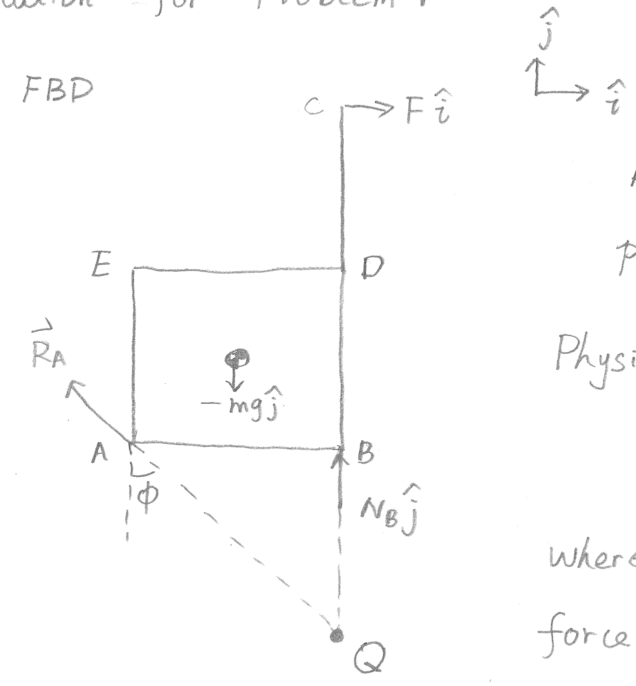
Solution for Problem 4

(a) FBD



$\vec{R}_A$ is the reaction force acting on point A by the ground.

Physics tells us that

$$\vec{R}_A = -f_A\,\hat{i} + N_A\,\hat{j}$$

where N_A is the supporting (normal) force and f_A is the friction force.

Friction law $\Rightarrow$ $f_A = \mu N_A$

$$\therefore \quad \vec{R}_A = -\mu N_A\,\hat{i} + N_A\,\hat{j}$$

The angle ϕ shown in the FBD is the "friction angle".

$$\tan\phi = \mu$$

We extend the line of $\vec{R}_A$ and locate the intersection of it with line DB at Q

Take AMB about Q:

$$\Sigma \vec{M}_{/Q} = \dot{\vec{H}}_{/Q}$$

$$\Rightarrow \underbrace{\cancel{\vec{r}_{A/Q} \times \vec{R}_A}}_{0} + \underbrace{\cancel{\vec{r}_{B/Q} \times N_B\hat{i}}}_{0} + \vec{r}_{C/Q} \times F\hat{i} + \vec{r}_{G/Q} \times (-mg\hat{j})$$

$$= \vec{r}_{G/Q} \times m\underbrace{\vec{a}_G}_{a_G\hat{i}}$$

Since $|BQ| = \dfrac{|AB|}{\tan\phi} = \dfrac{l}{\mu}$

$\therefore \ \vec{r}_{C/Q} = (\frac{l}{\mu} + 2l)\hat{j}$, $\vec{r}_{G/Q} = -\frac{l}{2}\hat{i} + (\frac{l}{\mu} + \frac{l}{2})\hat{j}$

$$\Rightarrow (\frac{l}{\mu} + 2l)\hat{j} \times F\hat{i} + (-\frac{l}{2}\hat{i} + (\frac{l}{\mu} + \frac{l}{2})\hat{j}) \times (-mg\hat{j})$$
$$= (-\frac{l}{2}\hat{i} + (\frac{l}{\mu} + \frac{l}{2})\hat{j}) \times m a_G \hat{i}$$

$$\Rightarrow \{ -F(\frac{l}{\mu} + 2l)\hat{k} + mg\frac{l}{2}\hat{k} = -m a_G(\frac{l}{\mu} + \frac{l}{2})\hat{k} \}$$

$$\frac{\{\ \} \cdot \hat{k}}{l} \cdot 2\mu \Rightarrow \quad -F(2 + 4\mu) + \mu mg = -m a_G(2 + \mu)$$

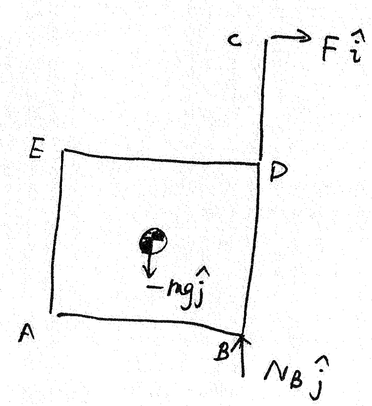
$$\Rightarrow \quad a_G = \frac{F(2+4\mu) - \mu mg}{(2+\mu)m}$$

In this way, we obtain the acceleration through a single equation!

