Your Name: RUINA

TA's name and Section time: ____________

# T&AM 203 Final Exam

## Wednesday Dec 16, 2008

Draft December 16, 2008

5 problems, $25^{+}$ points each, and 150 minutes minutes.

**Please follow these directions to ease grading and to maximize your score.**

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

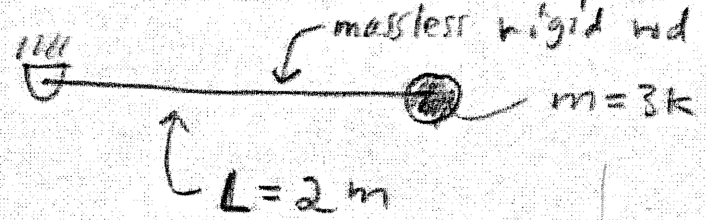
b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I. ) neat,
  II. ) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

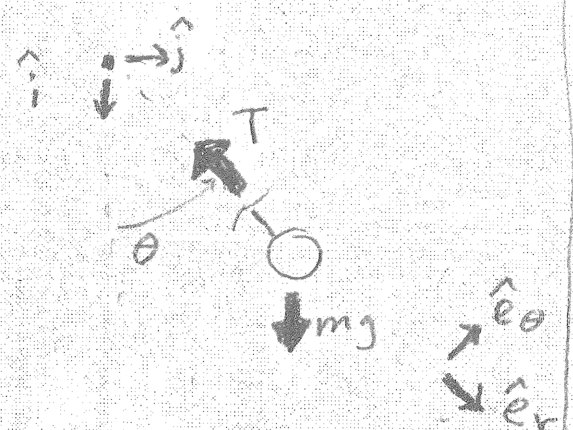
| | |
|---|---|
| Problem 1: | /25 |
| Problem 2: | /25 |
| Problem 3: | /25 |
| Problem 4: | /25 |
| Problem 5: | /25 |

1) A simple point mass pendulum is released from rest at the position shown.
Write Matlab code that would plot tension (in Newtons) vs time (in seconds) for 10 seconds.




$\downarrow g = 10\, m/s^2$

FBD:




LMB:

$$\sum \vec{F} = m\vec{a}$$

$$mg\hat{i} - T\vec{e}_r = m\left[ L\ddot{\theta}\hat{e}_\theta - L\dot{\theta}^2\hat{e}_r \right]$$

$$\{\vec{F} = m\vec{a}\} \cdot \hat{e}_\theta \Rightarrow$$

$$\Rightarrow mg\underbrace{\hat{i}\cdot\hat{e}_\theta}_{-\sin\theta} = mL\ddot{\theta}$$

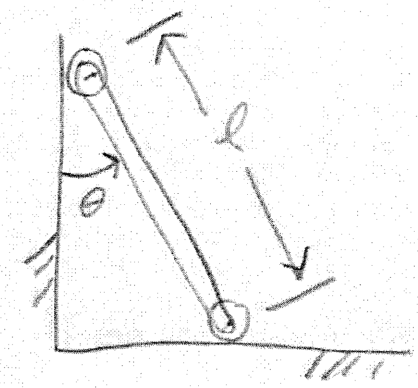
$$\Rightarrow \boxed{\ddot{\Theta} = \frac{-g}{L}\sin\theta \quad (1)}$$

$$\{\vec{F} = m\vec{a}\} \cdot \hat{e}_r \Rightarrow$$

$$mg\underbrace{\hat{i}\cdot\hat{e}_r}_{\cos\theta} - T = -mL\dot{\theta}^2$$

$$\Rightarrow \boxed{T = mL\dot{\theta}^2 + mg\cos\theta} \quad (2)$$

2) A uniform ladder is released from rest in the position shown. Just after release what is the ~~force of~~ the wall on the ladder?

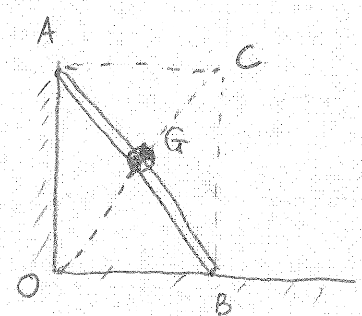



mass = m

$\downarrow g$

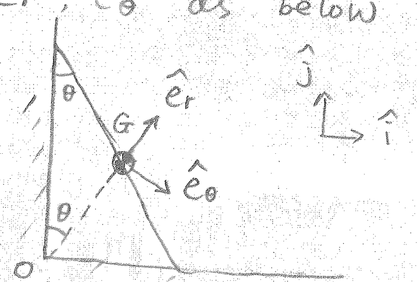
Solution:

Fact: center of mass of the ladder is doing circular motion.




