"SOLUTIONS"

[w/ comments on ODE solutions on pages 9-13]

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Section day and time: Wed. 10:10

T&AM 203 Prelim 1

Tuesday Feb 26, 2002

Draft February 26, 2002

3 problems, 100 points, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

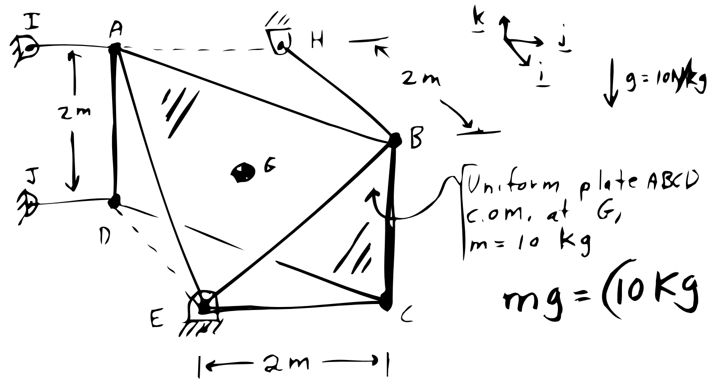
b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/50
TOTAL:	**/100**

1) **(25 pt)** Statics. The sign is held up by 6 bars. Find the tension in bar EB.



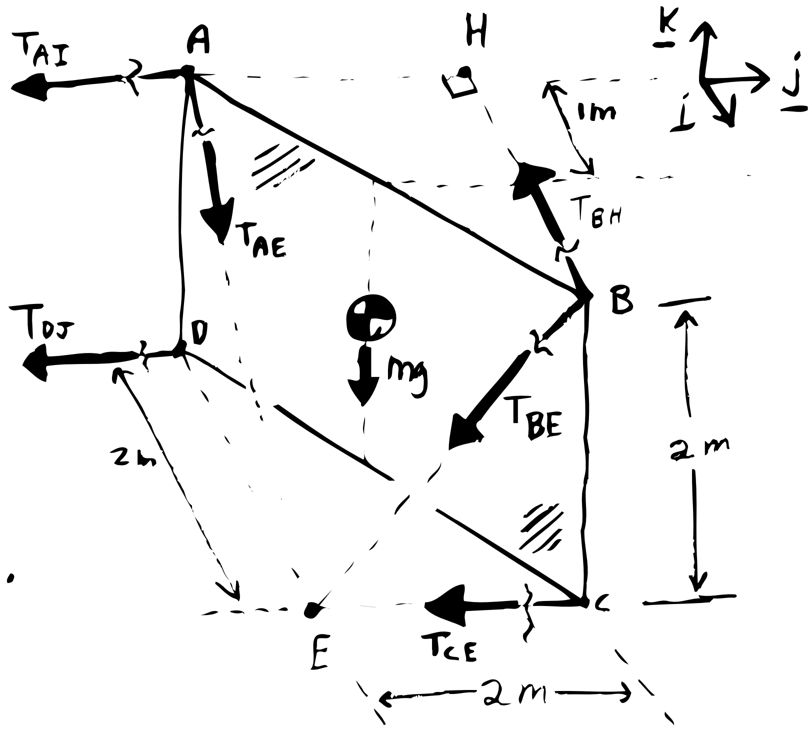
$mg = (10\,\text{kg})(10\,\text{N/kg}) = 100\,\text{N}$

Consider axis AH:

* T_{AI}, T_{DJ}, T_{CE} are $\parallel$ to axis.

* T_{BH} & T_{AE} intersect axis,

$\Rightarrow$ Only T_{BE} and mg contribute to moment about axis AH. But mg is known.



$$\sum M_{\substack{axis\\AH}} = \left(\sum \underline{M}_{/H}\right)\cdot \underline{j} = 0$$

$$\Rightarrow \quad \underbrace{100\,\text{Nm}}_{\text{moment of } mg \text{ about axis AH}} + \Big(\underline{r}_{B/H} \times \overbrace{T_{BE}\left(\frac{-\underline{j}-\underline{k}}{\sqrt{2}}\right)}^{\underline{T}_{BE}}\Big)\cdot \underline{j} = 0$$

