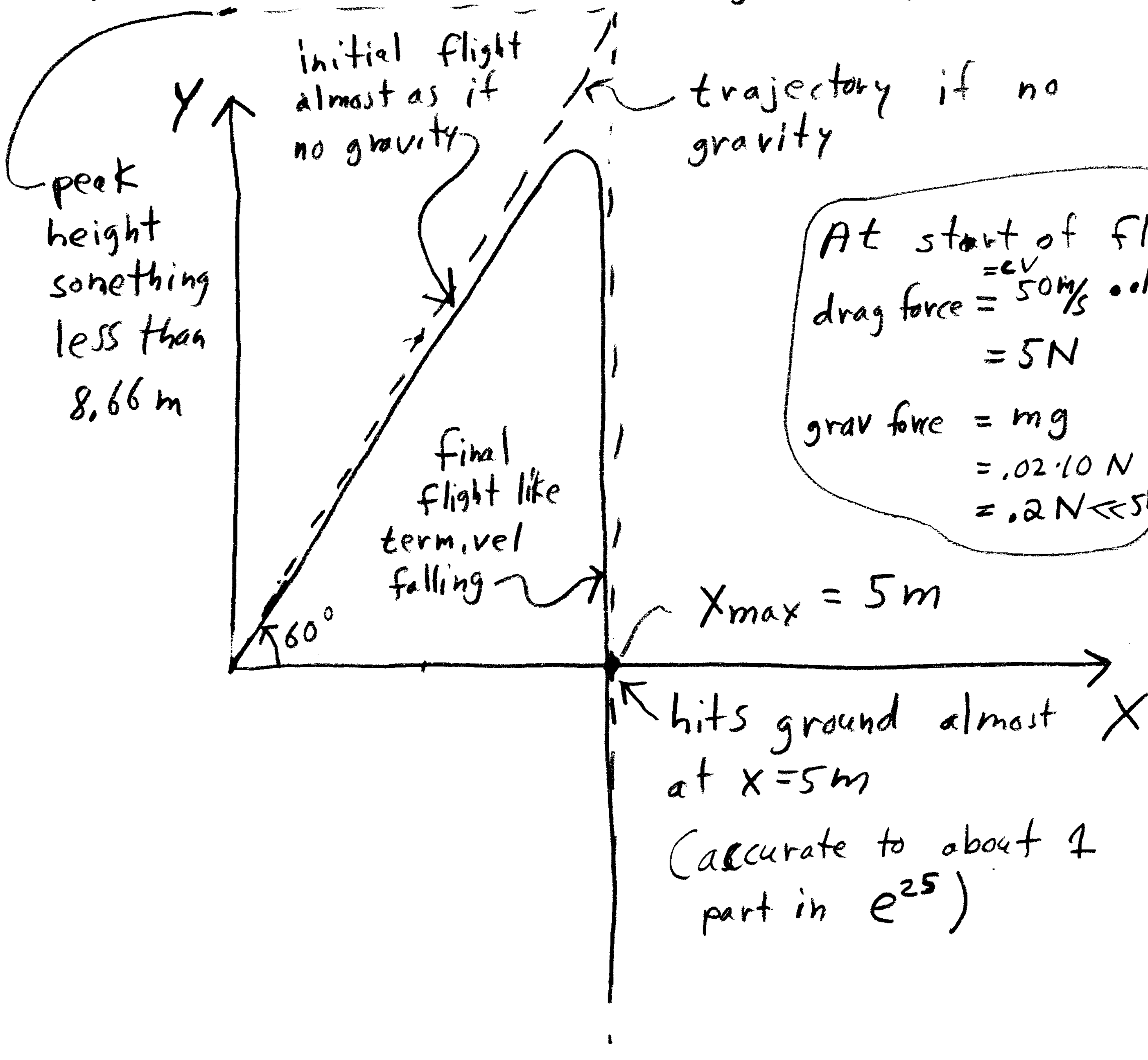
$\frac{mV_{y0}}{c} = \frac{\sqrt{3}}{2} \cdot 5\,m \approx 8.66\,m$

$\frac{gm}{c} = 2\,m/s$

(b)

$$\boxed{\underline{r}(t) = 5\,m\left(1 - e^{-5t/s}\right)\underline{i} + \left[\left(\frac{5\sqrt{3}}{2} - .4\right)\left(1 - e^{-5t/s}\right)m - 2(m/s)t\right]\underline{j}}$$

c) Plot. All we need from anal. soln. is $\underbrace{\text{eqn. 7}}_{\text{i.e. eqn for } x(t)}$.
(No need for soln. of inhomog. soln.)



At start of flight

drag force $= \overset{=cv}{50\,\text{m/s} \cdot .1\,\text{N/m/s}}$
$= 5\,N$

grav force $= mg$
$= .02 \cdot 10\,N$
$= .2\,N \ll 5N$

Some comments on ODE solns.

Solve $\dot{V}_x = -(c/m)V_x$

method 1: guess $V_x = e^{rt}$

plug in $r\cancel{e^{rt}} = -(c/m)\cancel{e^{rt}}$

$$r = -c/m$$

$\Rightarrow V_x = c_1 e^{-(c/m)t}$

c_1 ← arb. const., pick to sat. I.C.

method 2: (Edwards & Penny 1.4)

$$\frac{dV_x}{dt} = -(c/m)V_x$$

$$\Rightarrow \frac{dV_x}{V_x} = -(c/m)dx$$

$$\Rightarrow \int \frac{dV_x}{V_x} = -\int c/m\, dt$$

$$\Rightarrow \ln V_x = -\frac{c}{m}t + c_1'$$

$$\Rightarrow V_1 = e^{-(c/m)t + c_1'}$$

$$V_x = c_1 e^{-(c/m)t} \qquad (c_1 = e^{c_1'})$$

(again)

ODEs (cont'd)

<u>method 3</u>: Integrating factor
(Edwards & Penny 1.5)

$$\frac{dV_x}{dt} + \frac{c}{m} V_x = 0$$

$e^{c/m\, t}$ is the integrating factor.

$$\Rightarrow \quad e^{\frac{c}{m}t}\frac{dV_x}{dt} + e^{(c/m)t}\frac{c}{m}V_x = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{(c/m)t} V_x\right) = 0$$

$$\Rightarrow \quad e^{(c/m)t}\, V_x = C_1$$

$$\Rightarrow \quad V_x = C_1 e^{-(c/m)t}$$

(again)

<u>Solve</u> $\ddot{x} + c/m\, \dot{x} = 0$

<u>method 1</u>: guess $x = e^{rt}$

$$\Rightarrow \quad r^2 \cancel{e^{rt}} + r\frac{c}{m}\cancel{e^{rt}} = 0$$

$$\Rightarrow \quad r\left(r + \frac{c}{m}\right) = 0 \Rightarrow r = 0, -c/m$$

$$\Rightarrow \quad x(t) = C_1 e^{-\frac{c}{m}t} + C_2 e^{-0t}$$

$$= C_1 e^{-(c/m)t} + C_2$$

find using ICs.

This is $x(t)$ as found.

Solve $\dot{V}_y + (c/m)\, V_y = -g$

method 1: a) find homog. soln.

$$\dot{V}_y + (c/m) V_y = 0.$$

This is identical to prev. problem which we solved to get

$$\underbrace{V_{y_h}}_{\text{homog. soln.}} = c_1 e^{-(c/m)t}$$

b) Find any "particular" soln. of

$$\dot{V}_y + (c/m) y = -g.$$

As for spring-mass problem. Easiest guess is a constant. In this case you can get this physically by thinking of falling at terminal velocity.

guess $V_{y_p} = c_2$

$$\cancelto{0}{\dot{c}_2} + (c/m)\, c_2 = g$$

$$c_2 = gm/c$$

Solution is

$$V_y = V_{yh} + V_{yp}$$

$$= \underset{\text{pick to match I.C.}}{c_1} e^{-(c/m)t} + gm/c$$

method 2; (E 8P 1.4) (seperable eqn.)

$$\frac{dV_y}{dt} + \frac{c}{m}V_y = -g$$

$$\Rightarrow \frac{dV_y}{dt} = -\frac{c}{m}V_y - g$$

$$\Rightarrow \frac{dV_y}{V_y + \frac{gm}{c}} = -\frac{c}{m}\,dt$$

$$\Rightarrow \int \frac{dV_y}{V_y - \frac{gm}{c}} = -\int \frac{c}{m}\,dt$$

$$\Rightarrow \ln\left(V_y + \frac{gm}{c}\right) = -\frac{c}{m}t + c_1''$$

$$\Rightarrow V_y + \frac{gm}{c} = e^{-\frac{c}{m}t + c_1'}$$

$$\Rightarrow V_y = c_1 e^{-(c/m)t} - gm/c$$

$c_1 = e^{c_1'}$, pick to sat. I.C.

(again)

method 3; (E 8P 1.5) linear ODE using integrating factor

$$\frac{dV_y}{dt} + \frac{c}{m}V_y = -g$$

define $\rho(t) = e^{(c/m)t}$ integrating factor

mult. through by ρ

$$\frac{dV_y}{dt}e^{(c/m)t} + \frac{c}{m}V_y e^{(c/m)t} = -g e^{(c/m)t}$$

$$\frac{d}{dt}\left(v_y e^{(c/m)t}\right) = -g e^{(c/m)t}$$

$$\Rightarrow \quad d\left[v_y\, e^{(c/m)t}\right] = -g e^{(c/m)t}\, dt$$

$$\Rightarrow \quad \int d\left(v_y e^{(c/m)t}\right) = -\int g e^{c/m\, t}\, dt$$

$$\Rightarrow \quad v_y\, e^{(c/m)t} = -\frac{gm}{c} e^{(c/m)t} + C_1$$

$$\Rightarrow \quad v_y = \frac{-gm}{c} + c_1 e^{-(c/m)t}$$

pick to sat. I.C.

(again)

"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: Wed, 11:15

T&AM 203 Prelim 2

Tuesday April 16, 2002

Draft April 16, 2002

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	30/30
Problem 2:	40 /40
Problem 3:	30 /30
TOTAL:	100 /100

1) (30 pt) A front-wheel drive car has an engine with arbitrarily large power. It has a stiff suspension and light well-lubricated front wheels. It rides on level ground. Given:

- $m =$ total car mass;
- $I^G =$ the polar moment of inertia about an axis through the center of mass G and in the ussual zz direction (perpendicular to the plane on which a side view of the car is drawn);
- $h =$ the height of G above the ground;
- $d =$ the distance G is forward of the rear ground contact at A;
- $e =$ the distance G is behind the front ground contact at B;
- $\mu =$ the coefficient of friction between the wheels and ground;
- $g =$ the gravity constant. ~~In terms of some or all of m, I^G, g, h, d, e, and μ find the maximum possible forward acceleration on level ground.~~

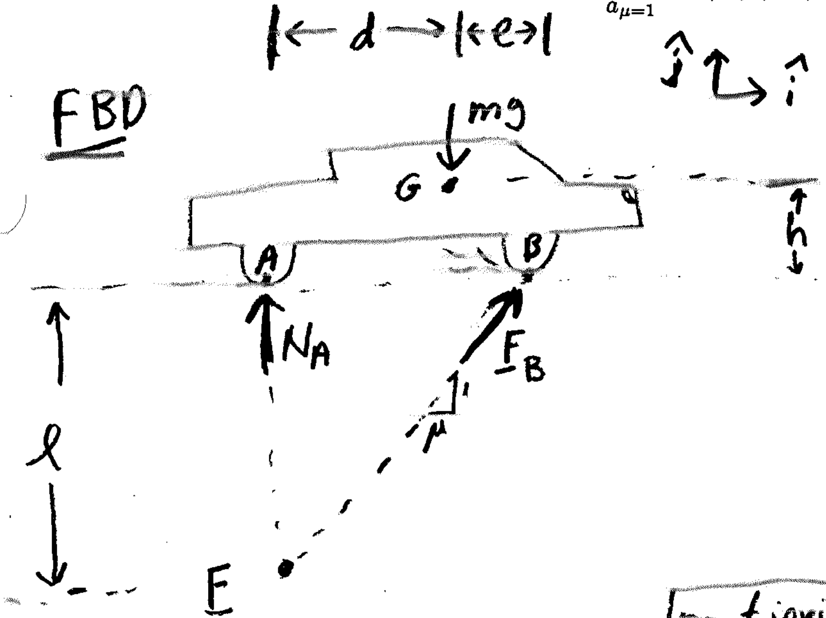
a) (20 points) Assuming the car does not tipe over, find the maximum possible forward acceleration a ($\underline{\mathbf{a}}_G = a\hat{\mathbf{i}}$). Answer in terms of some or all of m, I^G, g, h, d, e, and μ.

