"SOLUTIONS" Your Name: ANDY RUINA

T&AM 203 Final Exam

Friday December 17, 2004

Draft December 17, 2004

5 problems, 150 minutes (no extra time).

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors. If you cannot answer a problem with pencil and paper, you can get partial credit for a good Matlab solution.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

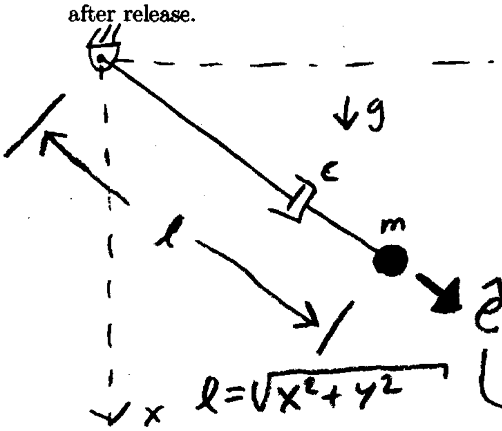
d) Even if not asked for, you can get partial credit by showing a Matlab solution to a problem you can't solve with pencil and paper.

Problem	1:	/20
Problem	2:	/20
Problem	3:	/20
Problem	4:	/20
Problem	5:	/20
TOTAL:		**/100**

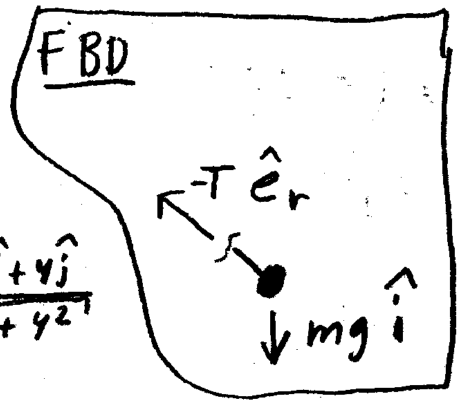
1) (20 pt) In the almost new sport of spongy jumping a spring is replaced by a dashpot c. Assume $m = 3\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, and $c = 7\,\mathrm{kg/s}$. The mass is released from rest at $x = 2\,\mathrm{m}$, $y = 3\,\mathrm{m}$.

a) (15 points) What are the equations of motion for this system (differential equations involving x and y and their derivatives)? (For this part of the problem please use m, c and g rather than their numerical values.

b) (5 points) *This part will only be graded if part (a) is almost entirely correct.* Write Matlab code that would give the arc-length of the center of mass trajectory over the first 5 seconds after release.



$$\hat{e}_r = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$$



$$\underline{\text{LMB}}: \quad \sum \underline{F}_i = m\underline{a} \Rightarrow -T\hat{e}_r + mg\hat{j} = m\underline{a}$$

$$\underline{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

$$T = c\dot{\ell} = c\left(\frac{\partial \ell}{\partial x}\dot{x} + \frac{\partial \ell}{\partial y}\dot{y}\right) = c\frac{1}{\sqrt{x^2+y^2}}(x\dot{x} + y\dot{y})$$

$$\Rightarrow \left\{\left(-c\frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}\right)\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} + mg\hat{i} = (\ddot{x}\hat{i} + \ddot{y}\hat{j})m\right\}$$

$$\{\}\cdot\hat{i} \Rightarrow \quad \boxed{\ddot{x} = \frac{-cx}{m}\frac{(x\dot{x}+y\dot{y})}{(x^2+y^2)} + g}$$

$$\{\}\cdot\hat{j} \Rightarrow \quad \boxed{\ddot{y} = \frac{-cy}{m}\frac{(x\dot{x}+y\dot{y})}{(x^2+y^2)}}$$

$$\dot{s} = |\underline{v}|$$

$\dot{V}_x = \ddot{x}$ (from above)

$\dot{V}_y = \ddot{y}$ (from above)

$\dot{x} = V_x$

$\dot{y} = V_y$

$z_1 = x$
$z_2 = y$
$z_3 = V_x$
$z_4 = V_y$
$z_5 = s$

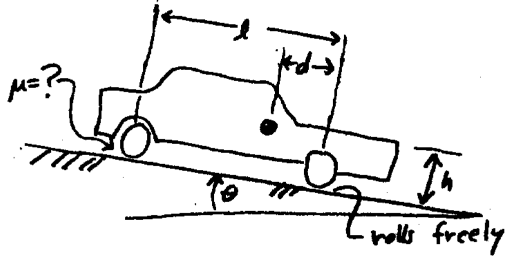
```
       x0   y0   v0    s0
z0= [2   3   0 0   0]';
tspan= [0   5];

[t  z] = ode23('spongy', tspan, z0);
arclength=  z(end,5);
```

```
function   zdot = spongy(t,z)
m = 3;  c = 7;  g = 10;
x = z(1);  y = z(2);
vx = z(3);  vy = z(4);

xdot = vx;
ydot = vy;
L2 = x^2 + y^2;
D  =   c*(x*vx + y*vy)/(m*L2);
vxdot = -x*D + g;
vydot = -y*D;
sdot = sqrt(vx^2 + vy^2);
zdot = [xdot, ydot, vxdot, vydot, sdot]';
```