$$\underline{r}_{B/H} = 2\,\text{m}\,\underline{i}$$

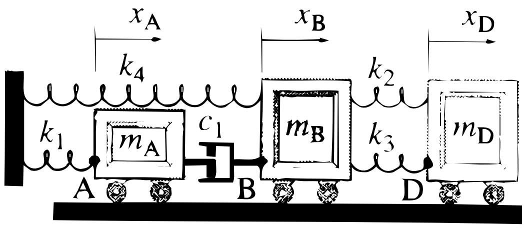
$$\Rightarrow \quad 100\,\text{Nm} + \left[2\,\text{m}\, T_{BE}(-\underline{k} + \underline{j})/\sqrt{2}\,\right]\cdot \underline{j} = 0$$

$$100\,\text{N}\cancel{\text{m}} + \sqrt{2}\,\cancel{\text{m}}\,T_{BE} = 0$$

$$\boxed{T_{BE} = \frac{-100}{\sqrt{2}}\,\text{N} \approx -70.7\,\text{N}}$$

(BE is in compression)

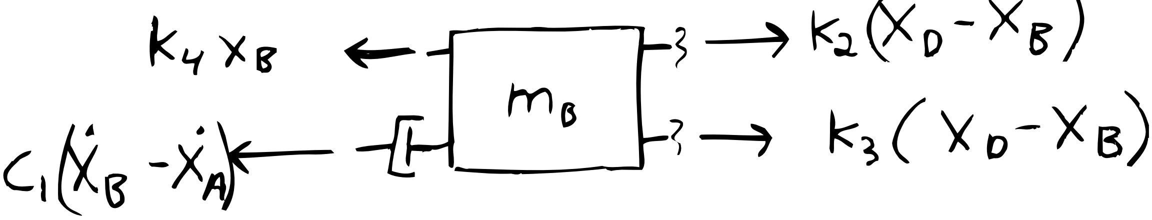
2) (25 pt.) In terms of some or all of $x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_D, k_1, k_2, k_3, k_4, m_A, m_B, m_D$ and c_1 find $\ddot{x}_B$. Assume the springs are relaxed when $x_A = x_B = x_D = 0$



FBD

$\rightarrow \underline{i}$

When m_B is at position x_B



LMB

$$\left\{ \sum \underline{F}_i = m_B \underline{a}_B \right\} \cdot \underline{i}$$

$$\Rightarrow -c_1(\dot{x}_B - \dot{x}_A) - k_4 x_B + k_2(x_D - x_B) + k_3(x_D - x_B) = m_B \ddot{x}_B$$

$$\boxed{\ddot{x}_B = \frac{1}{m_B}\left[-(k_2 + k_3 + k_4)x_B + (k_2 + k_3)x_D + c_1\dot{x}_A - c_1\dot{x}_B\right]}$$

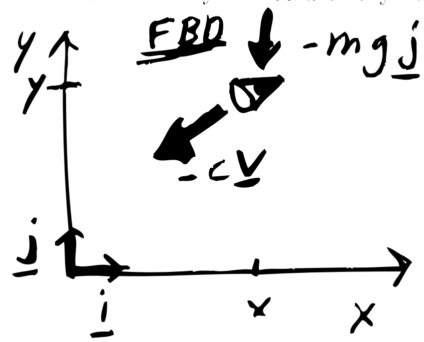
3) **(50 pt)** Trajectory. A 0.02 kg projectile (a badminton birdie, say) is launched from the origin at a 60° upwards angle at a speed of 50 m/s. The projectile stays near the earth so gravity $g = 10\text{m/s}^2$ is well approximated as constant (and all lines towards the center of the earth are effectively parallel). The air drag opposes motion and is proportional to speed with proportionality constant of $c = 0.1$ N/(m/s).

a) (20 pt) Write Matlab code to plot the trajectory, with the same vertical and horizontal scale, for 10 seconds [hints: FBD → LMB → first order ODEs → numerical solution → plotting].

b) (20 pt) Find, analytically, the position $\underline{\mathbf{r}}(t)$. [hints: same as above but use calculus instead of Matlab. The calculation has several steps (4 calculus problems, in one way of counting).]

c) (10 pt) More difficult. As accurately and neatly as you can, plot the trajectory. Label the units on the axis. The plot should go from when the projectile is launched until it hits the ground again. Key quantities to show are the peak height and the distance the projectile goes (which can be calculated very accurately). You can use the solution above or anything else you know or think. This will be graded on its correctness, not its agreement (or not) with the solution (b) above. But you should briefly rationalize your plot.