b) (5 points) (Good reasoning may or may not depend on part (a)) Assuming $d = e = h$, what is the largest value of μ that is possible without violating reasonable assumptions (assume that rubber with arbitrarily large μ could be made reasonably)? [Clearly state the reasonable assumptions that you are checking]. The answer is one of these

$$\mu_{max} = 0, .5, 1, \sqrt{2}, \sqrt{3}, 2, e, 2\sqrt{2}, 3, \pi, 4, 8, \text{ or } \infty.$$

c) (5 points) (Good reasoning may or may not depend on part (a)) Assuming $d = e = h$ and that the car does not tip over, by what ratio does the peak acceleration increase if μ is increased from $\mu = 1$ to $\mu = \infty$ (infinite coefficient of friction). The answer is one of these

$$\frac{a_{\mu=\infty}}{a_{\mu=1}} = 0, .5, 1, \sqrt{2}, \sqrt{3}, 2, e, 2\sqrt{2}, 3, \pi, 4, 8, \text{ or } \infty.$$



Geometry:

$$\frac{d+e}{\ell} = \frac{\mu}{1} \Rightarrow \ell = \frac{d+e}{\mu}$$

$$\sum \underline{M}_{/E} = \dot{\underline{H}}_{/E}$$

no tipping

$$-dmg\hat{k} = \underline{r}_{G/E} \times m\underline{a}_G + I^G \cancel{\alpha}^{\,0}\hat{k}$$

$$= ((\ell+h)\hat{j} + d\hat{i}) \times ma\hat{i}$$

$$= -\left(\frac{d+e}{\mu}+h\right) a\hat{k}m$$

$\{\}\cdot\hat{k} \Rightarrow$

$$dmg = \left(\frac{d+e}{\mu}+h\right) am$$

$$\boxed{a = \frac{dg}{\left(h+\frac{d+e}{\mu}\right)}}$$

$$\boxed{\underline{a}_G = \frac{dg\hat{i}}{\left(h+\frac{d+e}{\mu}\right)}}$$

(a)

(prob 1 cont'd)

(b) $\sum \underline{M}_{/B} = \dot{\underline{H}}_{/B}$

$$\left\{ \left(-N_A(d+e) + mge\right)\hat{k} = -mah\hat{k} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow N_A = \frac{m(ah+ge)}{d+e} > 0$$

so rear wheels have no lift off problem. Front wheels can't have a lift off problem because the drive ~~force~~ goes to zero as lifting proceeds.

$\Rightarrow \boxed{\mu = \infty \text{ is o.k.}}$ (b)

c) Could redo problem using assumptions, or just plug in soln. to (a): $(d=e=h)$

$$\boxed{\frac{a_{\mu=\infty}}{a_{\mu=1}} = \frac{\frac{dg}{\left(d+\frac{2d}{\infty}\right)}}{\frac{dg}{d+\frac{2d}{1}}} = \frac{\frac{g}{1}}{\frac{g}{3}} = 3}$$!

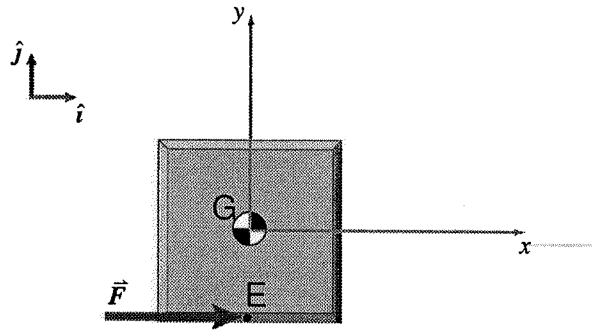
(Front wheel drive car (w/ $d=e=h$) can have accel. of $g/3$ if $\mu=1$ and g if $\mu=\infty$.)

2) (40 pt) A uniform 1kg plate that is one meter on a side is initially at rest in the position shown. A constant force $\underline{\mathbf{F}} = 1\,\mathrm{N}\,\hat{\imath}$ is applied at $t = 0$ and maintained henceforth. [If you need to calculate any quantity that you don't know, but can't do the calculation to find it, assume that the value is given.]

a) Find the position of G as a function of time (the answer should have numbers and units).

b) Find a differential equation, and initial conditions, that when solved would give θ as a function of time. θ is the counterclockwise rotation of the plate from the configuration shown.

c) Write MATLAB commands that would generate a drawing of the outline of the plate at $t = 1\,\mathrm{s}$. You can use hand calculations or Matlab for as many of the intermediate commands as you like. Add enough hand work so your MATLAB reasoning is clear.



$\Rightarrow$

$\underline{r}_G = \frac{F}{m} t^2/2\,\hat{\imath}$

$= \frac{1N}{1Kg}\frac{t^2}{2}\hat{\imath}$

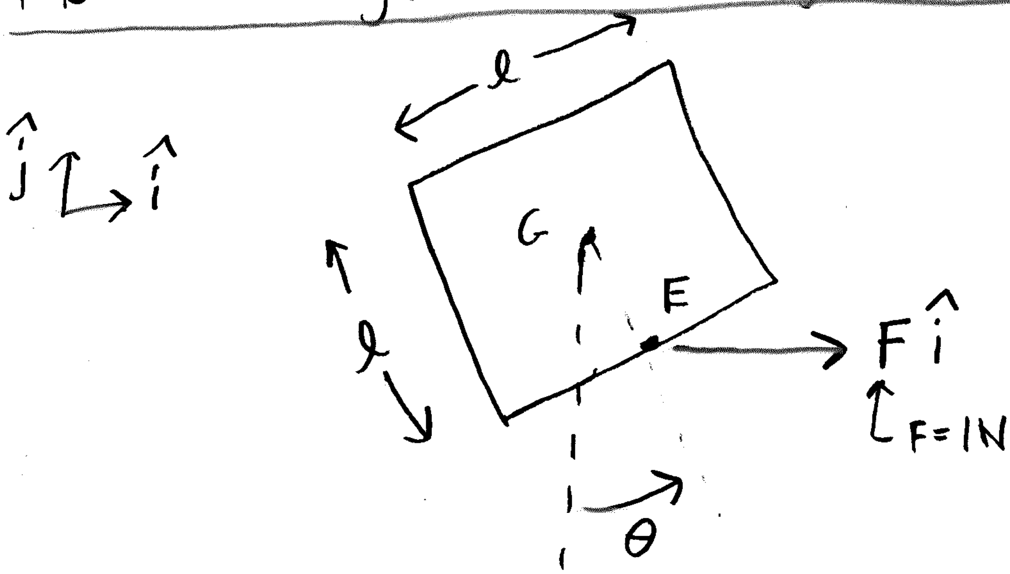
$\boxed{\underline{r}_G = \frac{t^2}{2}\hat{\imath}\ \frac{m}{sec^2}}$ (a)

or

$\underline{r}_G = \frac{t^2}{2}\hat{\imath}$

if all quantities in meters, kg, sec.

FBD in general configuration



LMB: $\sum \underline{F}_i = m\underline{a}_G$

$\{F\hat{\imath} = m a_{Gx}\hat{\imath} + m a_{Gy}\hat{\jmath}\}$

$\{\ \}\cdot\hat{\imath} \Rightarrow a_{Gx} = F/m \Rightarrow V_{Gx} = \frac{F}{m}t \Rightarrow X_G = \frac{F}{m}t^2/2$

$\{\ \}\cdot\hat{\jmath} \Rightarrow a_{Gy} = 0 \Rightarrow Y_G = 0$

b) $\sum \underline{M}_{/G} = \dot{\underline{H}}_{/G}$ (Prob. 2 cont'd)

$$\underline{r}_{E/G} \times F\hat{i} = I^G \alpha \hat{k}$$

$$\frac{l}{2}(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) \times F\hat{i} = I^G \ddot{\theta}\hat{k}$$

$$\Rightarrow \frac{lF}{2}\cos\theta = I^G\ddot{\theta}$$

$$\boxed{\ddot{\theta} = \frac{lF}{2I_G}\cos\theta}$$

$l = 1\,m$, $F = 1\,N$, $m = 1\,kg$

$$I^G = \int_{-l/2}^{l/2}\int_{-l/2}^{l/2}(x^2+y^2)\frac{m}{l^2}\,dx\,dy$$

$$I^G = \left(\frac{x^3}{3}l\Big|_{-l/2}^{l/2} + \frac{y^3}{3}l\Big|_{-l/2}^{l/2}\right)\frac{m}{l^2}$$

$$I^G = \left(\frac{l^4}{12} + \frac{l^4}{12}\right)\frac{m}{l^2} = \frac{ml^2}{6}$$

$$\Rightarrow \ddot{\theta} = \left(\frac{1\cdot 1}{2\cdot\frac{1}{6}}\cos\theta\right)s^{-2} \Rightarrow \boxed{\ddot{\theta} = 3\cos\theta\ s^{-2}}\ (b)$$

---

c)

all in consistent mKS units

```
pic = [1  -1  -1   1  1;
       1   1  -1  -1  1]/2;
θ0 = 0;  ω0 = 0;
[t z] = ode23('gosh', [0 1], [θ0 ω0])
```

$[0\ 1]$: tspan ; $[\theta_0\ \omega_0]$: z_0

```
θ1 = z(end,1);   % final rotation
R = [cos θ1   -sin θ1;
     sin θ1    cos θ1 ];
disp = [1/2  0]';   % displ. from part (a)
homo = [ R  disp;
        [0 0 1] ];   % homog. transf.
newpic = homo * [pic; ones(1,5)];
plot(newpic(1,:), newpic(2,:));
axis('equal')
```

ODE from (b)

```
function zdot = gosh(t,z)
θ = z(1);  ω = z(2);
θ̇ = ω;  ω̇ = 3cos θ;
zdot = [θ̇  ω̇]';
```

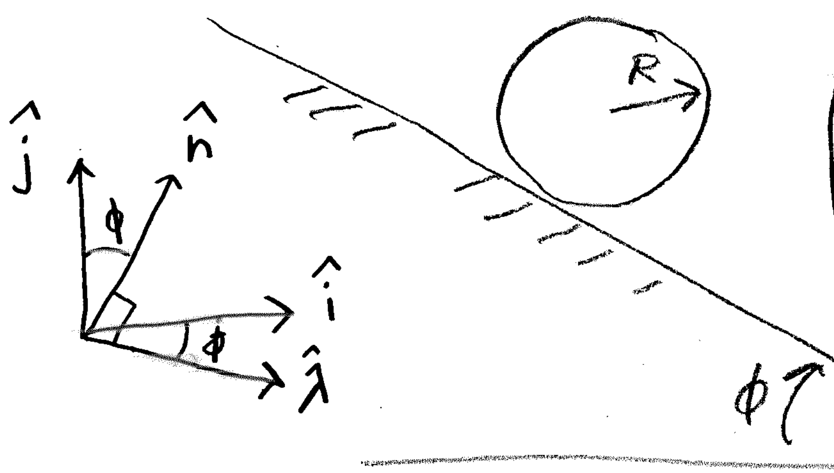
gosh.m

3) **(30 pt)** A uniform disk with mass m and radius R is released from rest to roll down a slope that is tipped ϕ from the horizontal. The local gravity constant is g.

a) (10 pt) Assume that slope is high, or friction coefficient small, so the disk slides down the slope. What is the acceleration of the center of the disk?

b) (10 pt) Assume the disk rolls, what is the angular acceleration of the disk?

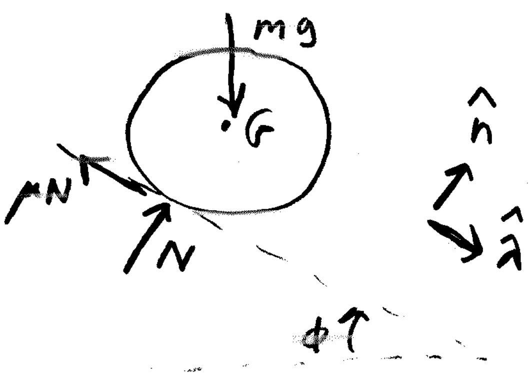
c) (10 pt) If $\mu = .5$, what is the biggest angle ϕ for which the disk will roll and not slide.



$$I^G = \int r^2 dm = \int_0^{2\pi}\int_0^R r^2 \frac{m}{\pi R^2} r\,dr\,d\theta$$
$$= \frac{2\pi m}{\pi R^2}\int_0^R r^3 dr$$
$$= mR^2/2$$

a) FBD

sliding



Kinematics

$$\underline{a}_G = a_G\hat{\lambda}$$

LMB

$$\sum \underline{F}_i = m\underline{a}_G$$

$$\{N\hat{n} - \mu N\hat{\lambda} - mg\hat{j} = m a_G\hat{\lambda}\}$$

$$\{\}\cdot\hat{n} \Rightarrow N - mg\underbrace{\hat{j}\cdot\hat{n}}_{\cos\phi} = 0$$

$$\Rightarrow N = mg\cos\phi$$

$$\{\}\cdot\hat{\lambda} \Rightarrow$$

$$-\mu N - mg\underbrace{\hat{j}\cdot\hat{\lambda}}_{-\sin\phi} = m a_G$$

$$\Rightarrow$$

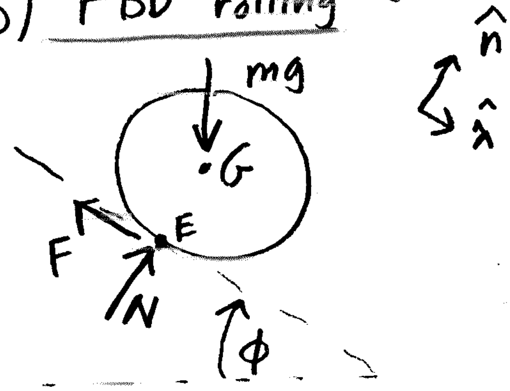
$$-\mu\cancel{m}g\cos\phi + \cancel{m}g\sin\phi = \cancel{m}a_G$$

$$a_G = g(\sin\phi - \mu\cos\phi)$$

$$\Rightarrow \boxed{\underline{a}_G = g(\sin\phi - \mu\cos\phi)\hat{\lambda}}$$

(a)

b) FBD rolling (prob 3 cont'd)



