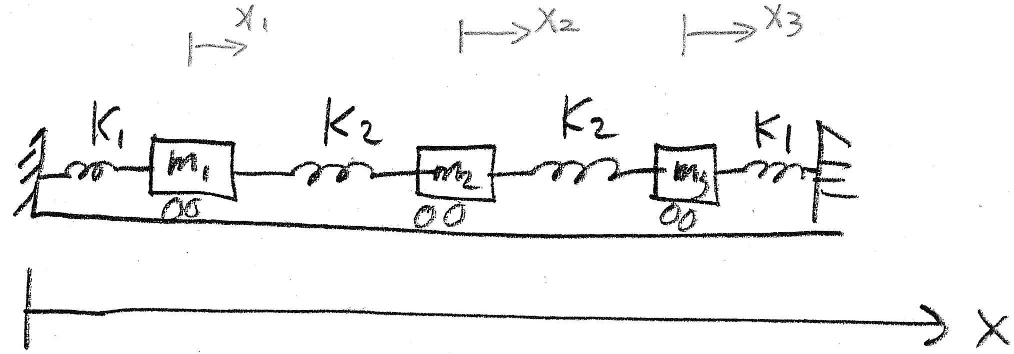
1) Given $m_1 = m_2 = m_3 = m$ and $K_2 = 2K$, $K_1 = K$

All springs are relaxed when $x_1 = x_2 = x_3 = 0$.

a) Given $x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, k$ & m;
$\ddot{x}_2 = ?$

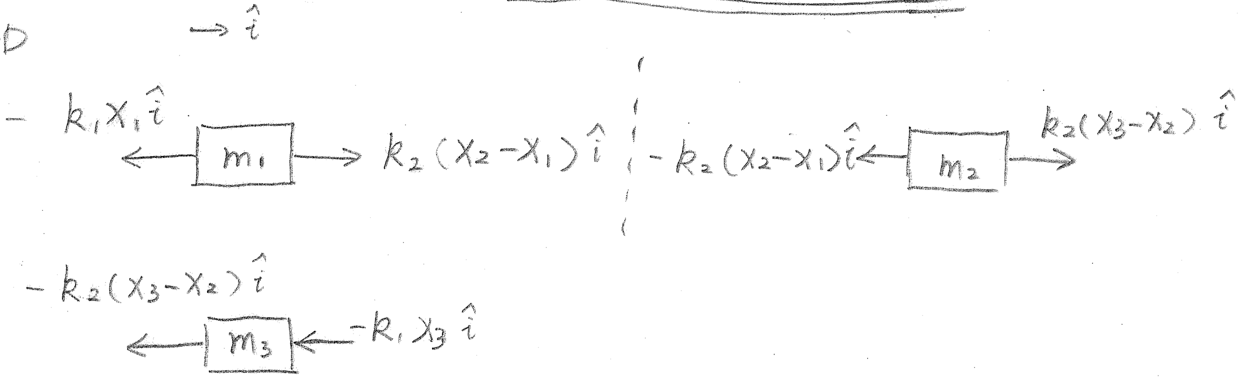
b) Find $x_1(t)$, $x_2(t)$, $x_3(t)$ for any (of your choosing) motion of the system besides $x_1 = x_2 = x_3 = 0$.



Solution:

a). First, draw Free Body Diagrams

FBD



Appy LMB to m_2:

$$\Sigma \vec{F} = m\vec{a} \Rightarrow -k_2(x_2 - x_1)\hat{i} + k_2(x_3 - x_2)\hat{i} = m_2 \ddot{x}_2 \hat{i} \quad \text{——— (1)}$$

$$(1) \cdot \hat{i} \Rightarrow m_2 \ddot{x}_2 = -k_2(x_2 - x_1) + k_2(x_3 - x_2)$$

$$= k_2 x_1 - 2k_2 x_2 + k_2 x_3$$

$$m_2 = m,\ k_2 = 2k \Rightarrow \boxed{\ddot{x}_2 = \frac{2k}{m} x_1 - \frac{4k}{m} x_2 + \frac{2k}{m} x_3}$$

b). Version 1:

1) $\boxed{\text{Page 2}}$

Apply LMB to m_1, m_2, m_3 and we get

m_1:

$$m_1\ddot{x}_1 = -(k_1+k_2)x_1 + k_2x_2 \Rightarrow \boxed{\ddot{x}_1 + \frac{3k}{m}x_1 - \frac{2k}{m}x_2 = 0} \quad ②$$

m_2:

$$m_2\ddot{x}_2 = k_2x_1 - 2k_2x_2 + k_2x_3 \Rightarrow \boxed{\ddot{x}_2 - \frac{2k}{m}x_1 + \frac{4k}{m}x_2 - \frac{2k}{m}x_3 = 0} \quad ③$$

m_3:

$$m_3\ddot{x}_3 = k_2x_2 - (k_2+k_1)x_3 \Rightarrow \boxed{\ddot{x}_3 - \frac{2k}{m}x_2 + \frac{3k}{m}x_3 = 0} \quad ④$$

Because of symmetry, we make a guess that $\boxed{x_2(t)=0}$ may be one solution.

Then

$$\left.\begin{aligned} ② &\Rightarrow \ddot{x}_1 + \frac{3k}{m}x_1 = 0 \\ ③ &\Rightarrow 0 - \frac{2k}{m}x_1 + 0 - \frac{2k}{m}x_3 = 0 \\ &\qquad \Rightarrow x_1 = -x_3 \\ ④ &\Rightarrow \ddot{x}_3 + \frac{3k}{m}x_3 = 0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x_1 &= A\cos\sqrt{\tfrac{3k}{m}}t + B\sin\sqrt{\tfrac{3k}{m}}t \\ x_3 &= -A\cos\sqrt{\tfrac{3k}{m}}t - B\sin\sqrt{\tfrac{3k}{m}}t \end{aligned}}$$

Since we are only looking for a special motion, we can

Let $B=0$, $A=1$

$$\therefore \boxed{x_1(t) = \cos\sqrt{\frac{3k}{m}}t, \quad x_2(t)=0, \quad x_3(t) = -\cos\sqrt{\frac{3k}{m}}t}$$

is one possible motion

The corresponding initial conditions are

$x_1|_{t=0} = 1$; $x_2|_{t=0} = 0$; $x_3|_{t=0} = -1$

$\dot{x}_1|_{t=0} = 0$; $\dot{x}_2|_{t=0} = 0$; $\dot{x}_3|_{t=0} = 0$

b) Version 2 (optional, just for your information)

Details on how to find the general solution

1) $\boxed{\text{Page 3}}$

Apply LMB to the three masses: m_1, m_2, m_3. The resulting equations are summarized below:

m_1:

$$m_1 \ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2 \Rightarrow \boxed{\ddot{x}_1 + \frac{3k}{m}x_1 - \frac{2k}{m}x_2 = 0} \quad \cdots (2)$$

m_2:

$$m_2 \ddot{x}_2 = k_2 x_1 - 2k_2 x_2 + k_2 x_3 \Rightarrow \boxed{\ddot{x}_2 - \frac{2k}{m}x_1 + \frac{4k}{m}x_2 - \frac{2k}{m}x_3 = 0} \quad \cdots (3)$$

m_3:

$$m_3 \ddot{x}_3 = k_2 x_2 - (k_2+k_1)x_3 \Rightarrow \boxed{\ddot{x}_3 - \frac{2k}{m}x_2 + \frac{3k}{m}x_3 = 0} \quad \cdots (4)$$

Write (2), (3), (4) in vector form:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{k}{m}\begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \cdots (5)$$

Assume a normal mode motion of m_1, m_2, m_3,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} A\cos(\omega t + \phi)$$

Phase, determined by IC's

Normal frequency

Normal mode Vector

Then

$$\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \left[-A\omega^2\cos(\omega t + \phi)\right]$$

∴ Equation ⑤ becomes

$$\underbrace{-A\omega^2 \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} A\cos(\omega t+\phi)}_{[\ddot{X}]} + \frac{k}{m}\begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} A\cos(\omega t+\phi)}_{[X]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This equation should be true for every t (time). The only chance is

$$-\omega^2 \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \underbrace{\frac{k}{m}\begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}}_{\text{matrix } [B]} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{\text{vector } V} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$[B][V] = \omega^2 [V] \qquad ⑥$$

⑥ shows $[V]$ is an eigenvector of $[B]$ coresponding to the eigenvalue ω^2.