$|GO| = \frac{l}{2}$ = constant

Define $\hat{e}_r$, $\hat{e}_\theta$ as below (at the instance of release)




$\hat{e}_r$ is along $\overrightarrow{OG}$, or, $\overrightarrow{OC}$

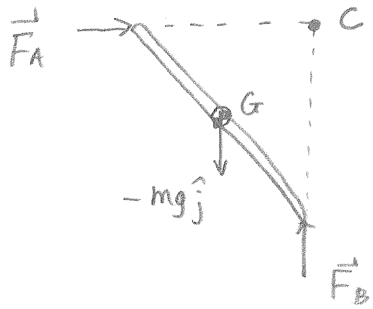
Acceleration of point G

$$\vec{a}_G = \frac{l}{2}\ddot{\theta}\,\hat{e}_\theta - \frac{l}{2}\dot{\theta}^2\,\hat{e}_r$$

At the instant of releasing, $\dot{\theta} = 0$

$$\Rightarrow \vec{a}_G = \frac{l}{2}\ddot{\theta}\,\hat{e}_\theta$$

FBD




$\vec{F_A}$ : reaction force from the vertical wall, $\vec{F_A} = F_A \hat{i}$

$\vec{F_B}$ : reaction force from the ground

$\vec{F_B} = F_B \hat{j}$

$I^G = \frac{ml^2}{12}$ (uniform rod)

AMB about C

$$\underbrace{\vec{r}_{G/C}}_{-\frac{l}{2}\hat{e}_r} \times (-mg\hat{j}) = \underbrace{\vec{r}_{G/C}}_{-\frac{l}{2}\hat{e}_r} \times (\underbrace{m\vec{a}_G}_{m\ddot{\theta}\frac{l}{2}\hat{e}_\theta}) + I^G \ddot{\theta}\hat{k}$$

$$\Rightarrow \quad mg\frac{l}{2}(\underbrace{\hat{e}_r \times \hat{j}}_{\sin\theta}) = \left(m\ddot{\theta}\frac{l^2}{4} + \frac{ml^2}{12}\ddot{\theta}\right)\hat{k}$$

$$\Rightarrow \quad mg\frac{l}{2}\sin\theta = \frac{ml^2}{3}\ddot{\theta} \quad \Rightarrow \quad \boxed{\ddot{\theta} = \frac{3g}{2l}\sin\theta} \qquad ①$$

LMB: $F_A\hat{i} + F_B\hat{j} - mg\hat{j} = m\vec{a}_G = m\frac{l}{2}\ddot{\theta}\hat{e}_\theta$ ... ②

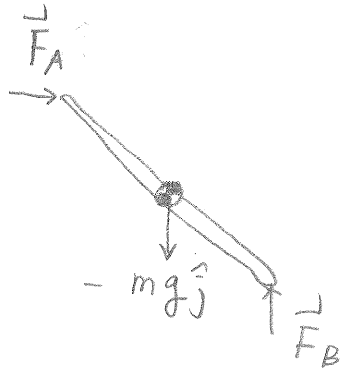
$\therefore$ ② $\cdot \hat{i}$ $\Rightarrow$ $F_A = m\frac{l}{2}\ddot{\theta}(\hat{e}_\theta \cdot \hat{i})$

$= m\frac{l}{2}\ddot{\theta}\cos\theta$

plug in $\ddot{\theta}$ using ① $\Rightarrow$ $F_A = \frac{3}{4}mg\sin\theta\cos\theta$

$\therefore$ Force of the wall : $\boxed{\vec{F_A} = \frac{3}{4}mg\sin\theta\cos\theta\,\hat{i}}$

Alternate method: FBD




$\vec{F}_A$: reaction force from the vertical wall, $\vec{F}_A = F_A\hat{i}$

$\vec{F}_B$: reaction force from the ground

$\vec{F}_B = F_B\hat{j}$

LMB:
$$F_A\hat{i} + F_B\hat{j} - mg\hat{j} = m\vec{a}_G = m\frac{l}{2}\ddot{\theta}\hat{e}_\theta \quad ①$$