(b.) No tipping condition, $N_A \geq 0$

On the verge, $N_A = 0$, $\therefore$ $\vec{R}_A = 0$

FBD:



$\{LMB\} \cdot \hat{j} \Rightarrow \quad N_B = mg$.

AMB about center of mass G

$$\Rightarrow \quad \Sigma \vec{M}_{/G} = \cancel{\dot{\vec{H}}_{/G}} \ 0$$

$$\Rightarrow \quad (\frac{3l}{2}\hat{j} + \frac{l}{2}\hat{i}) \times F\hat{i} + (\frac{l}{2}\hat{i} - \frac{l}{2}\hat{j}) \times N_B\hat{j} = 0$$

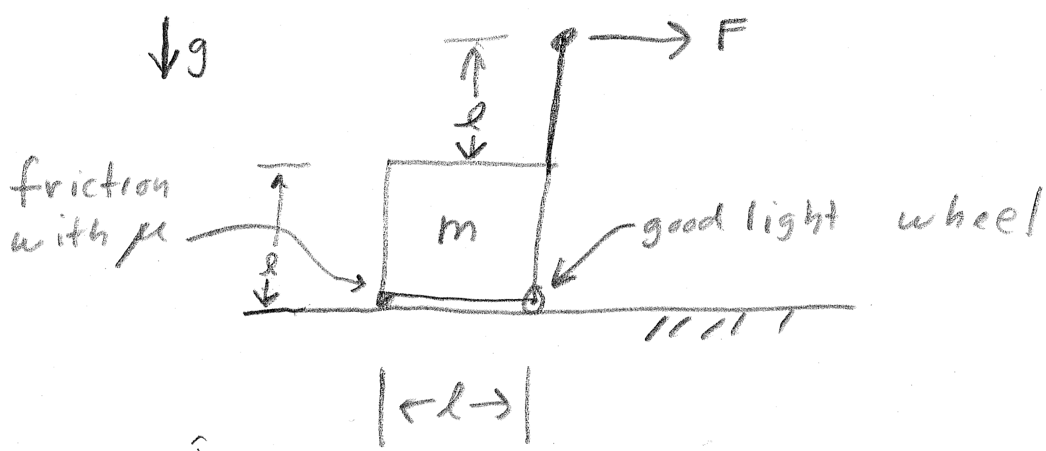
$$\Rightarrow \quad -F\frac{3l}{2} + N_B\frac{l}{2} = 0$$

$$\Rightarrow \quad F = \frac{N_B}{3} = \frac{mg}{3}$$

We know the suitcase tends to tip when F is large, so $F_{max} = \frac{mg}{3}$

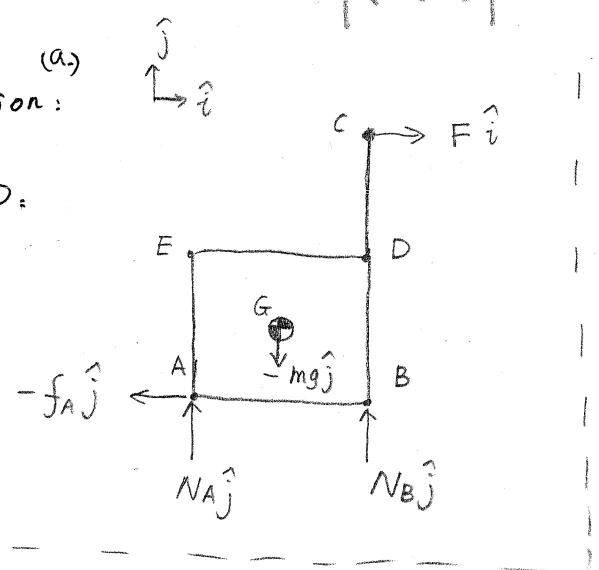
4) A uniform cubicle suitcase is pulled with force $F>0$. In terms of some or all of variables given a) find the acceleration of the suitcase (assuming it does not tip over).

b) What is max force for no tipping?



Solution: (a.)