Kinematics	$\underline{a}_G = -R\alpha\hat{\lambda}$
$R\alpha = -a_G$	$\underline{\alpha} = \alpha\hat{k}$

$\underline{\text{AMB}}_{/E} \Rightarrow \underline{r}_{G/E} \times (-mg\hat{j}) = \underline{r}_{G/E} \times m\underline{a}_G + I^G\alpha\hat{k}$

$\Rightarrow R\hat{n} \times (-mg\hat{j}) = R\hat{n} \times (m(-R\alpha\hat{\lambda})) + I^G\alpha\hat{k}$

$\Rightarrow \{-Rmg\sin\phi\,\hat{k} = R^2 m\alpha\,\hat{k} + I^G\alpha\hat{k}\}$

$\{\ \}\cdot\hat{k} \quad \Rightarrow \alpha = \dfrac{-Rmg\sin\phi}{\underbrace{I^G}_{mR^2/2} + mR^2} = \dfrac{-2Rg\sin\phi}{3R^2}$

$\Rightarrow \boxed{\text{ang. accel.} = \dfrac{-2g\sin\phi}{3R}\hat{k}}$ (b)

c) LMB

$\sum \underline{F}_i = m\underline{a}_G$

$\left\{N\hat{n} - F\hat{\lambda} - mg\hat{j} = m\underbrace{a_G}_{= -\alpha R = 2g\sin\phi/3}\hat{\lambda}\right\}$

$\{\ \}\cdot\hat{n} \Rightarrow N - mg\cos\phi = 0$

$N = mg\cos\phi$ (or for sliding)

$\{\ \}\cdot\hat{\lambda} \Rightarrow -F + mg\sin\phi = 2mg\sin\phi/3$

$\Rightarrow F = mg\sin\phi(1 - 2/3)$

$= mg\sin\phi/3$

critical slope when $\mu = \dfrac{F}{N} = \dfrac{mg\sin\phi/3}{mg\cos\phi}$

$\Rightarrow \tan\phi = 3\underbrace{\mu}_{0.5 \text{ given}} \Rightarrow \tan\phi = 3/2 \Rightarrow \boxed{\phi = \tan^{-1}(3/2)}$

"SOLUTIONS"

[w/ comments on ODE solutions on pages 9-13]

Your Name: Andy Ruina

Section day and time: Wed. 10:10

T&AM 203 Prelim 1

Tuesday Feb 26, 2002

Draft February 26, 2002

3 problems, 100 points, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

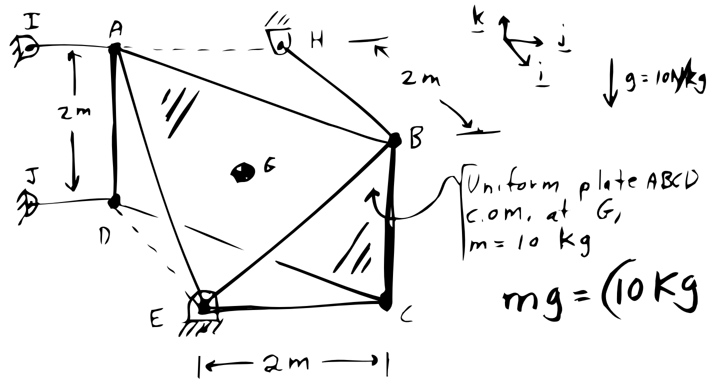
b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/50
TOTAL:	**/100**

1) **(25 pt)** Statics. The sign is held up by 6 bars. Find the tension in bar EB.



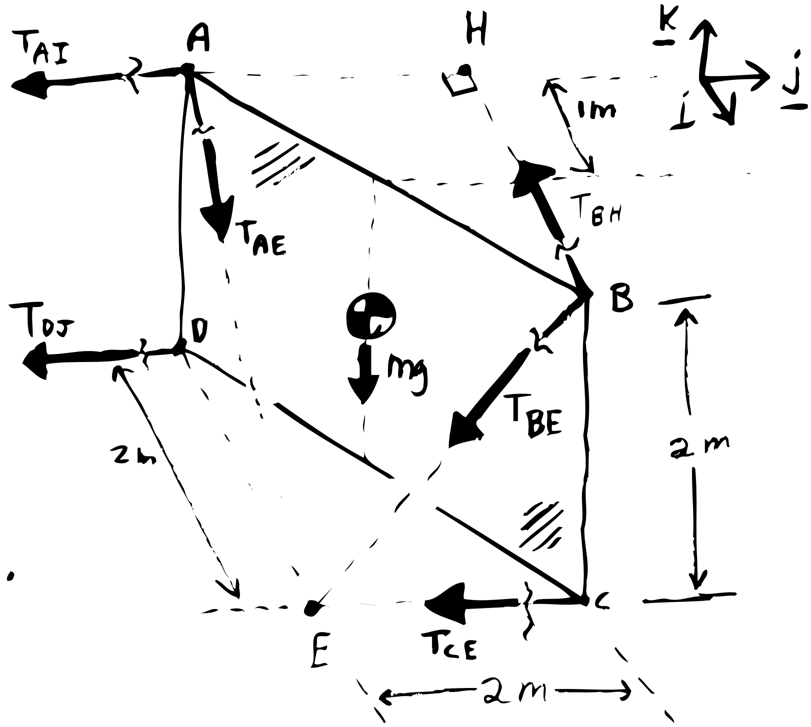
$$mg = (10\,kg)(10\,N/kg) = 100\,N$$

Consider axis AH:

* T_{AI}, T_{DJ}, T_{CE} are ∥ to axis.
* T_{BH} & T_{AE} intersect axis,

⇒ Only T_{BE} and mg contribute to moment about axis AH.

But mg is known.



$$\sum M_{\substack{axis\\AH}} = \left(\sum \underline{M}_{/H}\right)\cdot \underline{j} = 0$$

$$\Rightarrow \quad \underbrace{100\,Nm}_{\text{moment of mg about axis AH}} + \Bigg(\underline{r}_{B/H} \times \overbrace{T_{BE}\left(\frac{-\underline{j}-\underline{k}}{\sqrt{2}}\right)}^{\underline{T}_{BE}}\Bigg)\cdot \underline{j} = 0$$

$$\underline{r}_{B/H} = 2\,m\,\underline{i}$$

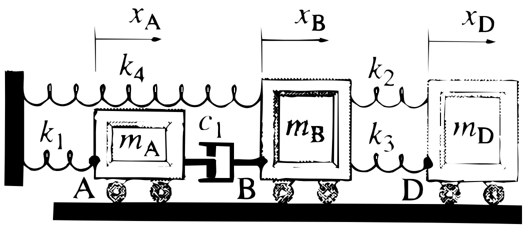
$$\Rightarrow \quad 100\,Nm + \left[2m\,T_{BE}(-\underline{k} + \underline{j})/\sqrt{2}\right]\cdot\underline{j} = 0$$

$$100\,N\cancel{m} + \sqrt{2}\,\cancel{m}\,T_{BE} = 0$$

$$\boxed{T_{BE} = \frac{-100}{\sqrt{2}}\,N \approx -70.7\,N}$$

(BE is in compression)

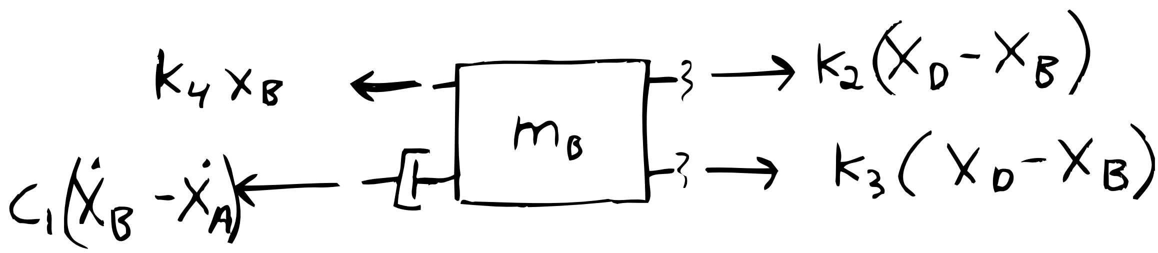
2) **(25 pt.)** In terms of some or all of $x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_D, k_1, k_2, k_3, k_4, m_A, m_B, m_D$ and c_1 find $\ddot{x}_B$. Assume the springs are relaxed when $x_A = x_B = x_D = 0$



FBD

$\rightarrow \underline{i}$

When m_B is at position x_B



LMB

$$\left\{ \Sigma \underline{F}_i = m_B \underline{a}_B \right\} \cdot \underline{i}$$

$$\Rightarrow -c_1(\dot{x}_B - \dot{x}_A) - k_4 x_B + k_2(x_D - x_B) + k_3(x_D - x_B) = m_B \ddot{x}_B$$

$$\boxed{\ddot{x}_B = \frac{1}{m_B}\left[-(k_2 + k_3 + k_4)x_B + (k_2 + k_3)x_D + c_1\dot{x}_A - c_1\dot{x}_B\right]}$$

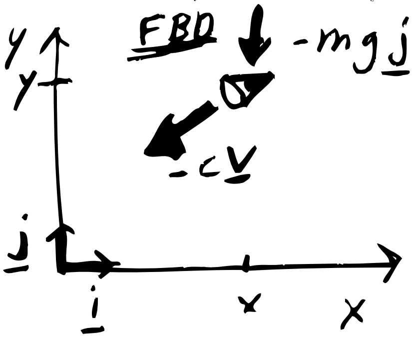
3) (50 pt) Trajectory. A 0.02 kg projectile (a badminton birdie, say) is launched from the origin at a 60° upwards angle at a speed of 50 m/s. The projectile stays near the earth so gravity $g = 10\text{m/s}^2$ is well approximated as constant (and all lines towards the center of the earth are effectively parallel). The air drag opposes motion and is proportional to speed with proportionality constant of $c = 0.1$ N/(m/s).

a) (20 pt) Write Matlab code to plot the trajectory, with the same vertical and horizontal scale, for 10 seconds [hints: FBD → LMB → first order ODEs → numerical solution → plotting].

b) (20 pt) Find, analytically, the position $\underline{\mathbf{r}}(t)$. [hints: same as above but use calculus instead of Matlab. The calculation has several steps (4 calculus problems, in one way of counting).]

c) (10 pt) More difficult. As accurately and neatly as you can, plot the trajectory. Label the units on the axis. The plot should go from when the projectile is launched until it hits the ground again. Key quantities to show are the peak height and the distance the projectile goes (which can be calculated very accurately). You can use the solution above or anything else you know or think. This will be graded on its correctness, not its agreement (or not) with the solution (b) above. But you should briefly rationalize your plot.