2D



given:

$$\begin{cases} m = .02\ \text{Kg} \\ c = .1\ \text{N/(m/s)} \\ g = 10\ \text{N/Kg} \\ \underline{v}_0 = V_0 \underline{\lambda}_0 \\ \quad = 50\ \text{m/s}\left(\cos 60^\circ \underline{i} + \sin 60^\circ \underline{j}\right) \end{cases}$$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\left\{ -c\underline{v} - mg\underline{j} = m(\ddot{x}\,\underline{i} + \ddot{y}\,\underline{j}) \right\}$$
$$\llcorner\ \underline{v} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j}$$

$$\{\}\cdot\underline{i} \Rightarrow \quad -c\dot{x} = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{-c}{m}\dot{x} \qquad (1)$$

$$\{\}\cdot\underline{j} \Rightarrow \quad -c\dot{y} - mg = m\ddot{y} \quad \Rightarrow \quad \ddot{y} = \frac{-c}{m}\dot{y} - g \qquad (2)$$

Define $V_x = \dot{x}$, $V_y = \dot{y}$

(1), (2) $\Rightarrow$

$$\boxed{\begin{aligned} \dot{x} &= V_x && (3) \\ \dot{y} &= V_y && (4) \\ \dot{V}_x &= -(c/m)V_x && (5) \\ \dot{V}_y &= -(c/m)V_y - g && (6) \end{aligned}}$$

4 coupled 1st order ODEs

I.C.'s

$$\boxed{\begin{aligned} X_0 &= 0 \\ Y_0 &= 0 \\ V_{x0} &= 50\cos 60^\circ\ \text{m/s} \\ V_{y0} &= 50\sin 60^\circ\ \text{m/s} \end{aligned}}$$

Matlab solution (a)

```
% Ruina trajectory soln., assume consistent units
x0=0; y0=0;
vx0 = 50*cos(60*pi/180); vy0= 50*sin(60*pi/180);
z0 = [x0 y0 vx0 vy0];
tspan = linspace [0, 10, 101];
[t z]= ode23('myrhs', tspan, z0);
x = z(:,1); y= z(:,2);
plot (x,y); xlabel('x'); ylabel('y'); title('trajectory');
axis ('equal')
```

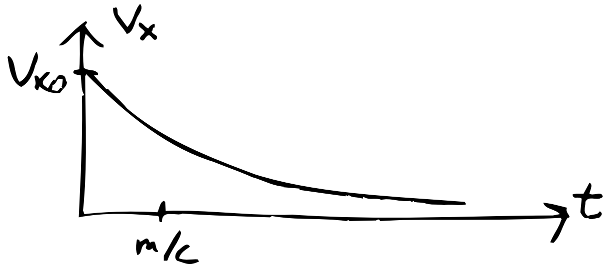
in file myrhs.m

```
function zdot = myrhs(t,z)
x = z(1); y= z(2); vx = z(3); vy = z(4);
c= .1; m= .02; g=10;
xdot = vx;
ydot = vy;
vxdot = -(c/m)*vx;
vydot = -(c/m)*vy - g;
zdot= [xdot ydot vxdot vydot]';
```

Analytic Soln. (prob 3 cont'd)

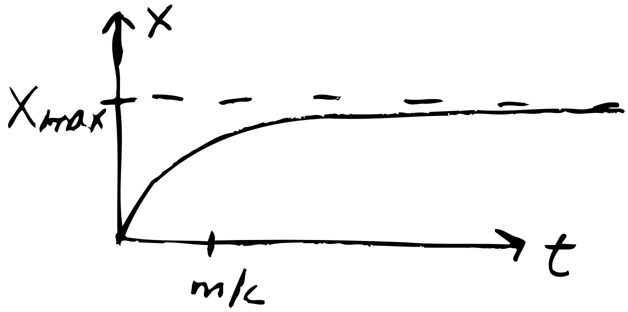
$$5 \Rightarrow \quad V_x = V_{xo}\, e^{-(c/m)\,t}$$

[See comments on pgs. 9-13 about ODE solutions.]



$$3 \Rightarrow \quad x = x_o + \int_0^t V_x(t')\,dt' = 0 + \int_0^t V_{xo}\, e^{-(c/m)\,t'}\,dt'$$

$$= -\frac{m V_{xo}}{c}\, e^{-(c/m)\,t'}\Big|_0^t = \underbrace{\frac{m V_{xo}}{c}\left(1 - e^{-(c/m)\,t}\right)}_{x(t)}$$



$$x_{max} \overset{7}{=} \frac{m V_{xo}}{c} = \frac{.02\,(kg)\; 25\,(m/s)}{.1\,N/(m/s)}$$