It turns out (using MATH 294). there are three sets of eigenvalues and eigenvectors.

$$\omega_A = \sqrt{\frac{3k}{m}}, \qquad [V]_A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\omega_B = \sqrt{\frac{(7+\sqrt{33})k}{2m}}, \qquad [V]_B = \begin{bmatrix} 1 \\ -\frac{1+\sqrt{33}}{4} \\ 1 \end{bmatrix}$$

$$\omega_C = \sqrt{\frac{(7-\sqrt{33})k}{2m}}, \qquad [V]_C = \begin{bmatrix} 1 \\ -\frac{1-\sqrt{33}}{4} \\ 1 \end{bmatrix}$$

Therefore, there are 3 normal modes

$$[X]_A = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} A \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_A\right) \quad \cdots ⑦$$

$$[X]_B = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1+\sqrt{33}}{4} \\ 1 \end{bmatrix} B \cos\left(\sqrt{\frac{(7+\sqrt{33})k}{2m}}\,t + \phi_B\right) \quad \cdots ⑧$$

$$[X]_C = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1-\sqrt{33}}{4} \\ 1 \end{bmatrix} C \cos\left(\sqrt{\frac{(7-\sqrt{33})k}{2m}}\,t + \phi_C\right) \quad \cdots ⑨$$

The general solution is a superposition of ⑦, ⑧ & ⑨

$$\boxed{\begin{aligned} x_1(t) &= A\cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_A\right) + B\cos\left(\sqrt{\frac{(7+\sqrt{33})k}{2m}}\,t + \phi_B\right) + C\cos\left(\sqrt{\frac{(7-\sqrt{33})k}{2m}}\,t + \phi_C\right) \\ x_2(t) &= -\frac{1+\sqrt{33}}{4} B\cos\left(\sqrt{\frac{(7+\sqrt{33})k}{2m}}\,t + \phi_B\right) - \frac{1-\sqrt{33}}{4} C\cos\left(\sqrt{\frac{(7-\sqrt{33})k}{m}}\,t + \phi_C\right) \\ x_3(t) &= -A\cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_A\right) + B\cos\left(\sqrt{\frac{(7+\sqrt{33})k}{2m}}\,t + \phi_B\right) + C\cos\left(\sqrt{\frac{(7-\sqrt{33})k}{2m}}\,t + \phi_C\right) \end{aligned}}$$

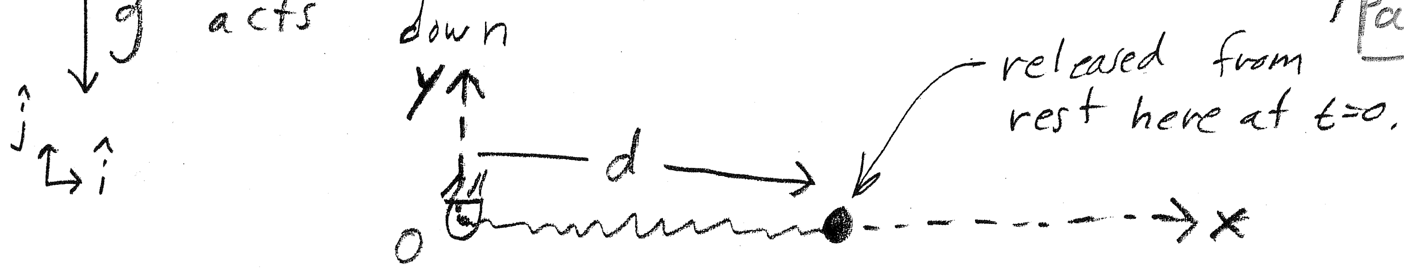
where A, B, C, ϕ_A, ϕ_B, ϕ_C are determined by initial conditions.

You answer must be a special case of the general solution by choosing some A, B, C, ϕ_A, ϕ_B, ϕ_C.