2D



given:
$$\left[\begin{aligned} m &= .02\ \text{Kg} \\ c &= .1\ \text{N/(m/s)} \\ g &= 10\ \text{N/Kg} \\ \underline{v}_0 &= v_0 \underline{\lambda}_0 \\ &= 50\ \text{m/s}\left(\cos 60° \underline{i} + \sin 60° \underline{j}\right) \end{aligned}\right.$$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\left\{ -c\underline{v} - mg\underline{j} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) \right\}$$
$$\quad \underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$$

$$\{\ \}\cdot\underline{i} \Rightarrow \quad -c\dot{x} = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{-c}{m}\dot{x} \qquad (1)$$

$$\{\ \}\cdot\underline{j} \Rightarrow \quad -c\dot{y} - mg = m\ddot{y} \Rightarrow \quad \ddot{y} = \frac{-c}{m}\dot{y} - g \qquad (2)$$

Define $v_x = \dot{x}$, $v_y = \dot{y}$

I.C.'s

$(1), (2) \Rightarrow$
$$\boxed{\begin{aligned} \dot{x} &= v_x & (3) \\ \dot{y} &= v_y & (4) \\ \dot{v}_x &= -(c/m)v_x & (5) \\ \dot{v}_y &= -(c/m)v_y - g & (6) \end{aligned}}$$

4 coupled 1st order ODEs

$$\boxed{\begin{aligned} x_0 &= 0 \\ y_0 &= 0 \\ v_{x0} &= 50\cos 60°\ \text{m/s} \\ v_{y0} &= 50\sin 60°\ \text{m/s} \end{aligned}}$$

(prob 3 cont'd)

Matlab solution (a)

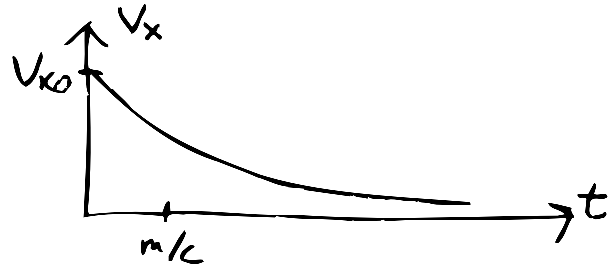
```
% Ruina trajectory soln., assume consistent units
x0=0; y0=0;
vx0 = 50*cos(60*pi/180); vy0= 50*sin(60*pi/180);
z0 = [x0 y0 vx0 vy0];
tspan = linspace [0, 10, 101];
[t z]= ode23('myrhs', tspan, z0);
x= z(:,1); y= z(:,2);
plot (x,y); xlabel('x'); ylabel('y'); title('trajectory');
axis ('equal')
```

in file myrhs.m

```
function zdot = myrhs(t,z)
x = z(1); y = z(2); vx = z(3); vy = z(4);
c=.1; m=.02; g=10;
xdot = vx;
ydot = vy;
vxdot = -(c/m)*vx;
vydot = -(c/m)*vy -g;
zdot=[xdot ydot vxdot vydot]';
```

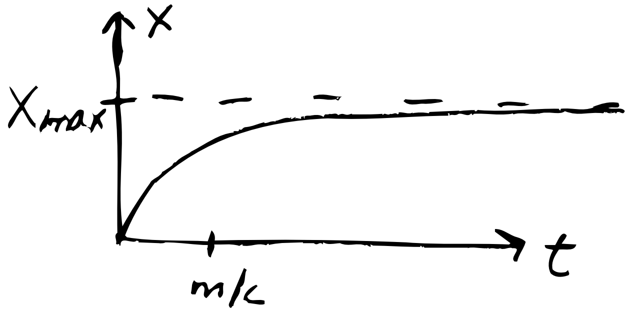
Analytic Soln. (prob 3 cont'd)

(5) $\Rightarrow \quad V_x = V_{x0}\, e^{-(c/m)\,t}$ [See comments on pgs. 9-13 about ODE solutions.]



(3) $\Rightarrow \quad x = x_0 + \int_0^t V_x(t')\,dt' = 0 + \int_0^t V_{x0}\, e^{-(c/m)\,t'}\,dt'$

$$= -\frac{mV_{x0}}{c}\, e^{-(c/m)\,t'}\Big|_0^t = \underbrace{\frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)}_{x(t)}$$



$$x_{max} \overset{(7)}{=} \frac{mV_{x0}}{c} = \frac{.02\,(kg)\; 25\,(m/s)}{.1\,N/(m/s)}$$

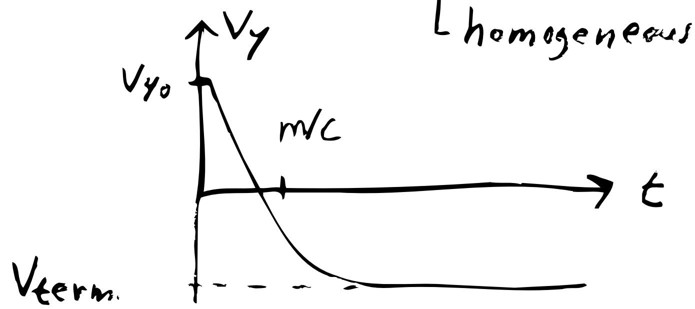
$$= \underbrace{5\,m}$$

[x expontially approaches 5 m w/ time

(6) $\Rightarrow \quad V_y = \underbrace{\left(V_{y0} + \frac{gm}{c}\right) e^{-\frac{c}{m}t}}\ \underbrace{-\, gm/c}$

↑ [partic. soln. from inspection (terminal vel.)

└ homogeneous soln., constant picked to sat. I.C.



V_{term} = terminal vel.

$= -gm/c$ (drag balances weight)

$= -10 \cdot .02/.1$ (m/s)

$= 2$ m/s

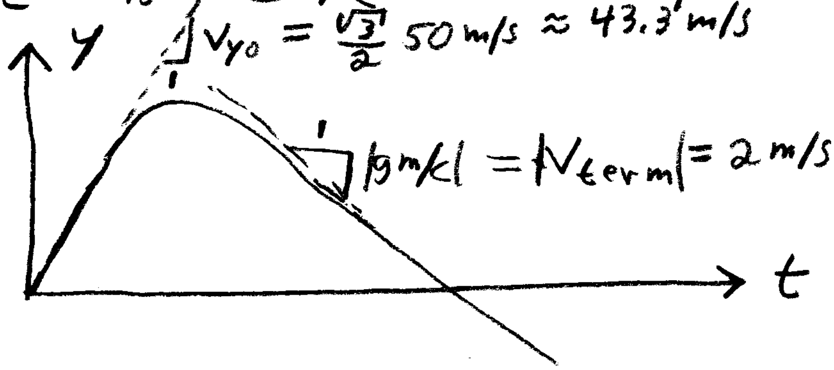
(4) $\Rightarrow \quad y = y_0 + \int_0^t v_y(t')\,dt'$

$$= 0 + \int_0^t \left[\left(V_{y0} + \frac{gm}{c}\right) e^{-(c/m)\,t'} - gm/c\right] dt'$$

(prob (3) cont'd)

$$y = \left[-\frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right) e^{-(c/m)t'} - \frac{gm}{c} t' \right]_0^{t'}$$

$$= \frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c} t$$



$$\underline{r}(t) = x\,\underline{i} + y\,\underline{j}$$

$$= \frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)\underline{i}$$

$$+ \left[\left(\frac{mV_{y0}}{c} + \frac{gm^2}{c^2}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c} t\right]\underline{j}$$

$c/m = (.1/.02)\, s^{-1} = 5/s$, $\quad \frac{gm^2}{c^2} = 10 \cdot \frac{(.02)^2}{(.1)^2}\, m = .4\, m$

$\frac{mV_{x0}}{c} = 5\, m$ (see 7)

$\frac{mV_{y0}}{c} = \frac{\sqrt{3}}{2} \cdot 5\, m \approx 8.66\, m$

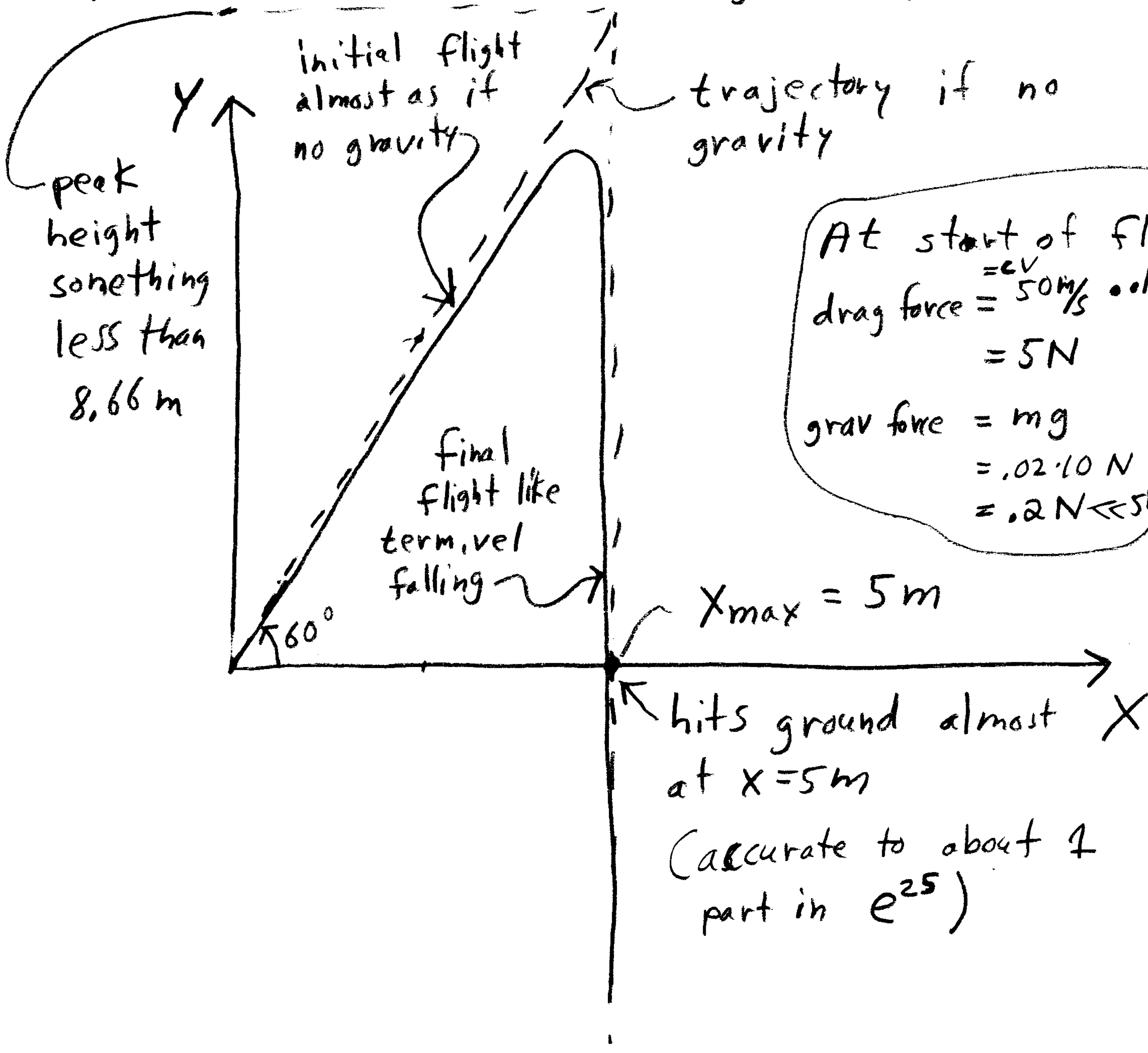
$\frac{gm}{c} = 2\, m/s$

(b)

$$\boxed{\underline{r}(t) = 5m\left(1 - e^{-5t/s}\right)\underline{i} + \left[\left(\frac{5\sqrt{3}}{2} - .4\right)\left(1 - e^{-5t/s}\right)m - 2(m/s)t\right]\underline{j}}$$

(prob. 3, cont'd)

c) Plot. All we need from anal. soln. is $\underbrace{\text{eqn. 7}}^{\text{i.e. eqn for } x(t)}$.
(No need for soln. of inhomog. soln.)



At start of flight

drag force $= \overset{=cv}{50\,\text{m/s} \cdot .1\,\text{N/(m/s)}}$

$= 5\,\text{N}$

grav force $= mg$

$= .02 \cdot 10\,\text{N}$

$= .2\,\text{N} \ll 5\,\text{N}$

Some comments on ODE solns.

Solve $\dot{V}_x = -(c/m)V_x$

method 1: guess $V_x = e^{rt}$

plug in $r\cancel{e^{rt}} = -(c/m)\cancel{e^{rt}}$

$r = -c/m$

$\Rightarrow \quad V_x = c_1 e^{-(c/m)t}$

c_1: arb. const., pick to sat. I.C.

method 2:
(Edwards & Penny 1.4)

$$\frac{dV_x}{dt} = -(c/m)V_x$$

$$\Rightarrow \frac{dV_x}{V_x} = -(c/m)\,dx$$

$$\Rightarrow \int \frac{dV_x}{V_x} = -\int c/m\,dt$$

$$\Rightarrow \ln V_x = -\frac{c}{m}t + c_1'$$

$$\Rightarrow \quad V_1 = e^{-(c/m)t + c_1'}$$

$$V_x = c_1 e^{-(c/m)t} \qquad (c_1 = e^{c_1'})$$

(again)

method 3: Integrating factor (Edwards & Penny 1.5)

$$\frac{dV_x}{dt} + \frac{c}{m}V_x = 0$$

$e^{c/m\, t}$ is the integrating factor.

$$\Rightarrow \quad e^{\frac{c}{m}t}\frac{dV_x}{dt} + e^{(c/m)t}\frac{c}{m}V_x = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{(c/m)t}V_x\right) = 0$$

$$\Rightarrow \quad e^{(c/m)t}\, V_x = C_1$$

$$\Rightarrow \quad V_x = C_1 e^{-(c/m)t}$$

(again)

Solve $\ddot{x} + c/m\,\dot{x} = 0$

method 1: guess $x = e^{rt}$

$$\Rightarrow \quad r^2 \cancel{e^{rt}} + r\frac{c}{m}\cancel{e^{rt}} = 0$$

$$\Rightarrow \quad r\left(r + \frac{c}{m}\right) = 0 \quad \Rightarrow \quad r = 0, -c/m$$

$$\Rightarrow \quad x(t) = c_1 e^{-\frac{c}{m}t} + c_2 e^{-0t}$$

$$= C_1 e^{-(c/m)t} + C_2$$

find using ICs.

This is $x(t)$ as found.