FBD:



Note:

1). Front wheel is "good light wheel". We can assume it is massless and has frictionless bearing. AMB of the wheel shows no friction force between front wheel and the ground.

2). Assume handel CD is massless, so the center of mass of the suitcase is at the center of square ABDE.

The back wheel is sliding on the ground. By friction law,

$$f_A = \mu N_A \qquad \cdots \qquad ①$$

LMB of the suitcase

$$\sum \vec{F} = m\,\vec{a}_G \quad \Rightarrow \quad F\hat{i} - f_A\hat{i} + N_A\hat{j} + N_B\hat{j} - mg\hat{j} = m\vec{a}_G \quad \cdots \; ②$$

$\because$ The suitcase is translating on the ground, (every point on it has the same acceleration)

$\therefore$ $\vec{a}_G = a\hat{i}$ and no rotation (angular velocity $\omega(t) = 0$).

$$② \cdot \hat{i} \Rightarrow \quad F - f_A = ma \qquad ②a$$

$$② \cdot \hat{j} \Rightarrow \quad N_A + N_B - mg = 0 \qquad ②b$$

AMB about center of mass G.

$\Rightarrow$

$$\vec{r}_{A/G} \times (-f_A\hat{i} + N_A\hat{j}) + \vec{r}_{B/G} \times (N_B\hat{j}) + \vec{r}_{C/G} \times F\hat{i} + \vec{r}_{G/G} \times (-mg\hat{j})$$

$$= \vec{r}_{G/G} \times m\,\vec{a}_G$$

$$\Rightarrow \left(-\frac{l}{2}\hat{i} - \frac{l}{2}\hat{j}\right) \times (-f_A\hat{i} + N_A\hat{j}) + \left(\frac{l}{2}\hat{i} + \frac{l}{2}\hat{j}\right) \times N_B\hat{j}$$

$$+ \left(\frac{l}{2}\hat{i} + \frac{3l}{2}\hat{j}\right) \times F\hat{i} = 0$$

$$\Rightarrow \quad -\frac{N_A l}{2}\hat{k} - \frac{f_A l}{2}\hat{k} + \frac{N_B l}{2}\hat{k} - \frac{3l}{2}F\hat{k} = 0 \qquad ③$$

$$③ \cdot \hat{k} \Rightarrow \quad -\frac{N_A l}{2} - \frac{f_A l}{2} + \frac{N_B l}{2} - \frac{3l}{2}F = 0$$

$$\Rightarrow \quad N_A + f_A - N_B + 3F = 0 \qquad ③a$$

$$②b + ③a \Rightarrow \quad 2N_A + f_A + 3F - mg = 0 \qquad ④$$

Plug ① in ④, $\Rightarrow$ $(2+\mu)N_A + 3F - mg = 0$

$$N_A = \frac{mg - 3F}{2 + \mu}$$

(2a) $\Rightarrow$ $$a = \frac{F - f_A}{m} = \frac{F - \mu N_A}{m} = \frac{1}{m}\left[F - \frac{\mu}{2+\mu}(mg - 3F)\right]$$

$$= \boxed{\frac{(2+4\mu)F - \mu mg}{(2+\mu)m}}$$

The acceleration of the suitcase is $\boxed{\vec{a} = \frac{(2+4\mu)F - \mu mg}{(2+\mu)m}\hat{i}}$

(b) The condition for the suitcase not to tip over is

$$N_A \geq 0$$

[(Imagine if you increase F. N_A decreases from positive value to 0 to balance angular momentum. If you keep increasing F, but N_A cannot be negative (the ground cannot adhere the rear wheel), then there will be some angular acceleration ($\dot{\omega}$), and the suitcase tips over]