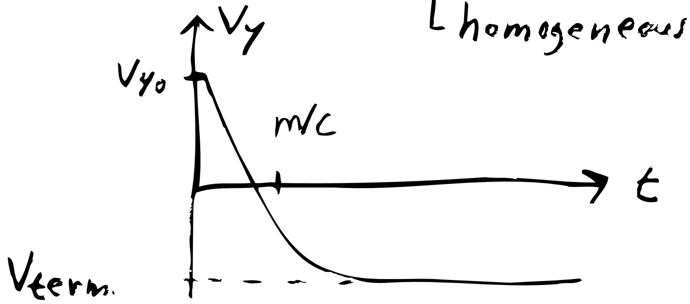
$$= \underbrace{5\,m}$$

[x expontially approaches 5 m w/ time

$$6 \Rightarrow \quad V_y = \underbrace{\left(V_{yo} + \frac{gm}{c}\right) e^{-\frac{c}{m}t}} \quad \underline{-\, gm/c}$$

↑ partic. soln. from inspection (terminal vel.)

└ homogeneous soln., constant picked to sat. I.C.



V_{term} = terminal vel.

$= -g\,m/c$ (drag balances weight)

$= -10 \cdot .02/.1\ (m/s)$

$= 2\ m/s$

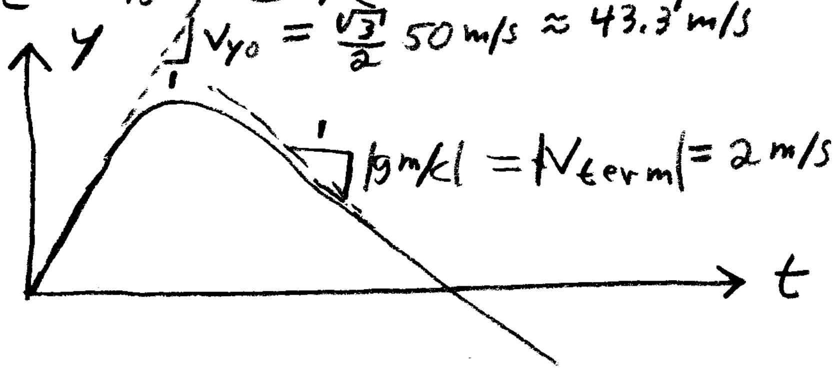
$$4 \Rightarrow \quad y = y_o + \int_0^t v_y(t')\,dt'$$

$$= 0 + \int_0^t \left[\left(V_{yo} + \frac{gm}{c}\right) e^{-(c/m)\,t'} - gm/c\right] dt'$$

(prob(3) cont'd)

$$y = \left[-\frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right)e^{-(c/m)t'} - \frac{gm}{c}t'\right]_0^{t'}$$

$$= \frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c}t$$



$$\underline{r}(t) = x\,\underline{i} + y\,\underline{j}$$

$$= \frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)\underline{i}$$

$$+ \left[\left(\frac{mV_{y0}}{c} + \frac{gm^2}{c^2}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c}t\right]\underline{j}$$

$c/m = (.1/.02)\,s^{-1} = 5/s$, $\frac{gm^2}{c^2} = 10 \cdot \frac{(.02)^2}{(.1)^2}\,m$