<u>Solve</u> $\dot{V}_y + (c/m)\, V_y = -g$

<u>method 1</u>: a) find homog. soln.

$$\dot{V}_y + (c/m) V_y = 0.$$

This is identical to prev. problem which we solved to get

$$\underbrace{V_{y_h}}_{\text{homog. soln.}} = c_1 e^{-(c/m)t}$$

b) Find <u>any</u> "particular" soln. of

$$\dot{V}_y + (c/m) Y = -g.$$

As for spring-mass problem. Easiest guess is a constant. In this case you can get this physically by thinking of falling at terminal velocity.

guess $V_{yp} = c_2$

$$\cancelto{0}{\dot{c}_2} + (c/m)\, c_2 = g$$

$$c_2 = gm/c$$

Solution is

$$V_y = V_{yh} + V_{yp}$$

$$= c_1 e^{-(c/m)t} + gm/c$$

c_1 ← pick to match I.C.

ODEs (cont'd)

method 2; (E 8P 1.4) (seperable eqn.)

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

$$\Rightarrow \frac{dV_y}{dt} = -\frac{c}{m} V_y - g$$

$$\Rightarrow \frac{dV_y}{V_y + \frac{gm}{c}} = -\frac{c}{m}\, dt$$

$$\Rightarrow \int \frac{dV_y}{V_y - \frac{gm}{c}} = -\int \frac{c}{m}\, dt$$

$$\Rightarrow \ln\left(V_y + \frac{gm}{c}\right) = -\frac{c}{m} t + c_1''$$

$$\Rightarrow V_y + \frac{gm}{c} = e^{-\frac{c}{m}t + c_1'}$$

$$\Rightarrow V_y = c_1 e^{-(c/m)t} - gm/c$$

$c_1 = e^{c_1'}$, pick to sat. I.C.

(again)

method 3; $\boxed{\text{(E 8P 1.5) linear ODE using integrating factor}}$

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

define $g(t) = e^{(c/m)t}$

integrating factor

mult. through by g

$$\frac{dV_y}{dt} e^{(c/m)t} + \frac{c}{m} V_y e^{(c/m)t} = -g e^{(c/m)t}$$

(ODEs cont'd)

$$\frac{d}{dt}\left(v_y e^{(c/m)t}\right) = -g e^{(c/m)t}$$

$$\Rightarrow \quad d\left[v_y \, e^{(c/m)t}\right] = -g e^{(c/m)t} \, dt$$

$$\Rightarrow \quad \int d\left(v_y e^{(c/m)t}\right) = -\int g e^{c/m\,t} \, dt$$

$$\Rightarrow \quad v_y \, e^{(c/m)t} = -\frac{gm}{c} e^{(c/m)t} + C_1$$

$$\Rightarrow \quad v_y \underset{\text{(again)}}{=} \frac{-gm}{c} + \underset{\text{pick to sat. I.C.}}{c_1} e^{-(c/m)t}$$

SOLUTIONS

Your Name: ____________________

TA name and section time: ____________

T&AM 203 Final Exam

Friday May 19, 2006, 2-4:30 PM

Draft May 15, 2006

5 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A} + \underline{\mathbf{v}}_{rel}$$

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}) + \dot{\underline{\boldsymbol{\omega}}} \times \underline{\mathbf{r}}_{B/A} + 2\underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}_{rel} + \underline{\mathbf{a}}_{rel}$$

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}$$

Problem	0:	/-125
Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	4:	/25
Problem	5:	/25
TOTAL:		

0) **-125 pt** In order to *not* get -125 points you need to sign your name below. If you do not sign your name you get *negative* 125 points. Whether or not you sign, any violations of the pledge below will be fully prosecuted under the Cornell policies concerning academic integrity. I (Andy) have prosecuted many such cases and no student I have accused has ever been found innocent or had a decision reversed on appeal.

Pledge: I realize that the regularly scheduled final might be identical to this test. No student taking the late final should have any more foreknowledge of the test than have students taking this early final now. Between now and 3 PM Friday May 19 I promise not to discuss any aspect of this test with anyone, or within earshot of anyone, with the exception of TAM 203 staff and other TAM 203 students who also took this early test (assuming I know and recognize them and saw them taking this test). That is, there should be *no possible means* by which any student in TAM 203 who is not taking the test with me now could learn by any direct or indirect way from me (for example though a third person overhearing me or reading my email or through my parents talking to their friends etc) anything about this test. For example, and these are only examples, no-one will get in any direct or indirect way from me the answers to any of these questions:

- Did I think the test was easy or hard, fair or unfair?
- Was there a Matlab question on the test?
- Did the test have a statics problem, a problem from the lab, a problem involving pulleys, etc?
- How many questions were on the test?
- Were any formulas given on the test?
- Did the test include material from the final homework?
- How well did I think I did on the test?

If anyone asks me any such questions or tries to get such information from me I will say that I am not allowed to even hint at the answers. If pressed further I will tell the person asking that such pressure is a violation of the rules of academic integrity. If pressed further I will tell 203 staff who was asking. If I know of any violations of this pledge I will promptly inform TAM 203 staff. By signing below I indicate that I understand and agree to the text above on this page.

Signed ______________________________
(sign clearly and legibly)

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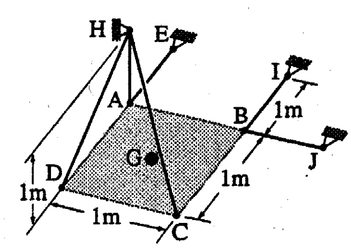
1) **(25 pt)** A uniform square horizontal rigid plate ABCD has weight mg and is held in place by 6 negligible-mass rods. You need not write long vector formulas if you can confidently justify your answers without them. Find the tension in bar HD.



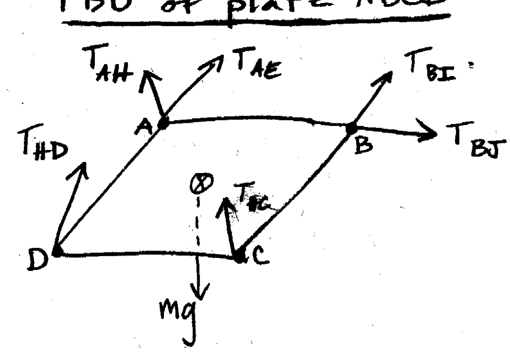
Taking moments about AC, the only force that doesn't pass through it is the force due to tension in bar HD.

Since for static equilibrium

$$\Sigma \underline{M}_{AC} = 0$$

$\Rightarrow$ The tension HD must be 0.

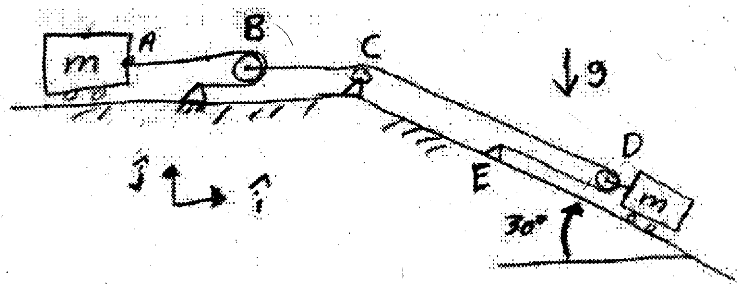
FBD of plate ABCD



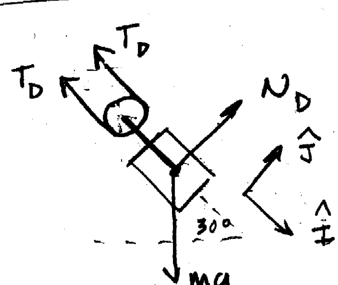
2) **(25 pt)** Make the usual assumptions about pulleys and the like.

a) (20 pt) In terms of some or all of m and g find $\mathbf{a}_D \cdot \hat{\mathbf{j}}$. That is, find the y component of the acceleration of point D.

b) (5 pt) Roughly speaking can you explain the answer to part (a). Hint: the answer to part (a) is a number multiplied by a symbol or symbols. That number is close to $2^{\pm n}$ where n is an integer. For example, if the answer to part (a) was $9m/g$ (it isn't) then we could say that answer was close to $2^3 m/g$ and we would have $n = 3$. Use words and/or diagrams to rationalize the appropriate value of n from part (a). That is, somehow the mechanics has in it, approximately, n factors of two. Can you identify each one of these factors. [A very good answer to this part can make up for lost points in part (a)].

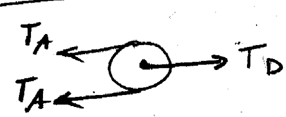


FBD for mass D



$(\Sigma F)\cdot\hat{I} = \boxed{mg\cos 60^\circ - 2T_D = m\ddot{X}_D}$

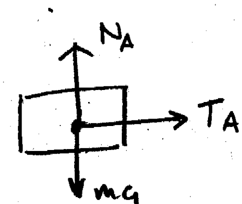
FBD for "massless" drum B



$\Sigma F = -2T_A + T_D = 0$

$\Rightarrow \boxed{T_D = 2T_A}$

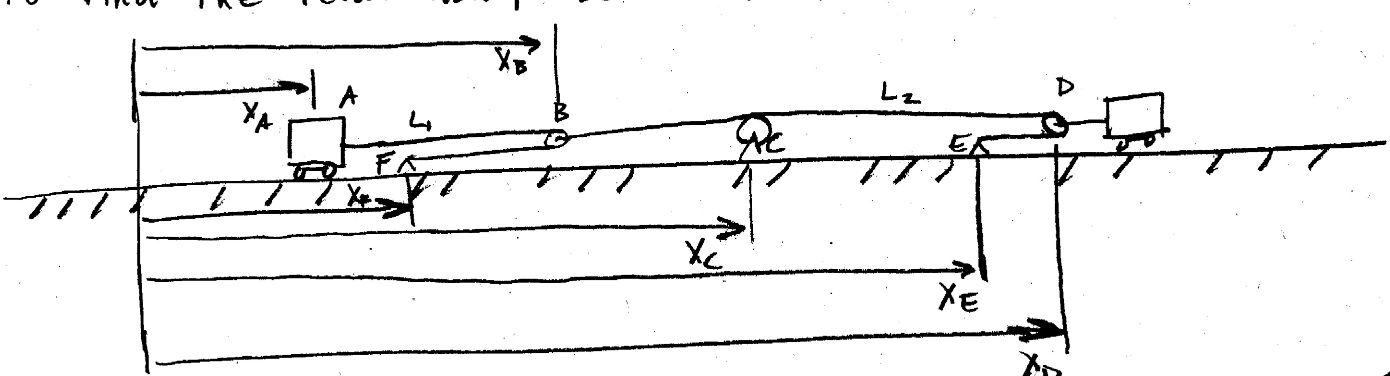
FBD for mass A



$(\Sigma F)\cdot\hat{i} = \boxed{T_A = m\ddot{X}_A}$

① $\boxed{mg\cos 60^\circ - 4m\ddot{X}_A = m\ddot{X}_D}$

To find the relationship between $\ddot{X}_A$ & $\ddot{X}_D$



$L_2 = X_D - X_E + X_D - X_C + X_C - X_B \Rightarrow \boxed{2\ddot{X}_D = \ddot{X}_B}$ ②

$L_1 = X_B - X_F + X_B - X_A \Rightarrow \boxed{2\ddot{X}_B = \ddot{X}_A} \longrightarrow \boxed{\ddot{X}_D = \frac{1}{4}\ddot{X}_A}$

Combining ① & ②

$$mg\cos 60^\circ - 4m(4\ddot{x}_D) = m\ddot{x}_D \implies \boxed{\ddot{x}_D = \frac{1}{17} g \cos 60^\circ}$$

The $\hat{j}$-component of $\underline{a_D}$ is then

$$(\underline{a_D})\cdot\hat{j} = -\frac{1}{17} g \cos 60^\circ \sin 30^\circ = -\frac{g}{17}\sin^2 30^\circ$$

$$= -\frac{g}{17}\left(\frac{1}{2}\right)^2 = \boxed{-\frac{1}{68} g \approx -2^{-6} g}$$

Thus we find that $\boxed{n=-6}$. This number can be explained as follows:

From the kinematics we found that mass A has 4 times the acceleration of mass D $\implies$ mass A has 16 times the energy & thus dominates the inertia of the 1-d.o.f. system. So basically we have the weight at D (w/ negligible mass) pulling on mass A.

1. Only half the weight at D is carried by the rope because of the slope, $\sin 30^\circ = 1/2$.
2. Only half of this weight is carried by rope BD because of the pulley at D.
3. Only half the tension of rope BD is carried by rope AB because of pulley B.

So far we have $\ddot{x}_A = T_A/m = \left(\frac{mg}{8}\right)/m = \frac{g}{8}$ from the force analysis. From kinematics we find:

4. Point B has half the acceleration of mass A because of pulley B.

5. Point D has only half the acceleration of B because of the pulley at D.

6. The $\hat{j}$-component of the acceleration of mass D is half the along-slope acceleration because of the slope, $\sin 30° = 1/2$.

Thus the acceleration of mass D is $\approx 2^6$ times smaller than it would be in free flight. It's actually smaller than that since we neglected the inertia of mass D completely, and that would slow the system down more.

3) (25 pt) A person with mass m stands still at the back of a stationary boat with mass M. Then at $t = 0$ she walks the length L of the boat over time T according to the equation

$$x_{p/b} = \frac{L(1 - \cos(\pi t/T))}{2}$$

where $x_{p/b}$ is how far she has moved relative to the boat. Then for $t > T$ she stands still in the front of the boat.

a) (5 pt) Make a plot of $x_{p/b}$ vs t (put t on the "x" axis). Label key points on the "x" and "y" axes in terms of m, M, T and L.

b) (10 pt) Make a plot of x_b vs t, labeling key points on the axis as for part (a). x_b is the absolute position of the boat relative to a fixed reference frame. Assume the boat moves frictionlessly on the water.

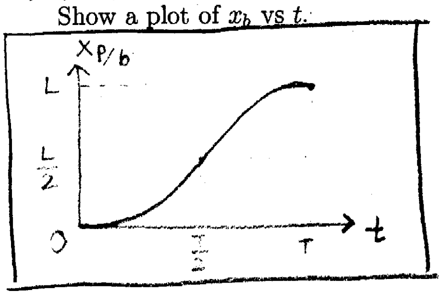
c) (5 pt) For parts (c & d) assume that the boat *has* friction with the water. The drag force is proportional to the boat speed:

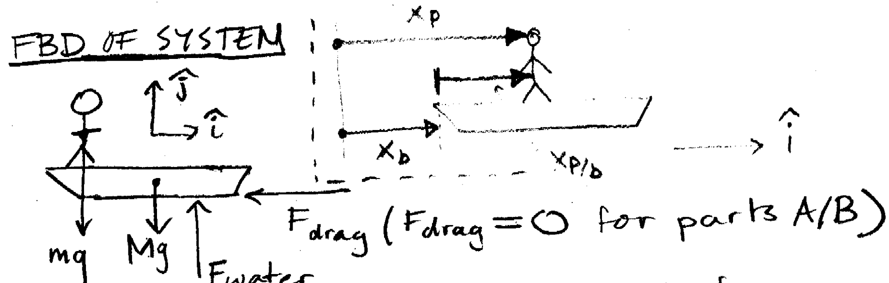
$$F_{drag} = cv_b.$$

Eventually, as $t \to \infty$, the boat speed tends to zero and the system comes to rest. What is the net impulse of the force of the water on the boat? That is, evaluate $\int_0^\infty F_{drag}\, dt$ (using basic mechanics principles this is a short calculation).

d) (5 pt) What is $x_b(\infty)$? That is, after all has come to rest how far will the boat have moved? Show a plot of x_b vs t.

a)





b) Taking man & boat as system, there is no external force on the system ⇒ Linear momentum of system is constant

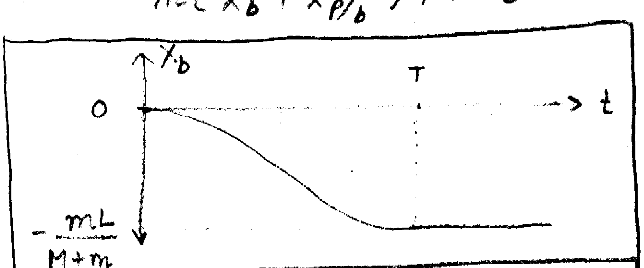
$\Rightarrow \quad \vec{P}(t<0) = \vec{P}(0<t<T) = \vec{P}(t>T)$

$\Rightarrow \quad 0 = m\vec{v}_p + M\vec{v}_b = 0$ note $\vec{v}_p = \vec{v}_b + \vec{v}_{p/b}$

$\Rightarrow \vec{x}_p = \vec{x}_b + \vec{x}_{p/b}$

integrating $\quad m\vec{x}_p + M\vec{x}_b = 0$

$m(\vec{x}_b + \vec{x}_{p/b}) + M\vec{x}_b = 0 \Rightarrow \boxed{x_b = -\frac{m}{M+m} x_{p/b}} \quad t \le T$



for $t > T$ the system comes to rest.

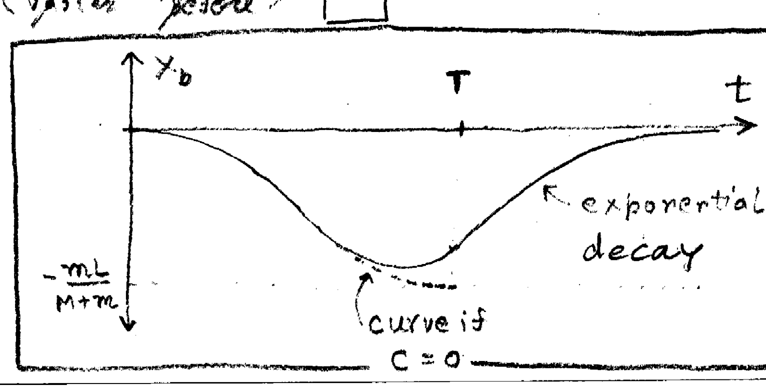
c) $\int_0^\infty F_{drag}\, dt = \Delta \vec{P}_{system} = (m+M)(V_{after}^{\,0} - V_{before}^{\,0}) = \boxed{0}$

d) $X_b(\infty) = X_b(\infty) - X_b(0)$

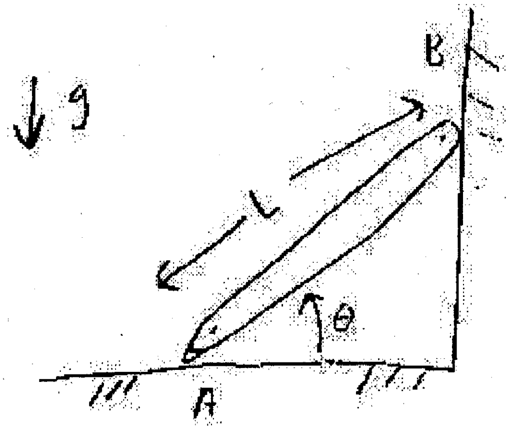
$= \int_0^\infty V_b\, dt$

$= \frac{1}{c}\int_0^\infty c V_b\, dt$ See part (c) above

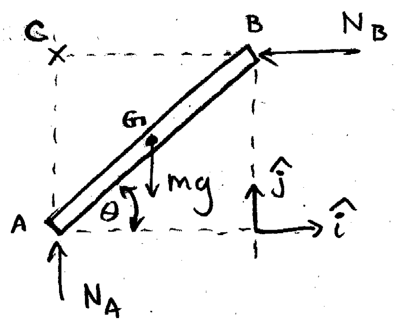
$= \frac{1}{c}\int_0^\infty F_{drag}\, dt = \boxed{0}$



4) **(25 pt)** A uniform ladder with mass m and length L slides on a slippery floor and against a slippery wall. It is released from rest at angle θ. Immediately after release find the angular acceleration of the rod. Answer in terms of some or all of θ, g, L, m, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. If you think you need I_G, I_A or I_B you can recall them or derive them or, for less credit, leave them in your final answer.



FBD of ladder AB



LMB

$$\Sigma \underline{F} = \boxed{-N_B\,\hat{\imath} + (N_A - mg)\,\hat{\jmath} = m\underline{a}_G}$$

AMB about point C

$$\Sigma \underline{M} = \boxed{\underline{r}_{G/C} \times -mg\,\hat{\jmath}} \;\boxed{= I_G\,\ddot{\theta}\,\hat{k} + \underline{r}_{G/C} \times m\underline{a}_G}$$

To find $\underline{a}_G$ we use the rigidity of AB to write

$$\boxed{\underline{a}_G = \underline{a}_A + \underline{a}_{G/A} = \underline{a}_A + \underline{\alpha} \times \underline{r}_{G/A} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{G/A}}$$

To find $\underline{a}_A$ we use rigidity & the fact that ends A & B are constrained to move along their respective walls.

$$\underline{a}_B = a_B\,\hat{\jmath} = \underline{a}_A + \underline{a}_{B/A} = a_A\,\hat{\imath} + \underline{\alpha} \times \underline{r}_{B/A} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{B/A}$$

$$\underline{\alpha} \times \underline{r}_{B/A} = \ddot{\theta}\,\hat{k} \times (L\cos\theta\,\hat{\imath} + L\sin\theta\,\hat{\jmath}) = -L\ddot{\theta}\sin\theta\,\hat{\imath} + L\ddot{\theta}\cos\theta\,\hat{\jmath}$$

$$\underline{\omega} \times \underline{\omega} \times \underline{r}_{B/A} = \dot{\theta}\hat{k} \times \dot{\theta}\hat{k} \times (L\cos\theta\,\hat{\imath} + L\sin\theta\,\hat{\jmath})$$

$$= -L\dot{\theta}^2\cos\theta\,\hat{\imath} - L\dot{\theta}^2\sin\theta\,\hat{\jmath}$$

$$\{\ \}\cdot\hat{\imath} \Rightarrow 0 = a_A - L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta$$

$$\Rightarrow \boxed{a_A = L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta}$$

Thus we have

$$\boxed{\begin{aligned}\underline{a}_G &= (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)\hat{\imath} + \left(-\frac{L\ddot{\theta}}{2}\sin\theta\,\hat{\imath} + \frac{L\ddot{\theta}}{2}\cos\theta\,\hat{\jmath}\right) \\ &\quad + \left(-\frac{L\dot{\theta}^2}{2}\cos\theta\,\hat{\imath} - \frac{L\dot{\theta}^2}{2}\sin\theta\,\hat{\jmath}\right) \\ &= \left(\frac{L\ddot{\theta}}{2}\sin\theta + \frac{L\dot{\theta}^2}{2}\cos\theta\right)\hat{\imath} + \left(\frac{L\ddot{\theta}}{2}\cos\theta - \frac{L\dot{\theta}^2}{2}\sin\theta\right)\hat{\jmath}\end{aligned}}$$

Plugging this back into AMB we get

$$-\frac{mgL}{2}\cos\theta\,\hat{k} = \frac{1}{12}mL^2\hat{k} + \left(\frac{L}{2}\cos\theta\,\hat{\imath} - \frac{L}{2}\sin\theta\,\hat{\jmath}\right) \times m\underline{a}_G$$

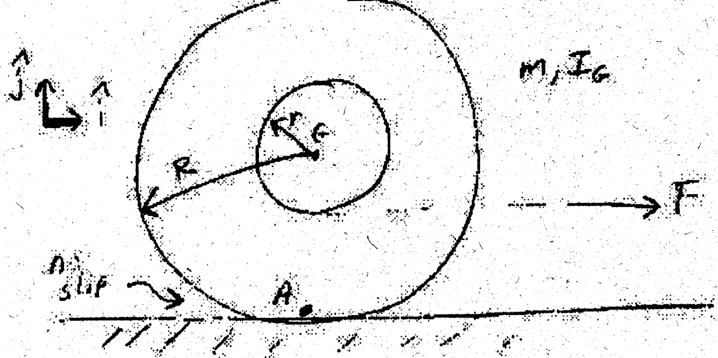
$$\{\ \}\cdot\hat{k} \Rightarrow -\frac{mgL}{2}\cos\theta = \underbrace{\frac{1}{12}mL^2\ddot{\theta} + \frac{1}{4}mL^2\ddot{\theta}}_{\frac{1}{3}mL^2\ddot{\theta}}$$

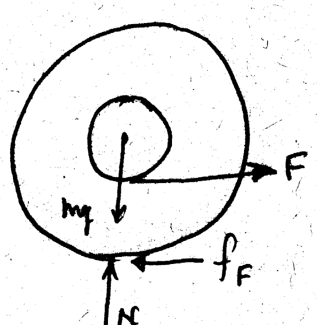
NOTE: The terms with $\dot{\theta}^2$ cancel out after the cross-product.

$$\Rightarrow \ddot{\theta} = -\frac{3g}{2L}\cos\theta \quad \Rightarrow \quad \boxed{\underline{\alpha} = -\frac{3g}{2L}\cos\theta\,\hat{k}}$$

<u>NOTE #1</u> - An alternative method would be to use conservation of energy. Set $T + V = E$ (constant) and differentiate w.r.t. time.

5) (25 pt) A spool (like the movie *Heat Treatment of Aluminum* shown in lecture), with outer radius R rolls without slip on a flat horizontal surface. The film is at a radius r and is being pulled with a horizontal force F. At the moment in question the velocity of the middle of the spool is $v\hat{\mathbf{i}}$. The mass of the spool is m and its moment of inertia about its center of mass is I_G. What is the acceleration of point A on the spool which is, at the instant in question, touching the ground. Answer in terms of some or all of m, I_G, r, R, g, v and F.





$$\underline{V}_A = \underline{V}_A \quad \text{no slip}$$

$$\Rightarrow \cancel{\underline{V}_{Ground}}^{\,0} + \cancel{\underline{V}_{A/Ground}}^{\,0} = \underline{V}_G + \underline{V}_{A/G}$$

$$\Rightarrow \underline{0} = \underline{V}_G + \underline{\omega} \times \underline{r}_{A/G}$$

$$\Rightarrow \underline{0} = v\hat{i} + \omega\hat{k} \times (-R\hat{j})$$

$$\Rightarrow \underline{0} = v\hat{i} + \omega R\hat{i}$$

$$\{\ \}\cdot\hat{i} \Rightarrow \omega = -\frac{v}{R} = \dot{\theta}$$

$$\underline{a}_G = -R\ddot{\theta}\,\hat{i}$$

$$\underline{a}_A = \underline{a}_G + (\overset{\alpha}{\ddot{\theta}}\hat{k}) \times \underset{-R\hat{j}}{\underline{r}_{A/G}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{r}_{A/G})}_{-\omega^2 \underline{r}_{A/G}}$$

$$\Rightarrow \underline{a}_A = \cancel{-R\ddot{\theta}\hat{i}} + \cancel{R\ddot{\theta}\hat{i}} + \omega^2 R\hat{j}$$

$$\Rightarrow \boxed{\underline{a}_A = \frac{v^2}{R}\hat{j}}$$

Hence the answer doesn't depend on using momentum balance.