2) (20 pt) A car going down a hill of slope θ (measured from the horizontal) puts on its rear brakes, causing the rear wheels to skid. The negligible-mass front wheels roll freely. The car moment of inertia about its center of mass G is I and its mass is m. The wheels are a distance ℓ apart (front to back distance). On level ground G is a height h above the ground and a distance d behind the front wheel. In terms of some or all of m, I, g, d, h, ℓ and θ find the minimum coefficient of friction μ needed to slow the car down.



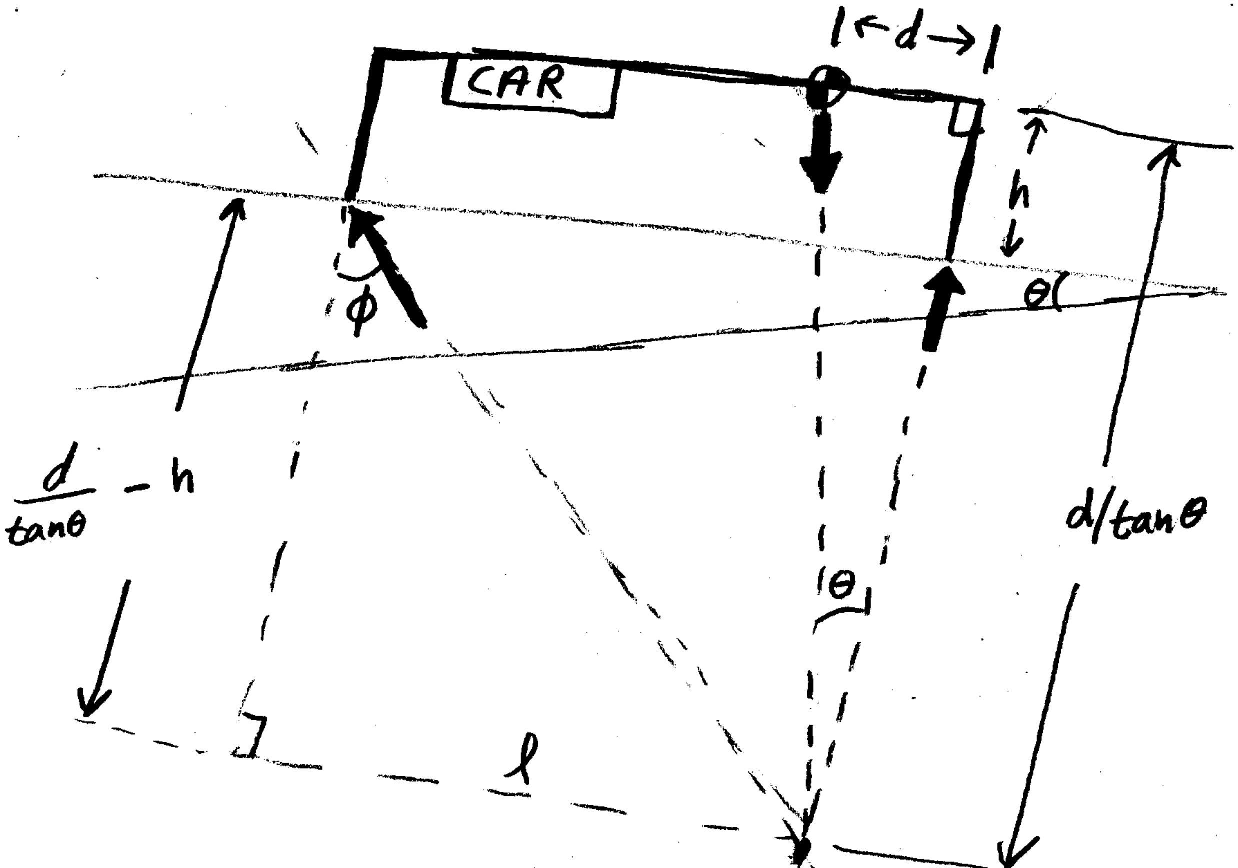
at critical μ

$a = 0 \Rightarrow$ statics

3-force body

$\Rightarrow$ all forces intersect at one point

FBD



$$\tan\phi = \boxed{\mu = \frac{\ell}{\frac{d}{\tan\theta} - h}}$$

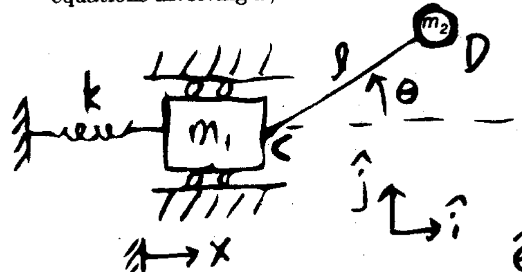
Sanity checks

$\theta \to 0 \Rightarrow \mu \to 0$ ✓

$h = 0,\ d = \ell \Rightarrow \mu = \tan\theta$ ✓

$d \to (\tan\theta) h \Rightarrow \mu \to \infty$ ✓

3) **(20 pt)** m_1 slides horizontally with x measuring the stretch of the spring k from its unstretched length. Point mass m_2 is at the end of a massless rod of length ℓ the other end of which is hinged on m_1. Neglect gravity. Find differential equations that govern the motion of the two masses (differential equations involving x, θ and some or all of m_1, m_2 and ℓ, K.

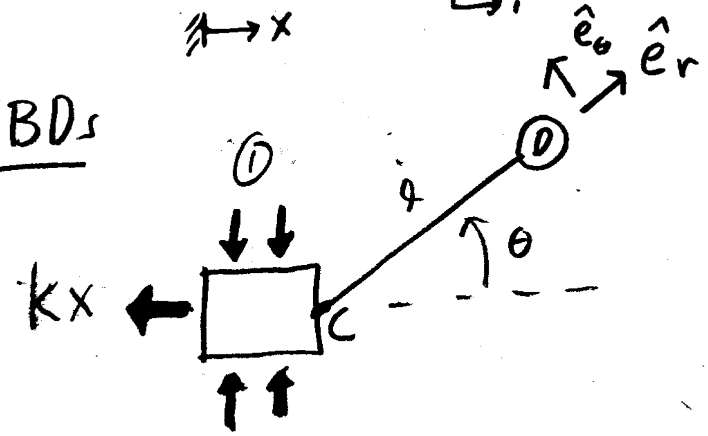


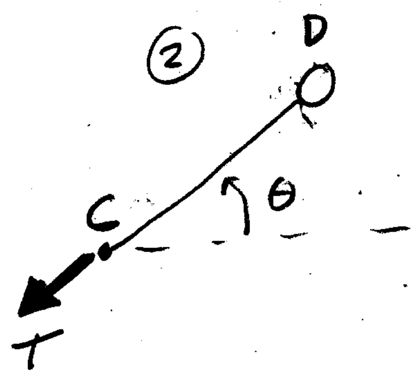
$\hat{e}_\theta \cdot \hat{i} = -\sin\theta$

$\hat{e}_r \cdot \hat{i} = \cos\theta$

$\hat{e}_r \times \hat{i} = -\sin\theta$

FBDs





FBD 1 $\quad \left\{ \sum \underline{F}_i = m \underline{a}_i \right\} \cdot \hat{i}$

$$-kx = m_1 \ddot{x} + m_2 \left(\ddot{x} + \left(\ell \ddot{\theta} \hat{e}_\theta - \ell \dot{\theta}^2 \hat{e}_r \right) \cdot \hat{i} \right)$$

$$\boxed{-kx = (m_1 + m_2)\ddot{x} - \ell \ddot{\theta} \sin\theta - \ell \dot{\theta}^2 \cos\theta} \quad ①$$

FBD 2 $\quad \underline{AMB}_{/C} \Rightarrow$

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$= \underline{a}_C + \underline{a}_{D/C}$$

$$\underline{0} = \underline{r}_{D/C} \times (m \underline{a}_D)$$

$$= \ell \hat{e}_r \times m \left[\left(\ell \ddot{\theta} \hat{e}_\theta - \ell \dot{\theta}^2 \hat{e}_r \right) + \ddot{x} \hat{i} \right]$$