E.g., if you choose $B = C = 0$, $\phi_A = 0$, $A = 1$, your answer is
$x_1(t) = \cos\left(\sqrt{\frac{3k}{m}}\,t\right)$, $x_2(t) = 0$, $x_3(t) = -\cos\left(\sqrt{\frac{3k}{m}}\,t\right)$

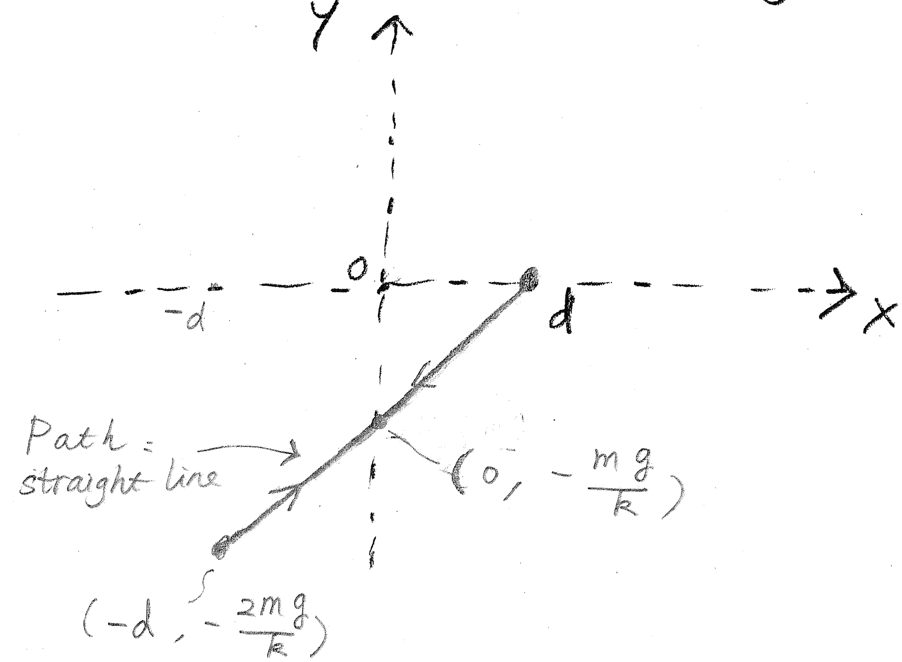
d) A mass m hangs from a zero-rest-length spring following $T = k\ell$ ($\ell_0 = 0$). Gravity g acts down

$\boxed{\text{Page 1}}$



a) Find $\vec{r}$ in terms of some or all $m, g, k, d, \hat{\imath}, \hat{\jmath}, t$

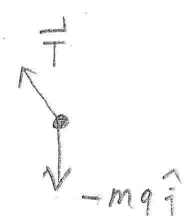
b) Draw the particle path neatly, labeling key pts.



Solution:

(a.) First, draw FBD of the particle when it is at $\vec{r} = x\hat{\imath} + y\hat{\jmath}$.

FBD:



The spring force vector

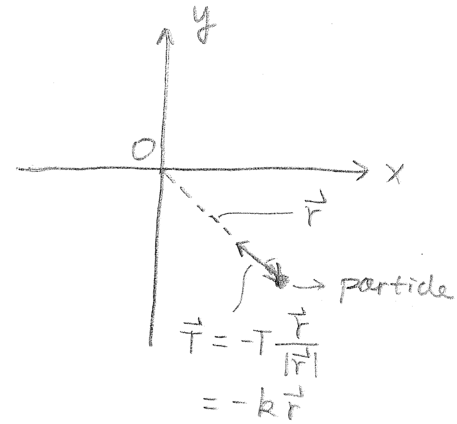
$$\vec{T} = -T\frac{\vec{r}}{|\vec{r}|}$$ since it points the origin

$$T = k\underbrace{\ell}_{\text{strech of the spring}} = k|\vec{r}|$$

2) Page 2

$\therefore \quad \vec{T} = -k|\vec{r}|\frac{\vec{r}}{|\vec{r}|} = -k\vec{r}$

$= -k(x\hat{i} + y\hat{j})$



Apply LMB to the particle

$$\Sigma \vec{F} = m\vec{a}$$

$$\Rightarrow \quad \vec{T} - mg\hat{j} = m\ddot{\vec{r}} \quad \Rightarrow \quad \boxed{-k(x\hat{i}+y\hat{j}) - mg\hat{j} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})} \quad \cdots (1)$$

$$(1)\cdot\hat{i} \Rightarrow \quad m\ddot{x} = -kx$$

$$(1)\cdot\hat{j} \Rightarrow \quad m\ddot{y} = -ky - mg$$

$$\therefore \quad \boxed{\begin{aligned} &\ddot{x} + \frac{k}{m}x = 0 \\ &\ddot{y} + \frac{k}{m}y + g = 0 \end{aligned}} \quad \cdots (2)$$

The general solution to (2) is

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$y(t) = C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right) - \frac{mg}{k}$$

A, B, C, D are constants determined by inititial conditions.