"Solutions"

Your Name: ANDY RUINA

TA name and section time: —

T&AM 203 Prelim 1

Tuesday February 28, 2006

Draft February 26, 2006

3 problems, 25 points each, and 90^{+} minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/25

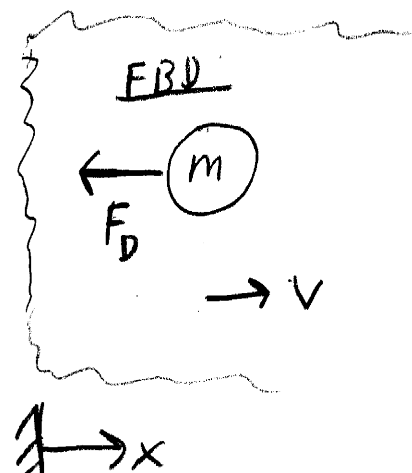
1) **(25 pt)** A shot pellet mass m hits a bird or person's skin at speed v_0. Neglect gravity. Assume m, v_0 and c (below) are given. Assume one dimensional motion in, say, the x direction.

a) (15 points) Assume that the force of the flesh on the pellet is $-cv$, that is the drag force resists motion and is proportional to the speed. How far does the pellet go before it comes to rest? (Please re-read the rules at the front of the exam.) $\Rightarrow F_D = cV$

b) (7 points) Assume that the force of the flesh on the pellet is $-c|\mathbf{v}|\mathbf{v}$, that is the drag force resists motion and is proportional to the speed squared. How far does the pellet go before it comes to rest (the answer is perhaps surprising). $\Rightarrow F_D = cV^2$

c) (3 points) Given that quadratic drag (b above) is a much more accurate model than linear drag (a above) for fast moving things in air, water and flesh, why does the calculation in b give a patently ridiculous answer? How could you change the calculation to make it more accurate? (It might be possible to get this problem right without getting b right.)

(assume $V \geq 0$)



$V = \dot{X}$

$\sum F = m a \Rightarrow \boxed{m\ddot{X} = -F_D}$ ①

a) $m\ddot{x} = -cV$

$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} = 0$

$\Rightarrow \dot{V} + \frac{c}{m}V = 0$

$\Rightarrow V = V_0 e^{-(c/m)t}$

$\Rightarrow x = -V_0 \frac{m}{c} e^{-(c/m)t} + C_1$

$x(0) = 0 \Rightarrow \quad C_1 = \frac{mV_0}{c}$

$\Rightarrow x = \frac{mV_0}{c}\left(1 - e^{-(c/m)t}\right)$

$\boxed{\underset{t\to\infty}{X =} \quad mV_0/c}$ (a)

b) ① $\Rightarrow m\dot{V} = -cV^2$

$\Rightarrow \frac{dV}{V^2} = \frac{-c}{m}dt$

$-1/V = -\frac{c}{m}t + C_1$

$V(0) = V_0 \Rightarrow \quad C_1 = -1/V_0$

$\Rightarrow V = \dfrac{1}{\frac{1}{V_0} + \frac{c}{m}t}$

$\Rightarrow dx = \dfrac{V_0\,dt}{1 + cV_0 t/m}$

$X = \frac{m}{c}\ln(1 + cV_0 t/m) + C_1$

$X(0) = 0 \Rightarrow \quad C_1 = 0$

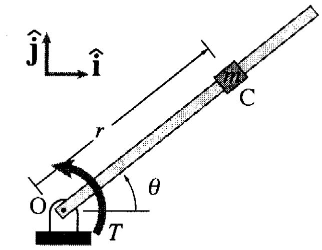
$\Rightarrow x = m/c \, \ln(1 + cV_0 t/m)$

$\boxed{\underset{t\to\infty}{X =} \quad \infty}$ (b)!

c) because eventually the pellet goes slowly. other neglected forces then dominate the quadratic drag.
Some fixes
i) $F_D = C_1V^2 + C_2V$
or ii) $F_D = C_1V^2 + C_2$

Both of these slow the pellet down enough at low speeds to predict finite (and more accurate) penetration

2) (25 pt) A torque T varies in time as it must in order to rotate a rigid rod at constant rate of $\dot{\theta} = 2$ rad/s. A bead slides on the rod. At the start $t = 0$, $\theta = 0$, $r = 1$m and $\dot{r} = 0$. Neglect gravity and friction.

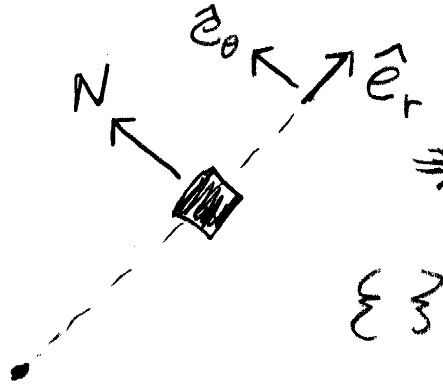


a) (15 points) What is the radius when $\theta = 2\pi$?

b) (5 points) What is the speed $|\underline{\mathbf{v}}|$ when $\theta = 2\pi$?

c) (5 points) When $\theta = 9\pi/4$ what is the direction of $\underline{\mathbf{v}}$. A *very* simple answer is desired which is not exact, but is accurate to within a degree or less.

FBD



$\underline{F} = m\underline{a}$

$$\Rightarrow \left\{ N\hat{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\cancel{\ddot{\theta}}^{\,0} + 2\dot{r}\dot{\theta})\hat{e}_\theta \right\}$$

$$\{\ \}\cdot\hat{e}_r \Rightarrow \ddot{r} - r\dot{\theta}^2 = 0$$

↑ a constant,

$$\Rightarrow r = c_1 e^{\dot{\theta}t} + c_2 e^{-\dot{\theta}t}$$

$$\dot{r} = c_1\dot{\theta}e^{\dot{\theta}t} - c_2\dot{\theta}e^{-\dot{\theta}t}$$

note: $\dot{\theta}t = \theta$ ↑ const.

$$\dot{r}(0) = 0 \Rightarrow c_1 = c_2 \Rightarrow r = r_0(e^{\theta} + e^{-\theta})/2$$

$$r(0) = r_0 \Rightarrow c_1 + c_2 = r_0$$

$$\dot{r} = r_0\dot{\theta}(e^{\theta} - e^{-\theta})/2$$

a) $$r(2\pi) = \underbrace{r_0}_{1m}(e^{2\pi} + e^{-2\pi})/2 = \boxed{[(e^{2\pi} + e^{-2\pi})/2]\ m \approx \frac{e^{2\pi}}{2}\ m}\ (a)$$

b) $$\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \overbrace{r_0\dot{\theta}\frac{(e^{\theta} - e^{-\theta})}{2}}^{v_r}\hat{e}_r + \overbrace{r_0\dot{\theta}\frac{(e^{\theta} + e^{-\theta})}{2}}^{v_\theta}\hat{e}_\theta$$

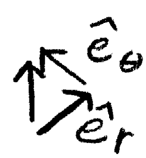
$$|\underline{V}| = \sqrt{V_r^2 + V_\theta^2} = \frac{r_0\dot{\theta}}{2}\sqrt{(e^{2\theta} - 2 + e^{-2\theta}) + (e^{2\theta} + 2 + e^{-2\theta})}$$

$$= \frac{r_0\dot{\theta}}{2}\sqrt{2(e^{2\theta} + e^{-2\theta})}$$

$$= \frac{2}{2}\sqrt{2e^{4\pi} + 2e^{-4\pi}}\ m/s = \boxed{2\sqrt{\cosh(4\pi)}\ m/s}\ (b)$$

c) for large θ, $e^{-\theta}/e^{\theta} \Rightarrow \underline{V} \approx \frac{r_0\dot{\theta}}{2}e^{\theta}(\hat{e}_r + \hat{e}_\theta)$ (c)

at $\theta = 9\pi/4$ "=" $\pi/4$ $\quad \hat{e}_r + \hat{e}_\theta =$



$= \boxed{\underline{V} \text{ in } \hat{j} \text{ direction}}$

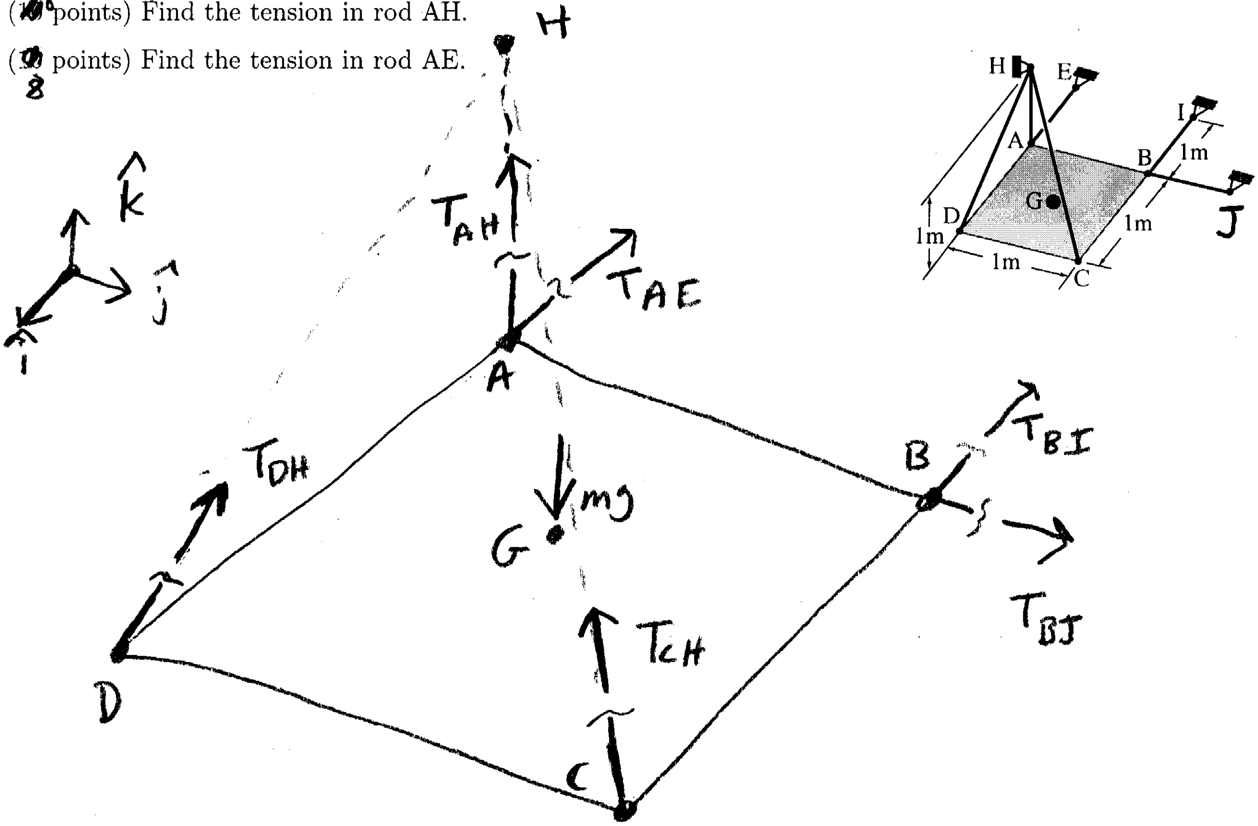
(approx)

3) (25 pt) A uniform square horizontal rigid plate ABCD has weight mg and is held in place by 6 negligible-mass rods. You need not write long vector formulas if you can confidently justify your answers without them.

a) (~~10~~ 9 points) Use moment balance about axis AH to find the tension in rod BI.

b) (~~10~~ 8 points) Find the tension in rod AH.

c) (~~5~~ 8 points) Find the tension in rod AE.



a) All forces have lines of action $\parallel$ to or intersecting AH except T_{BI} $\Rightarrow$ $T_{BI}\cdot(1m)=0 \Rightarrow \boxed{T_{BI}=0}$ (a)

b) $\sum M_{CD}=0$: only T_{AH} and mg contribute. T_{AH} has twice the lever arm $\Rightarrow \boxed{T_{AH} = mg/2}$ (b)

c) $\sum M_{BH}=0$: only mg & T_{AE} contribute

$$\left\{\sum \underline{M}_{/B}\right\}\cdot \underline{r}_{BH} = 0 \Rightarrow 0 = \left\{\underbrace{\underline{r}_{BG}}_{\frac{\hat{i}-\hat{j}}{2}m} \times -mg\,\hat{k} + \underbrace{\underline{r}_{BA}}_{-\hat{j}m} \times T_{AE}(-\hat{i})\right\}\cdot(-\hat{j}+\hat{k})$$

$$\Rightarrow 0 = \left[mg\left(\frac{\hat{j}-\hat{i}}{2}\right) + T_{AE}\,\hat{k}\right]\cdot(-\hat{j}+\hat{k}) = \frac{mg}{2} + T_{AE} \Rightarrow \boxed{T_{AE} = \frac{-mg}{2}}$$ (c)

Your Name: STAFF

"SOLUTIONS"

TA name and section time: ____________

T&AM 203 Prelim 2

Tuesday April 18, 2006, 2006

Draft April 18, 2006

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	4:	/25
Problem	5:	/25
Problem	6:	/25

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A} + \underline{\mathbf{v}}_{rel} \qquad \underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}) + \dot{\underline{\boldsymbol{\omega}}} \times \underline{\mathbf{r}}_{B/A} + 2\underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}_{rel} + \underline{\mathbf{a}}_{rel}$$

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}$$

ρ = radius of curvature

1) (29 pt) A car with mass m_1 moving at v_1 crashes into the rear of a stationary car with mass m_2 and sticks to it. The duration of the crash is Δt after which the cars move together. Give all answers in terms of m_1, m_2, v_1 and Δt. [Later work may not be graded if it depends on incorrect earlier work].

a) (15 points) Please re-read the rules at the front of the exam. How fast are the cars moving after the crash?