AMB about center of mass G:

$$\vec{r}_{A/G} \times F_A\hat{i} + \vec{r}_{B/G} \times F_B\hat{j} = I^G\dot{\vec{\omega}} = I^G\ddot{\theta}\hat{k}$$

$$\Rightarrow \quad -F_A\frac{l}{2}\cos\theta + F_B\frac{l}{2}\sin\theta = I^G\ddot{\theta} \quad ②$$

$① \cdot \hat{i} \Rightarrow \quad F_A = m\frac{l}{2}\ddot{\theta}(\hat{e}_\theta\cdot\hat{i}) = m\frac{l}{2}\ddot{\theta}\cos\theta \quad ③$

$① \cdot \hat{j} \Rightarrow \quad F_B - mg = m\frac{l}{2}\ddot{\theta}(\hat{e}_\theta\cdot\hat{j}) = -m\frac{l}{2}\ddot{\theta}\sin\theta \quad ④$

$②, ③ \Rightarrow$
$$\boxed{-F_A\frac{l}{2}\cos\theta^2 + F_B\frac{l}{2}\cos\theta\sin\theta = \frac{2I^G}{ml}F_A}$$

$③, ④ \Rightarrow$
$$\boxed{F_A\sin\theta + F_B\cos\theta = mg\cos\theta}$$

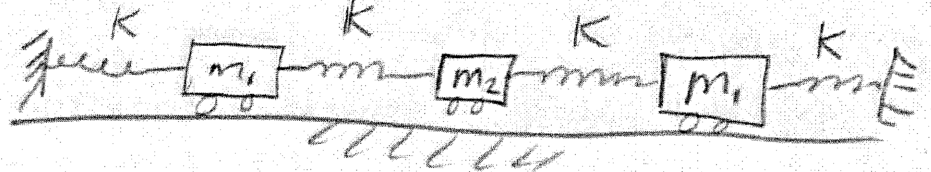
Solve for $F_A$, $F_B$ (with $I^G = \frac{ml^2}{12}$) (uniform ladder)

$$\Rightarrow \quad \boxed{\vec{F}_A = \frac{3}{4}mg\cos\theta\sin\theta\,\hat{i}} \quad \rightarrow \text{force from vertical wall}$$

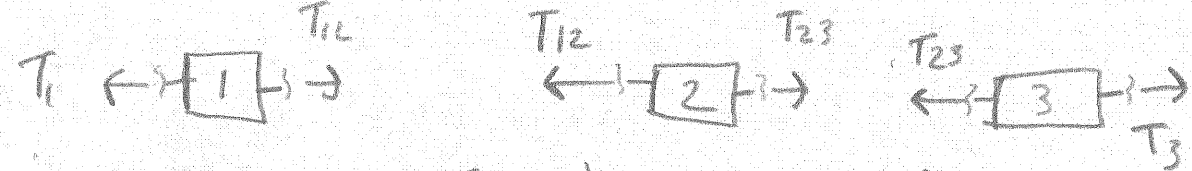
$$\vec{F}_B = mg\left(1 - \frac{3}{4}\sin^2\theta\right)\hat{j} \quad \rightarrow \text{force from ground}$$

3) The positions of the 3 masses are measured that $x_1 = x_2 = x_3 = 0$ is one solution of the equations of motion. Find any other solution:

$x_1(t) = ?$
$x_2(t) = ?$
$x_3(t) = ?$




## FBDs




$T_1 = Kx_1$ $\qquad$ $T_{12} = K(x_2 - x_1)$ $\qquad$ $T_{23} = K(x_3 - x_2)$

$T_3 = -Kx_3$

## LMB

$$m_1\ddot{x}_1 = T_{12} - T_1$$
$$= K(x_2 - x_1) - Kx_1$$
$$= -2Kx_1 + Kx_2$$

$$m_2\ddot{x}_2 = T_{23} - T_{12}$$
$$= K(x_3 - x_2) - K(x_2 - x_1)$$
$$= Kx_1 - 2Kx_2 + Kx_3$$

$$m_3\ddot{x}_3 = T_3 - T_{23}$$
$$= K x_3 - k(x_3 - x_2)$$
$$= Kx_2 - 2Kx_3$$

$$\Rightarrow \quad \begin{bmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \\ m_1\ddot{x}_3 \end{bmatrix} = \begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (1)$$

by inspection (symmetry of system) one normal mode is $(1, 0, -1)$.