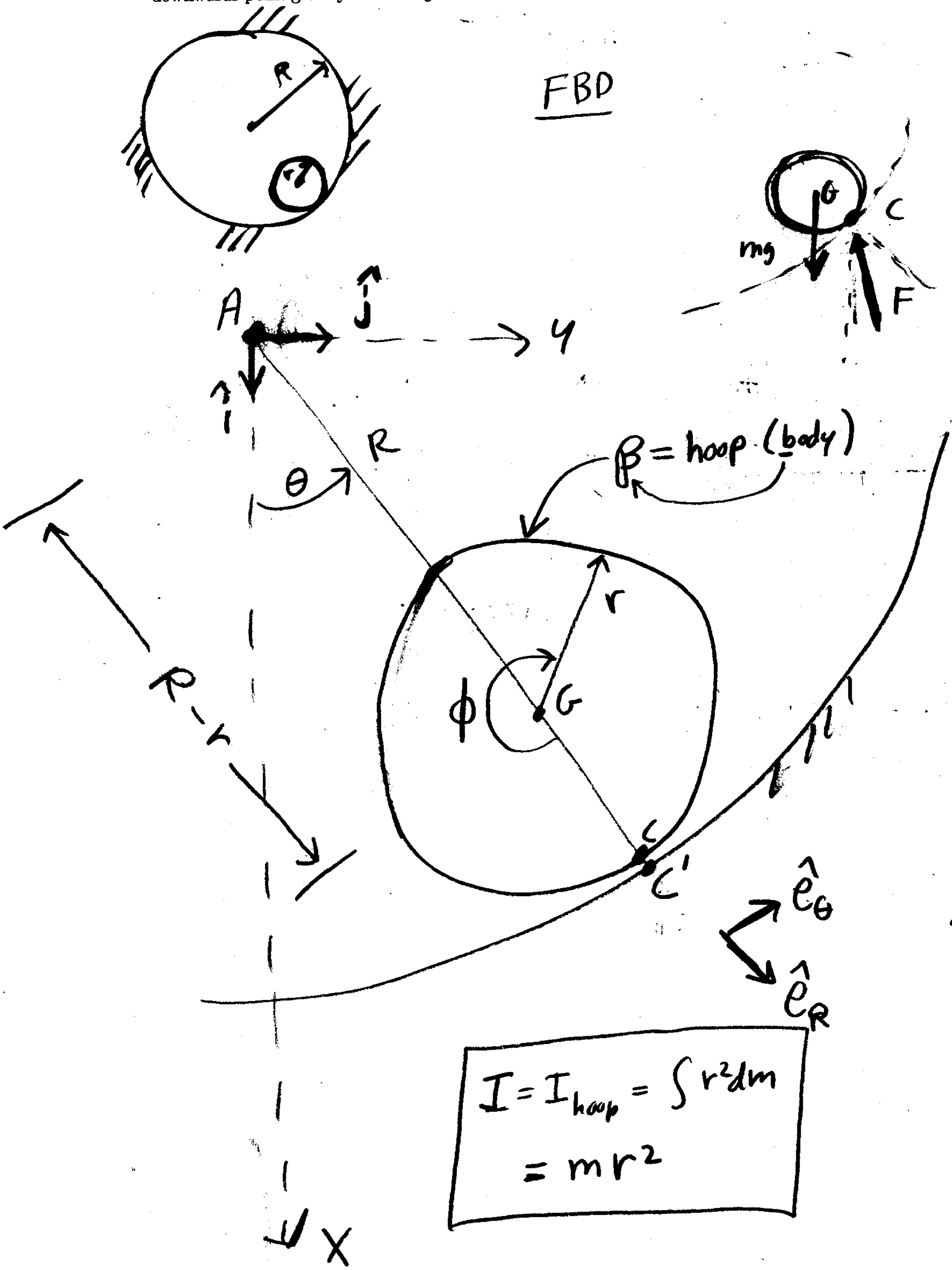
$$\left\{ \quad = \ell m \ddot{x} (-\sin\theta) \hat{k} + \ell^2 m \ddot{\theta} \hat{k} \right\}$$

$\{\ \}\cdot\hat{k} \Rightarrow$ $$\boxed{\ddot{\theta} - \frac{\ddot{x}}{\ell}\sin\theta = 0}$$ (2)

[looks just like the pendulum eqn. but with $-\ddot{x}$ instead of g.
Why? pendulum can't distinguish g from an accelerating frame.]

(1) & (2) are 2 coupled 2nd order ODEs for θ, x.