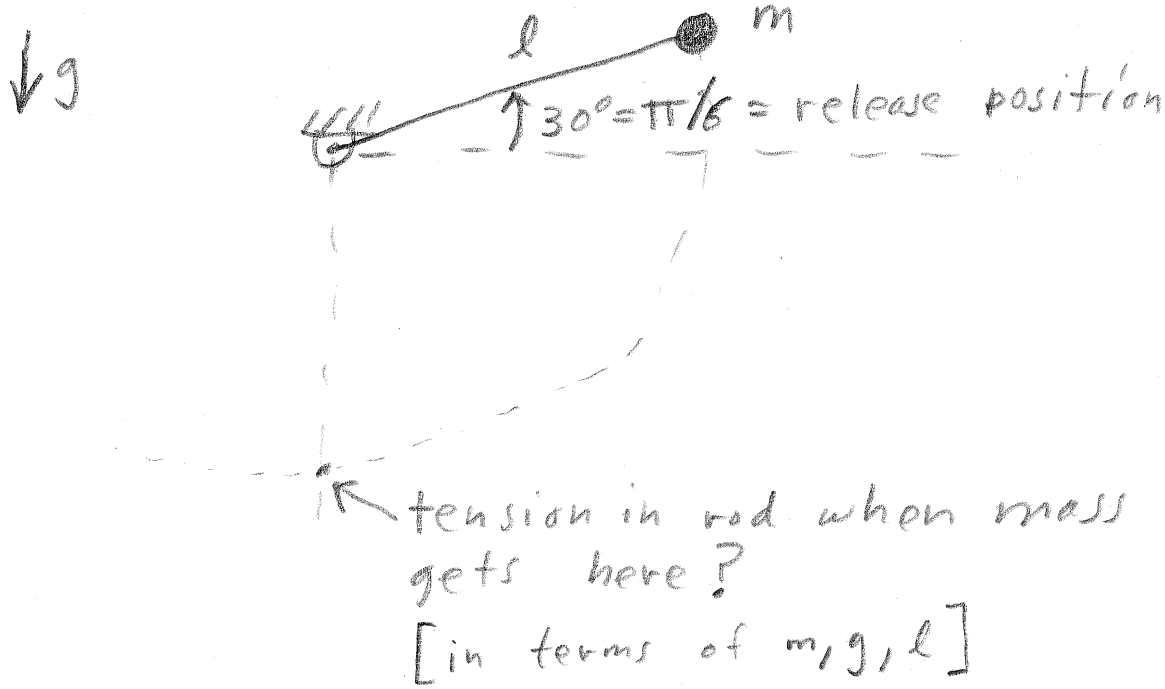
$\because$ From a) $$N_A = \frac{mg - 3F}{2+\mu}$$

$\therefore$ $$N_A \geq 0 \Rightarrow mg - 3F \geq 0$$

$$\Rightarrow F \leq \frac{mg}{3}$$

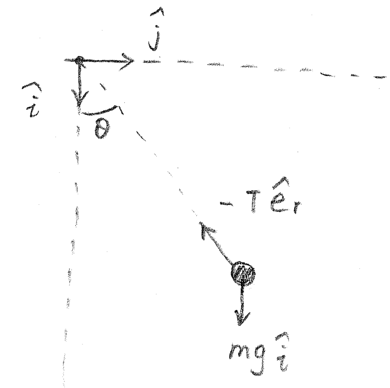
The max force for no tipping is $\boxed{F_{max} = \frac{mg}{3}}$

5) A point mass is connected to the end of a hinged massless rigid rod which is released from rest at the position shown. What is the tension when it swings to the bottom? [Full credit for Matlab commands which would yield the correct answer]



Solution:

FBD



Rigid rod $\Rightarrow$ ℓ is a constant

m is doing circular motion.

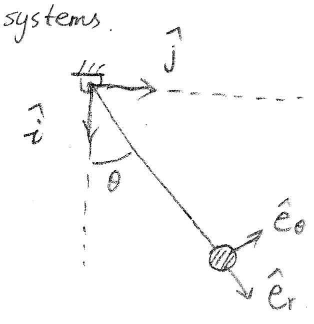
At any angle θ,

$\angle$MB $\Rightarrow$

$$-T\hat{e}_r + mg\hat{i} = m\vec{a} \quad \cdots \quad ①$$

where $-T\hat{e}_r$ is the tensile force

coordinate systems.



Acceleration $\vec{a}$ is

$$\vec{a} = \ell\ddot{\theta}\,\hat{e}_\theta - \ell\dot{\theta}^2\,\hat{e}_r$$

$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

$\therefore \quad ① \cdot \hat{e}_r \Rightarrow \quad -T + mg\cos\theta = -m\,l\,\dot{\theta}^2 \quad \Rightarrow \boxed{T = m\,l\,\dot{\theta}^2 + mg\cos\theta}$

$① \cdot \hat{e}_\theta \Rightarrow \quad m\,l\,\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$

When the mass swings to the bottom, $\theta = 0$

$\therefore \quad T_{bot} = T(\theta=0) = m\,l\,(\dot{\theta}|_{\theta=0})^2 + mg$

Now what is $\dot{\theta}|_{\theta=0}$?