At $t=0$,

$$x|_{t=0} = d, \qquad \dot{x}|_{t=0} = 0$$

$$y|_{t=0} = 0, \qquad \dot{y}|_{t=0} = 0$$

So according to initial conditions,

$$A = d, \quad B = 0, \quad C = \frac{mg}{k}, \quad D = 0$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= d\cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ y(t) &= \frac{mg}{k}\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{mg}{k} \end{aligned}} \quad (3)$$

$$\therefore \boxed{\begin{aligned} \vec{r} &= x\,\hat{i} + y\,\hat{j} \\ &= d\cos\left(\sqrt{\frac{k}{m}}\,t\right)\hat{i} + \left[\frac{mg}{k}\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{mg}{k}\right]\hat{j} \end{aligned}}$$

b). From (3), if we eliminate $\cos\left(\sqrt{\frac{k}{m}}\,t\right)$, i.e. substitute

$\cos\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{x}{d}$ into the expression of $y(t)$, we get

$\boxed{y = \frac{mg}{kd}x - \frac{mg}{k}}$ is an equation for LINE!

So the path is a straight line, intersecting y axis

at $-\frac{mg}{k}$.

Note $x = d\cos\left(\sqrt{\frac{k}{m}}\,t\right)$,

$\therefore -d \le x \le d$

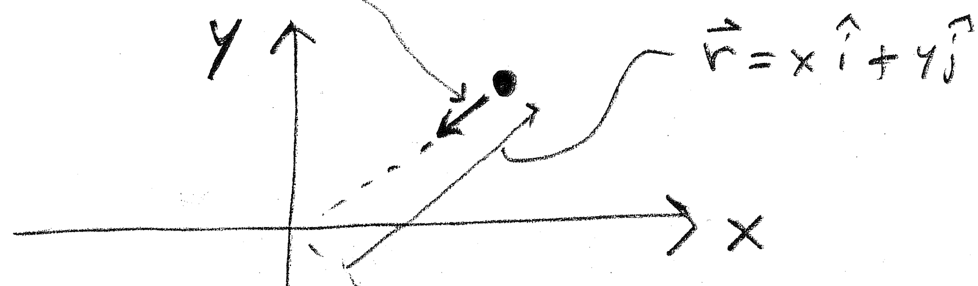
So the line is bounded in $x \in [-d, d]$.

The path is shown on the previous page.

5) A particle of mass m is attracted towards the origin with a central quadratic force

3) $\boxed{\text{Page 1}}$

$$F = k r^2$$



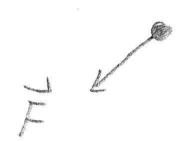
a) Write the equations of motion for this particle in first order form.

b) Evaluate, by any means, $\frac{d}{dt}(\vec{r} \times \dot{\vec{r}})$ in terms of (some or all of) x, y, k, m.

c) Does conservation of energy apply to this system? If so what, in terms of $x, y, \dot{x}, \dot{y}$, k & m is conserved?

Solution

a). Draw FBD of the particle



$$(\vec{r} = x\hat{i} + y\hat{j})$$

$$\vec{F} = -kr^2 \frac{\vec{r}}{|\vec{r}|} = -kr^2 \frac{\vec{r}}{r} = -kr\vec{r}$$

$$= -k\sqrt{x^2+y^2}\, x\,\hat{i} - k\sqrt{x^2+y^2}\, y\,\hat{j}$$

Apply LMB to the particle.