4) (20 pt) Consider a hollow thin-walled pipe with mass m and radius r. The friction coefficient μ is large enough so the pipe rolls without slip inside a rigid stationary hollow pipe with radius R. Find the period of small oscillations (near the bottom) in terms of some or all of m, r, R, μ and the downwards-point gravity constant g.



$$\boxed{I = I_{hoop} = \int r^2 dm = mr^2}$$

Kinematics:

rolling contact $\Rightarrow$ $r\phi = R\theta$ ①

$$\underline{\omega} = \underline{\omega}_{hoop} = \underline{\omega}_\beta = (\dot{\theta} - \dot{\phi})\hat{k} = \left(\dot{\theta} - \frac{R}{r}\dot{\theta}\right)\hat{k}$$

$$\underline{\omega}_\beta = (1 - R/r)\dot{\theta}\hat{k}$$

$$\underline{a}_G = \ddot{\theta}(R-r)\hat{e}_\theta - \dot{\theta}^2(R-r)\hat{e}_R$$

$AMB_{/C}$:

$$\sum \underline{M}_{/C} = \underline{\dot{H}}_{/C}$$

$$\underline{r}_{G/C} \times mg\hat{i} = \underline{r}_{G/C} \times m\underline{a}_G + I\underline{\dot{\omega}}_\beta$$

$$r\sin\theta\, mg\,\hat{k} = (-r\hat{e}_R) \times [\ddot{\theta}(R-r)\hat{e}_\theta - \dot{\theta}^2(R-r)\hat{e}_R]$$
$$+ I\ddot{\theta}(1-R/r)\hat{k}$$

$$\{ \quad = -r(R-r)\ddot{\theta}\hat{k}m + I\ddot{\theta}(1-R/r)\hat{k} \}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad rmg\sin\theta = -\left[r(R-r)m + \left(\frac{R}{r}-1\right)I\right]\ddot{\theta}$$

$$= -\underbrace{(I + mr^2)}_{I = mr^2}\left(\frac{R}{r}-1\right)\ddot{\theta}$$

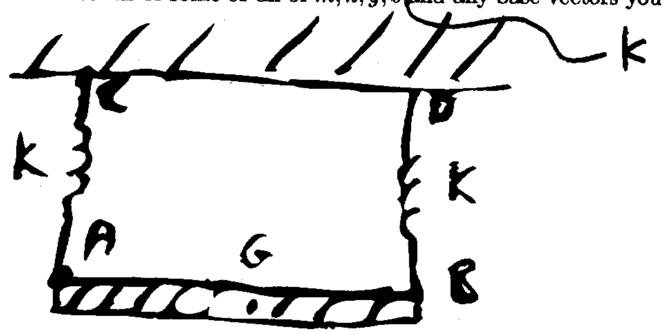
$$\Rightarrow \quad \ddot{\theta} + \frac{\cancel{r}\cancel{m}g}{2\cancel{m}r^{\cancel{2}}(R/r-1)}\sin\theta = 0$$

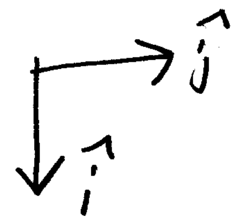
$$\underset{\theta\to 0}{\Rightarrow} \quad \ddot{\theta} + \underbrace{\left[\frac{g}{2(R-r)}\right]}_{\lambda^2}\theta = 0$$

$$\Rightarrow \theta = A\sin(\lambda t) + B\cos\lambda t$$

$$\lambda t_p = 2\pi \Rightarrow \boxed{t_p = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2(R-r)}{g}}}$$

5) (20 pt) A uniform bar AB with length ℓ and mass m is hanging (gravity constant $= g$) in stationary equilibrium from two identical springs k. Suddenly but gently spring A~~B~~ D is cut by a laser beam. Immediately after the cut, what is the acceleration of the rod center at G. Answer in terms of some or all of m, k, g, ℓ and any base vectors you define with clear sketches.

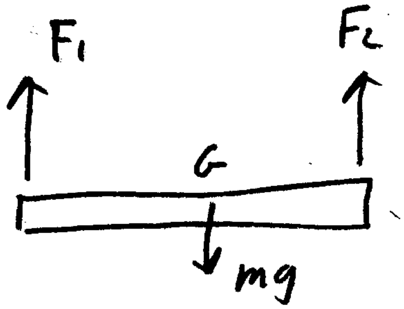




FBDs

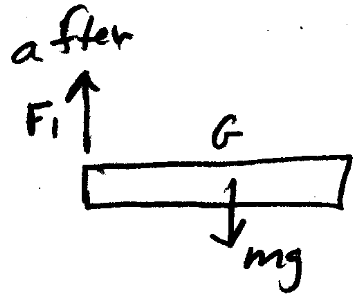
before

(statics)



$\Rightarrow F_1 = mg/2$

after



$$\sum \underline{F}_i = m\underline{a}_G$$

$$-F_1\hat{i} + mg\hat{i} = m\underline{a}_G$$

$$\left(-\frac{mg}{2} + mg\right)\hat{i} = m\underline{a}_G$$

$$\boxed{\underline{a}_G = \frac{g}{2}\hat{i}}$$