Check

$$(1) \Rightarrow \quad m_1 \begin{bmatrix} \ddot{x} \\ 0 \\ -\ddot{x} \end{bmatrix} = \begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix} \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$$

$$= -2k \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$$

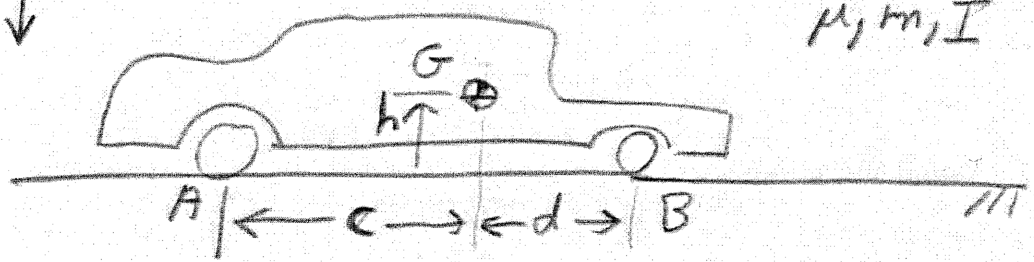
$$\Rightarrow \quad m_1\ddot{x} = -2kx \quad \Rightarrow \quad x = A\cos\left(\sqrt{2k/m_1}\, t\right)$$

$$\Rightarrow \quad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\cos\sqrt{\frac{2k}{m_1}}\, t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

this is a soln. for any A

4) a) A car has front wheel braking. What is the maximum deceleration?

$g \downarrow$ $\mu, m, I$

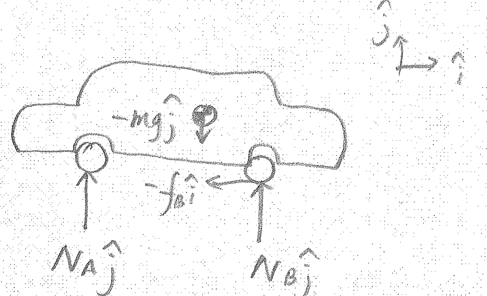



b) For $h=0$ & $c=d$ the deceleration for front wheel braking is the same as for rear wheel braking. Can you name any other combination of $m, g, I, c, d, \mu$ & $h$ so that this is so?

Solution:

a). Assume ideal wheels (massless frictionless)

FBD for front wheel breaking




Friction law

$f_B = \mu N_B$

LMB: $-f_B\hat{i} + N_A\hat{j} + N_B\hat{j} - mg\hat{j} = m\vec{a}_G = m(-a_G\hat{i})$

LMB $\cdot \hat{i}$ $\Rightarrow$ $-f_B = -m a_G$

LMB $\cdot \hat{j}$ $\Rightarrow$ $N_A + N_B = mg$

AMB about G $\Rightarrow$

$$-f_B h + N_B d - N_A c = 0$$

using $f_B = \mu N_B$, we get

$$N_B = \frac{mgc}{c+d-\mu h}$$

$$\therefore \quad a_G = \frac{f_B}{m} = \frac{\mu N_B}{m} = \frac{\mu c}{c+d-\mu h} g$$

$\therefore$ Maximum deceleration $\boxed{a_G = \frac{\mu c}{c+d-\mu h} g}$

b). If we break the rear wheel, do the similar calculation as that in (a), we get the deceleration

$$\boxed{a_G^{rear} = \frac{\mu d}{c+d+\mu h} g}$$

If $c=d$ & $h=0$, we have

$a_G^{front} = \frac{\mu}{2} g$, $a_G^{rear} = \frac{\mu}{2} g$ ✓ check!

Other conditions:

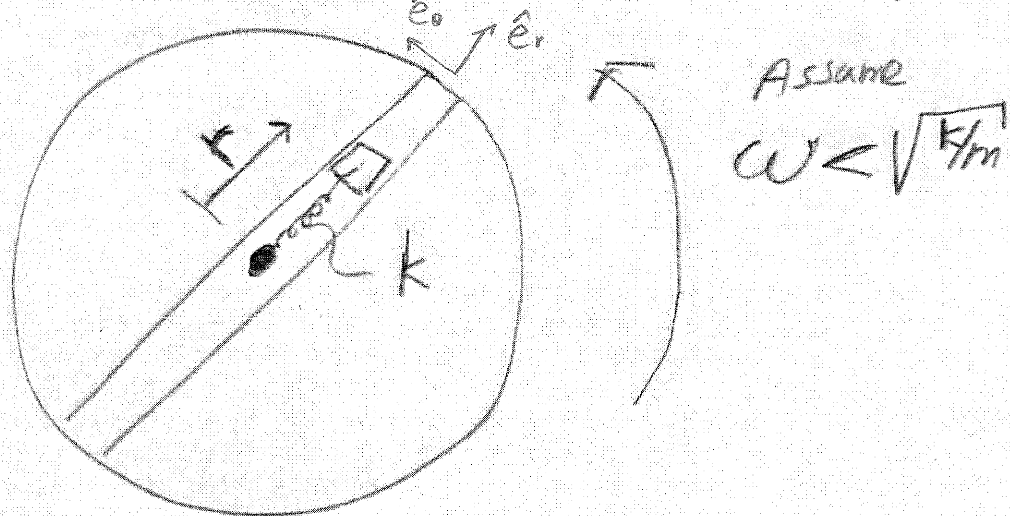
Let $a_G^{rear} = a_G^{front}$

$$\Rightarrow \quad \frac{\mu d}{c+d+\mu h} g = \frac{\mu c}{c+d-\mu h} g$$

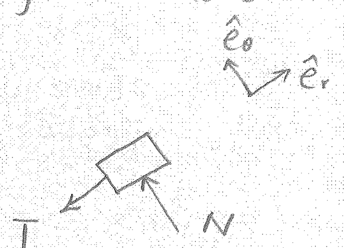
$$\Rightarrow \quad \boxed{d-c = \mu h}$$

5) A mass m slides in a frictionless slot in a turntable that turns at constant rate $\omega$. The spring has no tension when $r=r_0$. Find the general solution

$r(t) = ?$ (in terms of $m, k, \omega, r_0$ & any free constants you need)




<u>FBD</u> of the mass




T: spring tension

N: reaction force from the slot wall

At time t, we use basis vectors $\hat{e}_r$, $\hat{e}_\theta$ shown in the graph.

Acceleration of the mass is

$$\vec{a} = \ddot{r}\,\hat{e}_r - \omega^2 r\,\hat{e}_r + 2\omega\dot{r}\,\hat{e}_\theta$$

<u>LMB</u> $\Rightarrow$

$$\{ -T\hat{e}_r + N\hat{e}_\theta = m\ddot{r}\,\hat{e}_r - m\omega^2 r\,\hat{e}_r + 2m\omega\dot{r}\,\hat{e}_\theta \} \quad ①$$

① $\cdot\ \hat{e}_r \Rightarrow \quad -T = m\ddot{r} - m\omega^2 r$

The spring tension $\quad T = k(r - r_0)$

$$\Rightarrow \quad m\ddot{r} - m\omega^2 r = -k(r - r_0)$$

$$\Rightarrow \quad \boxed{\ddot{r} + \left(\frac{k}{m} - \omega^2\right) r = \frac{k}{m} r_0} \qquad ②$$

This equation shows $r(t)$ is a simple harmonic oscillating function.

solution :

$$r(t) = r_p(t) + r_H(t)$$

where $r_p(t)$ is a particular solution to ②

$r_H(t)$ is the general solution to the homogeneous equation $\quad \ddot{r} + \left(\frac{k}{m} - \omega^2\right) r = 0$

$$r_p(t) = \frac{\frac{k}{m} r_0}{\frac{k}{m} - \omega^2} = \frac{k r_0}{k - m\omega^2}$$

Since $\omega < \sqrt{\frac{k}{m}}$, $\quad \frac{k}{m} - \omega^2 > 0$

$$r_H(t) = C_1 \cos\left(\sqrt{\frac{k}{m} - \omega^2}\ t\right) + C_2 \sin\left(\sqrt{\frac{k}{m} - \omega^2}\ t\right)$$

$\therefore$ The general solution of $r(t)$ is

$$r(t) = C_1 \cos\left(\sqrt{\frac{k}{m} - \omega^2}\ t\right) + C_2 \sin\left(\sqrt{\frac{k}{m} - \omega^2}\ t\right) + \frac{k r_0}{k - m\omega^2}$$

$$\text{or} \quad = C_1 \cos\left(\sqrt{\frac{k}{m} - \omega^2}\ t + \phi\right) + \frac{k r_0}{k - m\omega^2}$$

where $C_1, C_2$ or $C_1, \phi$ are determined by initial conditions