$\frac{mV_{x0}}{c} = 5\,m$ (see (7)) $= .4\,m$

$\frac{mV_{y0}}{c} = \frac{\sqrt{3}}{2} \cdot 5\,m \approx 8.66\,m$

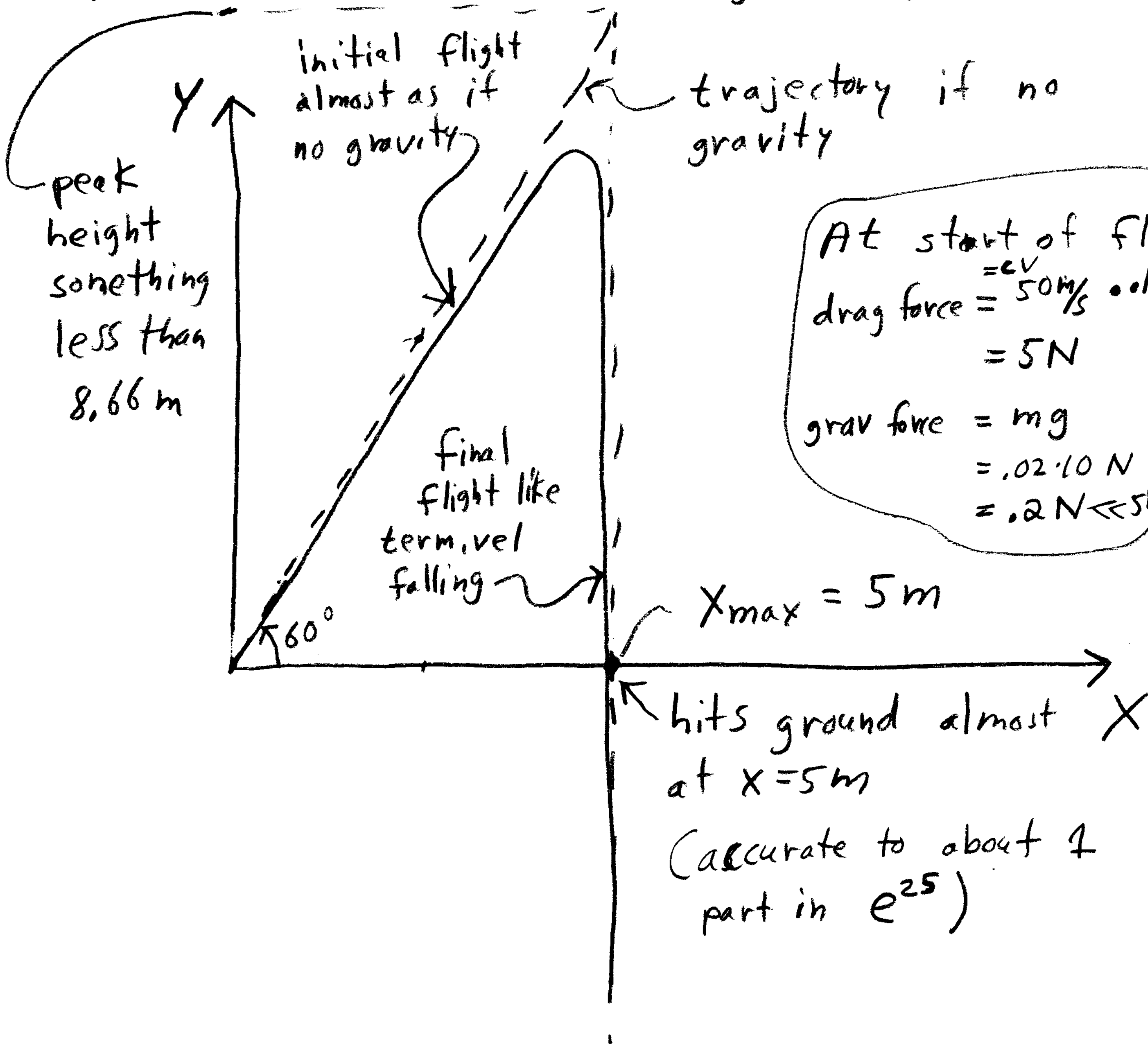
$\frac{gm}{c} = 2\,m/s$

(b)

$$\boxed{\underline{r}(t) = 5m\left(1 - e^{-5t/s}\right)\underline{i} + \left[\left(\frac{5\sqrt{3}}{2} - .4\right)\left(1 - e^{-5t/s}\right)m - 2(m/s)t\right]\underline{j}}$$

i.e eqn for $x(t)$

c) Plot. All we need from anal. soln. is eqn. 7.
(No need for soln. of inhomog. soln.)



At start of flight

drag force $= \overset{=cv}{50 m/s \cdot .1 N/m/s}$
$= 5N$

grav force $= mg$
$= .02 \cdot 10\ N$
$= .2N \ll 5N$

Some comments on ODE solns.