Can do it using energy conservation or direct integral. They are essentially the same.

i) energy conservation

Use the bottom point as the datum for gravitational potential energy.

At bottom point, kinetic energy $E_k^b = \frac{1}{2} m (\dot{\theta}|_{\theta=0})^2 l^2$

potential energy $E_p^b = 0$

At top point, kinetic energy $E_k^t = 0$

potential energy $E_p^t = mg\,(l + l\sin 30^\circ) = \frac{3}{2} mgl$

$E_k^b + E_p^b = E_k^t + E_p^t \quad \Rightarrow \quad \frac{1}{2} m (\dot{\theta}|_{\theta=0})^2 l^2 = \frac{3}{2} mgl$

$\therefore \quad (\dot{\theta}|_{\theta=0})^2 = \frac{3g}{l}$

$\therefore \quad T_{bot} = m\,l\,(\dot{\theta}|_{\theta=0})^2 + mg = m\,l\,\frac{3g}{l} + mg = 4mg$

$\therefore$ Tension is $\boxed{4mg}$ at the bottom.

ii) Direct integration

$\because \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$

$\therefore \quad \frac{d}{dt}(\dot{\theta}) + \frac{g}{l}\sin\theta = 0 \quad \Rightarrow \quad \dot{\theta}\frac{d}{dt}(\dot{\theta}) + \frac{g}{l}\dot{\theta}\sin\theta = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2\right) + \frac{d}{dt}\left(-\frac{g}{l}\cos\theta\right) = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\right) = 0 \quad \Rightarrow \quad \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = C$

At the beginning, $\theta = \frac{2\pi}{3}$, $\dot{\theta} = 0$

$\Rightarrow \quad C = -\frac{g}{l}\cos\left(\frac{2\pi}{3}\right) = \frac{g}{2l}$

At the bottom, $\theta = 0$, $\dot{\theta} = \dot{\theta}|_{\theta=0}$

$\Rightarrow \quad \frac{1}{2}\left(\dot{\theta}|_{\theta=0}\right)^2 - \frac{g}{l}\cos 0 = C = \frac{g}{2l}$

$\Rightarrow \quad \left(\dot{\theta}|_{\theta=0}\right)^2 = \frac{3g}{l}$

Similarly, $T_{bot} = ml\left(\dot{\theta}|_{\theta=0}\right)^2 + mg = \boxed{4mg}$

For Matlab:

1) Best if you did some non-dimensionalization first, e.g.

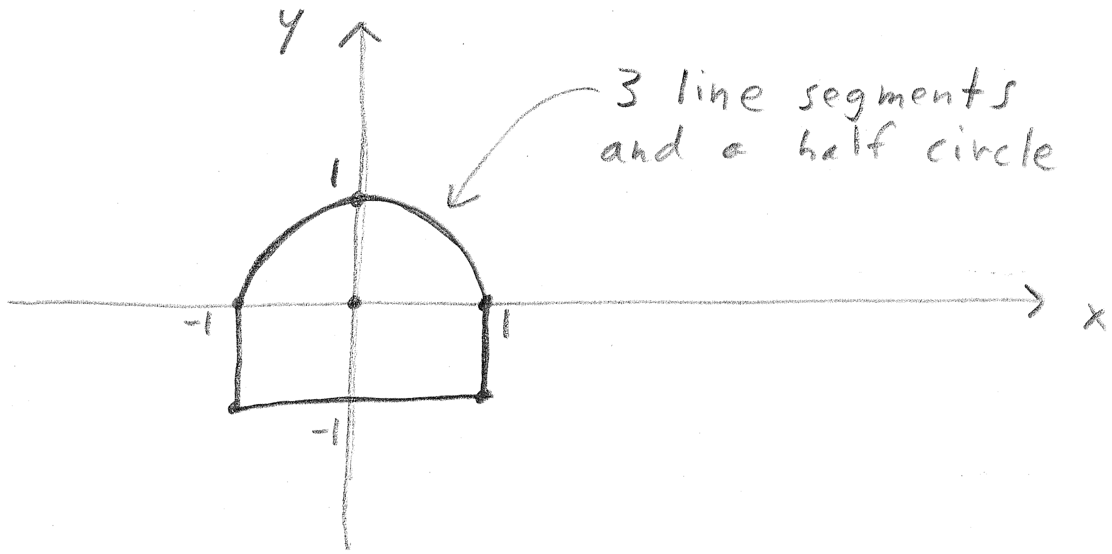
$\bar{t} = t\sqrt{\frac{g}{l}}, \quad \bar{T} = \frac{T}{mg} = \frac{l}{g}\dot{\theta}^2 + \cos\theta = \left(\frac{d\theta}{d\bar{t}}\right)^2 + \cos\theta$

Your ode is $\frac{d^2\theta}{d\bar{t}^2} - \cos\theta = 0$.