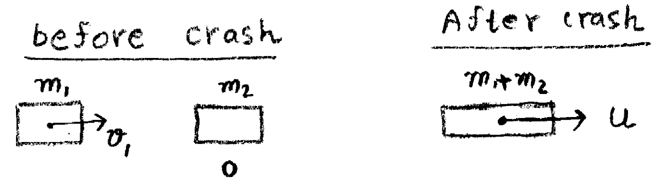
b) (5 points) What is F (the force of car 2 on car 1) during the crash, assuming F is constant in this time interval.

c) (3 points) Given v_1 and m_1 consider a range of cars that might be hit by car 1. For what mass m_2 car is its acceleration during the crash the biggest compared to all other possible cars? (Answers of the form $m_2 \to 0$ or $m_2 \to \infty$ are acceptable.)

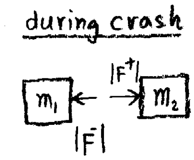
d) (3 points) Like part (c), for what mass m_2 car is its final kinetic energy maximum (that is, more than the kinetic energy of any other car with a different m_2)?

e) (3 points) Like parts (c) and (d) for what mass m_2 car is the total crash energy dissipation maximum (that is, more dissipation than for all other m_2)?

a) Taking m_1, m_2 as system



momentum before crash = momentum after crash



$$\Rightarrow \quad m_1 v_1 + 0 = (m_1 + m_2) u \quad \Rightarrow \quad \boxed{u = \frac{m_1 v_1}{m_1 + m_2}}$$

b) For mass m_1

$$I = F\Delta t = \Delta P = m_1 u - m_1 v_1 = \frac{m_1^2 v_1}{m_1 + m_2} - m_1 v_1 = -\frac{m_1 m_2 V_1}{m_1 + m_2}$$

$$\boxed{F^- = -\frac{m_1 m_2 V_1}{(m_1 + m_2)\Delta t}}$$

$\because |F^-| = |F^+|$

c) acceleration of m_2 during crash $= \dfrac{F^+}{m_2} = \dfrac{m_1 V_1}{(m_1 + m_2)\Delta t}$

this is max when $m_1 + m_2$ is min

$\Rightarrow \boxed{m_2 \to 0}$

d) Final KE of $m_2 = \dfrac{m_2 u^2}{2} = \dfrac{1}{2} m_2 \left(\dfrac{m_1 v_1}{m_1 + m_2}\right)^2 = \dfrac{m_1^2 v_1^2}{2}\left[\dfrac{m_2}{(m_1 + m_2)^2}\right]$

for max KE $\quad \dfrac{d(KE)}{dm_2} = 0 \quad \Rightarrow \quad (m_1 + m_2)^2 - 2(m_1 + m_2) = 0$

$\Rightarrow \quad (m_1 + m_2)(m_2 - m_1) = 0$

$\Rightarrow \quad \boxed{m_2 = m_1}$

e) Total crash energy dissipation $= \text{Initial}_{KE} - \text{Final}_{KE} = \frac{1}{2} m_1 v_1^2 - \frac{1}{2}(m_1 + m_2) u^2$

$$= \frac{1}{2} m_1 v_1^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1 v_1}{m_1 + m_2}\right)^2$$

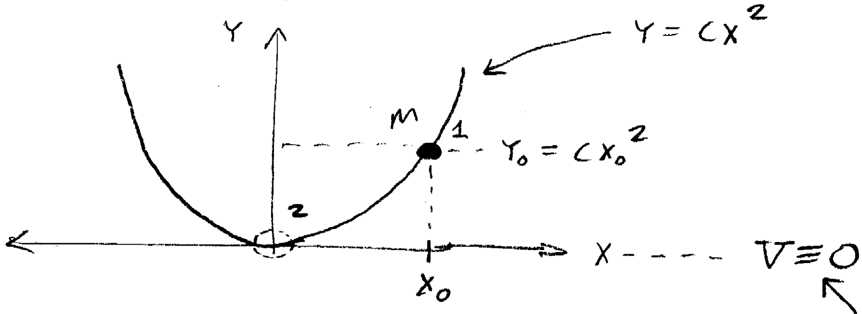
for max dissipation

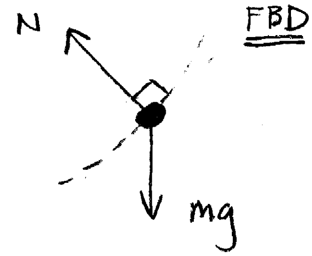
$\dfrac{1}{2}\dfrac{m_1^2 v_1^2}{m_1 + m_2}$ is min $\quad \Rightarrow \quad m_1 + m_2$ is max

$\Rightarrow \quad \boxed{m_2 \to \infty}$

2) (27 pt) A particle with mass m slides with no friction in a parabolic trough that is described with the equation $y = cx^2$. Equivalently you could think of a bead on a wire. Gravity g points in the negative y direction. The bead is released from rest at $x = x_0$. Find the force of the trough/wire on the mass/bead when it reaches $x = y = 0$. Answer in terms of some or all of x_0, c, g, m, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

The problem is set-up as





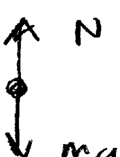
There is NO FRICTION. Use CONSERVATION OF ENERGY.

$$T_1 + V_1 = T_2 + V_2 \Rightarrow 0 + \cancel{m}g(cx_0^2) = \frac{1}{2}\cancel{m}V_2^2$$

$(x,y)=(x_0, cx_0^2)$ $\quad$ $(x,y)=(0,0)$

$$\Rightarrow \boxed{V_2 = x_0\sqrt{2gc}}$$

When $x=0, y=0$ the FBD is



$$\Rightarrow (\Sigma \underline{F}) \cdot \hat{n} = \boxed{N - mg = m\frac{V_2^2}{\rho|_{x=0}}}$$

Solving for N gives

$$N = mg + m\frac{2gcx_0^2}{\rho|_{x=0}} = \boxed{mg\left(1 + \frac{2cx_0^2}{\rho|_{x=0}}\right)}$$

Finally from the equation on the front page of the test

$$\rho|_{x=0} = \left.\frac{(1+y'^2)^{3/2}}{y''}\right|_{x=0} = \frac{(1+0)^{3/2}}{2c} = \boxed{\frac{1}{2c}}$$

Thus

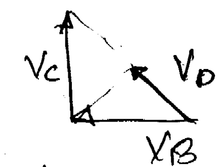
$$\boxed{N = mg\left(1 + 4c^2x_0^2\right)}$$

3) (27 pt) A motor drives link $\mathcal{A}$ at given constant $\omega_{\mathcal{A}}$. All three links are equal length ℓ. All questions concern the velocities and accelerations when the system is passing through the configuration shown.

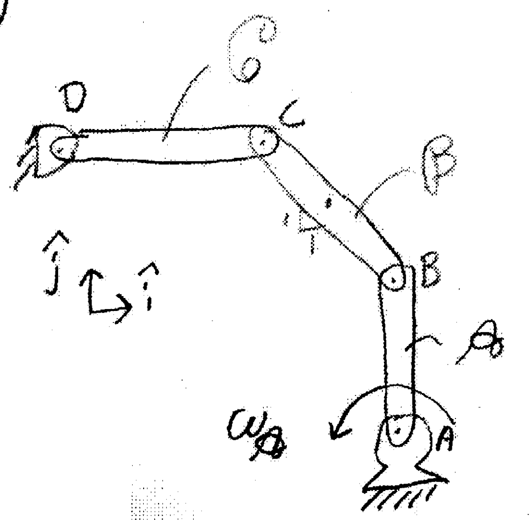
a) (9 points) What is the angular velocity of link $\mathcal{B}$.

b) (8 points) On figure (b) draw in, as accurately as you can, the velocities of points B, C and D. The velocity of point B is drawn for you. This problem will be graded independently of problem (a) and your reasoning can be based on equations or any thing else.

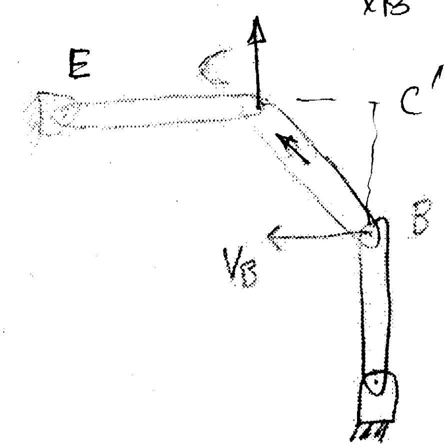
c) (8 points) Write out, but do not solve, one or more vector equations from which you could find the angular acceleration of $\mathcal{B}$. Clearly indicate which terms are known and which unknown in your equation(s) and explain how the number of equations match the number of unknowns. Expressions like $\underline{\mathbf{r}}_{B/A}$ should be evaluated in terms of ℓ, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.



a)



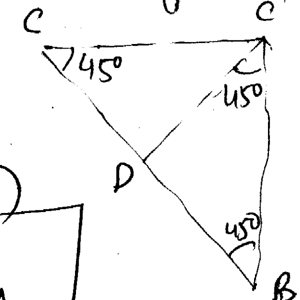
b)



Method 2

V_B is given to be towards left.
C can only move up or down. if it moves down the rod CB needs to compress or buckle which is not possible (rigid body).
Hence V_c is upward.

Drawing perpendiculars to V_c and V_B we get c' to be the instantaneous center of rotation



$r_{cc'} = r_{c'B} = \ell/\sqrt{2}$

$V_c = \omega_B r_{cc'} = \omega_B r_{c'B} = V_B$

Part (a)

$$\Rightarrow \boxed{\omega_B = \frac{V_B}{r_{c'B}} = \frac{\omega_A \ell}{\ell/\sqrt{2}} = \sqrt{2}\,\omega_A}$$

$$\boxed{V_B = \omega_A l}$$ (rotation about point A)

V_D needs to be perpendicular to ~~$r_{C/D}$~~ $\vec{r}_{C/D}$ hence it is along the rod BC.

$$V_D = \omega_B r_{C/D} = \sqrt{2}\,\omega_A \cdot \frac{l}{2}$$

$$\boxed{V_D = \frac{\omega_A l}{\sqrt{2}}}$$

$$\boxed{V_C = V_B = \omega_A l}$$

~~a_E ... $= a_A + a_{C/E} + a_{B/C} + a_{B/A}$~~

$$\underline{a}_E \Rightarrow \underline{0} = \cancel{\underline{a}_A}^{\,0} + \underline{a}_{B/A} + \underline{a}_{C/B} + \underline{a}_{E/C}$$

$\underline{\alpha}_A \times \underline{r}_{B/A} + \underline{\omega}_A \times (\underline{\omega}_A \times \underline{r}_{B/A})$

$\underline{\alpha}_B \times \underline{r}_{B/A} + \underline{\omega}_B \times (\underline{\omega}_B \times \underline{r}_{C/B})$

to be determined

$\underline{\alpha}_C \times \underline{r}_{E/C} + \underline{\omega}_C \times (\underline{\omega}_C \times \underline{r}_{E/C})$

$\underline{r}_{B/A} = l\hat{j}$

$\underline{r}_{C/B} = \frac{l}{\sqrt{2}}(-\hat{i} + \hat{j})$

$\underline{r}_{D/C} = -l\hat{i}$

$\underline{\alpha}_A = 0$

hence we get 2 eqns for two unknowns α_B and α_C