Solve $\dot{V}_x = -(c/m)V_x$

method 1: guess $V_x = e^{rt}$

plug in $r\cancel{e^{rt}} = -(c/m)\cancel{e^{rt}}$

$r = -c/m$

$\Rightarrow V_x = c_1 e^{-(c/m)t}$

c_1 ← arb. const., pick to sat. I.C.

method 2:
(Edwards & Penny 1.4)

$$\frac{dV_x}{dt} = -(c/m)V_x$$

$$\Rightarrow \frac{dV_x}{V_x} = -(c/m)dx$$

$$\Rightarrow \int \frac{dV_x}{V_x} = -\int c/m\, dt$$

$$\Rightarrow \ln V_x = -\frac{c}{m}t + c_1'$$

$$\Rightarrow V_1 = e^{-(c/m)t + c_1'}$$

$$V_x = c_1 e^{-(c/m)t} \qquad (c_1 = e^{c_1'})$$

(again)

ODEs (cont'd)

method 3; Integrating factor
(Edwards & Penny 1.5)

$$\frac{dV_x}{dt} + \frac{c}{m} V_x = 0$$

$e^{c/m\,t}$ is the integrating factor.

$$\Rightarrow \quad e^{\frac{c}{m}t}\frac{dV_x}{dt} + e^{(c/m)t}\frac{c}{m}V_x = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{(c/m)t}V_x\right) = 0$$

$$\Rightarrow \quad e^{(c/m)t}\,V_x = C_1$$

$$\Rightarrow \quad V_x = C_1 e^{-(c/m)t}$$

(again)

Solve $\ddot{x} + c/m\,\dot{x} = 0$

method 1; guess $x = e^{rt}$

$$\Rightarrow \quad r^2 \cancel{e^{rt}} + r\frac{c}{m}\cancel{e^{rt}} = 0$$

$$\Rightarrow \quad r\left(r + \frac{c}{m}\right) = 0 \quad \Rightarrow \quad r = 0, -c/m$$

$$\Rightarrow \quad x(t) = c_1 e^{-\frac{c}{m}t} + c_2 e^{-0t}$$

$$= C_1 e^{-(c/m)t} + C_2$$

find using ICs.

This is $x(t)$ as found.

<u>Solve</u> $\dot{V}_y + (c/m)\, V_y = -g$

<u>method 1</u>: a) find homog. soln.

$$\dot{V}_y + (c/m) V_y = 0.$$

This is identical to prev. problem which we solved to get

$$\underbrace{V_{y_h}}_{\text{homog. soln.}} = c_1 e^{-(c/m)t}$$

b) Find <u>any</u> "particular" soln. of

$$\dot{V}_y + (c/m) Y = -g.$$

As for spring-mass problem. Easiest guess is a constant. In this case you can get this physically by thinking of falling at terminal velocity.

guess $V_{y_p} = c_2$

$$\cancelto{0}{\dot{c}_2} + (c/m)\, c_2 = g$$

$$c_2 = gm/c$$

Solution is

$$V_y = V_{yh} + V_{yp}$$

$$= c_1 e^{-(c/m)t} + gm/c$$

pick to match I.C.

method 2;
(E&P 1.4)
(separable eqn.)

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

$$\Rightarrow \frac{dV_y}{dt} = -\frac{c}{m} V_y - g$$

$$\Rightarrow \frac{dV_y}{V_y + \frac{gm}{c}} = -\frac{c}{m}\, dt$$

$$\Rightarrow \int \frac{dV_y}{V_y - \frac{gm}{c}} = -\int \frac{c}{m}\, dt$$

$$\Rightarrow \ln\left(V_y + \frac{gm}{c}\right) = -\frac{c}{m} t + c_1'$$

$$\Rightarrow V_y + \frac{gm}{c} = e^{-\frac{c}{m}t + c_1'}$$

$$\Rightarrow V_y = c_1 e^{-(c/m)t} - gm/c$$

$c_1 = e^{c_1'}$, pick to sat. I.C.

(again)

method 3;
(E&P 1.5)
linear ODE using integrating factor

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

define $\rho(t) = e^{(c/m)t}$

integrating factor

mult. through by ρ

$$\frac{dV_y}{dt} e^{(c/m)t} + \frac{c}{m} V_y e^{(c/m)t} = -g e^{(c/m)t}$$

(ODEs cont'd)

$$\frac{d}{dt}\left(v_y e^{(c/m)t}\right) = -g e^{(c/m)t}$$

$$\Rightarrow \quad d\left[v_y e^{(c/m)t}\right] = -g e^{(c/m)t} dt$$

$$\Rightarrow \quad \int d\left(v_y e^{(c/m)t}\right) = -\int g e^{c/m\, t} dt$$

$$\Rightarrow \quad v_y e^{(c/m)t} = -\frac{gm}{c} e^{(c/m)t} + c_1$$

$$\Rightarrow \quad v_y = \frac{-gm}{c} + c_1 e^{-(c/m)t}$$

pick to sat. I.C.

(again)