In this way, you don't need to input value for g and l and should get $\bar{T} = 4$ at the bottom.

2) Should provide code to detect when the mass reaches the bottom. Use "event" control in ode45 to detect when $\theta = 0$ or other methods.

6) Write MATLAB commands to draw this picture rotated 63° about the origin.



Solution:

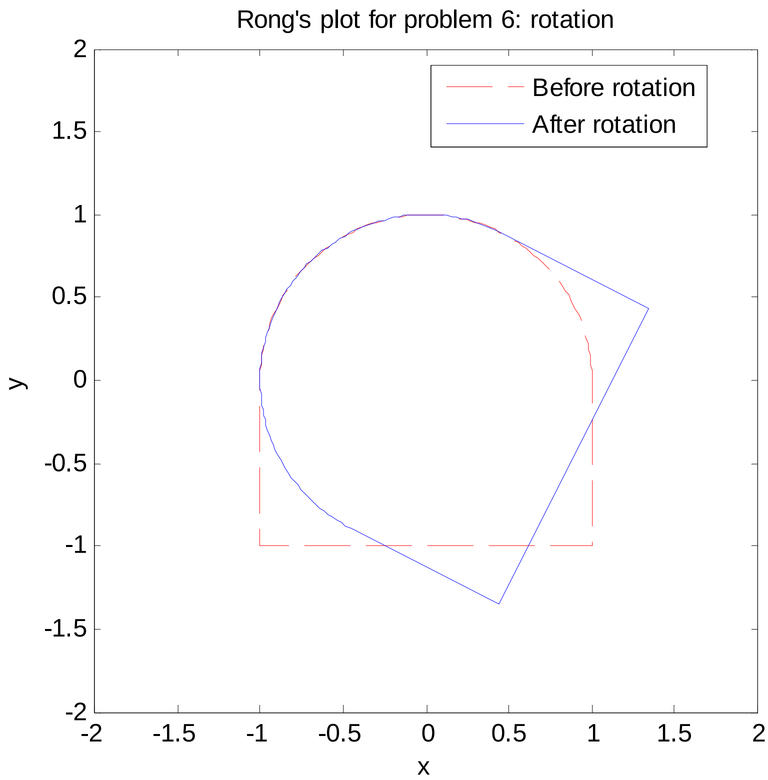
Rotation matrix

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the rotation angle, $\theta = 63°$.

See attached Matlab code:

```
function prelimquestion6

% draw the picture of a given object after rotation by 63 degrees about the
% origin. The object consists of 3 line segments and a half circle.

% Creat rotation matrix
theta=63;                          % Angle of rotation;
thetar=theta*pi/180;      % Convert degree to rad

R=[cos(thetar) -sin(thetar);
     sin(thetar)    cos(thetar)]; %Define rotation matrix

% Creat the position vectors before rotation.
t=linspace(0,pi,100);    % t is the parameter of the half circle.
xcircle=cos(t);
ycircle=sin(t);               % Creat x,y coordinates of points on the half circle.
                                       % The radius of the circle is 1.
x=[xcircle,-1,1,1];
y=[ycircle,-1,-1,0];      % Append the half circle to the 3 line segments

rref=[x;y];                     % Creat the matrix representing the object before rotation.

% Calculate positon after rotation
r=R*rref;

% Plot the object
plot(rref(1,:),rref(2,:),'--r');      %plot the object before rotation;
hold on;
plot(r(1,:),r(2,:));                            %plot the object after rotation;
hold off;

xlabel('x');ylabel('y');
title('Rong''s plot for problem 6: rotation');
legend('Before rotation','After rotation');
axis equal;
axis([-2 2 -2 2]);
```


Rong's plot for problem 6: rotation
Before rotation
After rotation
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x