3) $\boxed{\text{Page 2}}$

$$\sum \vec{F} = m\vec{a} \Rightarrow \vec{F} = m\ddot{\vec{r}}$$

$$\Rightarrow -kr\vec{r} = m\ddot{\vec{r}}$$

$$\text{or} \quad \boxed{\left(-k\sqrt{x^2+y^2}\, x\hat{i} - k\sqrt{x^2+y^2}\, y\hat{j}\right) = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})} \quad ①$$

$$① \cdot \hat{i} \Rightarrow m\ddot{x} = -kx\sqrt{x^2+y^2}$$

$$② \cdot \hat{j} \Rightarrow m\ddot{y} = -ky\sqrt{x^2+y^2}$$

$$\Rightarrow \quad \ddot{x} = -\frac{k}{m}x\sqrt{x^2+y^2}, \quad \ddot{y} = -\frac{k}{m}y\sqrt{x^2+y^2}$$

Let $V_x = \dot{x}$, $V_y = \dot{y}$

$\therefore$ $\dot{V}_x = \ddot{x}$, $\dot{V}_y = \ddot{y}$

$\therefore$ The equations of motion in first order form are

$$\boxed{\begin{aligned} \dot{x} &= V_x \\ \dot{V}_x &= -\frac{k}{m}x\sqrt{x^2+y^2} \\ \dot{y} &= V_y \\ \dot{V}_y &= -\frac{k}{m}y\sqrt{x^2+y^2} \end{aligned}}$$

or you can use the vector form

$$\boxed{\begin{aligned} \dot{\vec{r}} &= \vec{V} \\ \dot{\vec{V}} &= -\frac{k}{m}|\vec{r}|\,\vec{r} \end{aligned}}$$

b). $\frac{d}{dt}(\vec{r} \times \dot{\vec{r}})$ (product rule for cross product)

$$= \cancel{\dot{\vec{r}} \times \dot{\vec{r}}} + \vec{r} \times \ddot{\vec{r}}$$

$\vec{0}$

3) Page 3

use LMB, $m\ddot{\vec{r}} = m\underbrace{\vec{a}}_{\text{acceleration}} = \vec{F} = -kr\vec{r}$

$\therefore \ddot{\vec{r}} = -\frac{k}{m} r \vec{r}$

$\therefore \vec{r} \times \ddot{\vec{r}} = \vec{r} \times (-\frac{k}{m} r \vec{r}) = -\frac{k}{m} r (\cancel{\vec{r} \times \vec{r}})^{\vec{0}}$

$\therefore \boxed{\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \vec{0}}$ (This is true at any instance of time)

c). Conservation of energy applies to this system.

Reason: All forces in this system are conservative.

In this system, there is only one force $\vec{F} = -k\sqrt{x^2+y^2}\,x\hat{i} - k\sqrt{x^2+y^2}\,y\hat{j}$.

Since $\vec{\nabla} \times \vec{F} = \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{k}$ ↳ unit vector in z direction

$$= -\frac{kxy}{\sqrt{x^2+y^2}} - \left(-\frac{kxy}{\sqrt{x^2+y^2}}\right) = 0$$

$\vec{F}$ is conservative.

$E_k + E_p$ should be a constant during the motion

E_k: kinetic energy of the particle

E_p: potential energy of $\vec{F}$

$$E_K = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

For E_p, choose origin as the zero - pontential energy point.

E_p at a point $\vec{r}_0$ can be calculated as

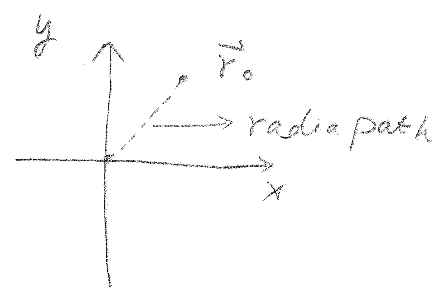
$$E_p(\vec{r}_0) = -\int_0^{\vec{r}_0} \vec{F} \cdot d\vec{r}$$

$$= -\int_0^{\vec{r}_0} (-kr\vec{r}) \cdot d\vec{r}$$

$$= \int_0^{\vec{r}_0} kr\vec{r} \cdot d\vec{r}$$

Remeber, since $\vec{F}$ is conservative, this integral is path independent. For simplicity, we pick the radial path, along which $d\vec{r} = \frac{\vec{r}_0}{|\vec{r}_0|} dr$, and $\vec{r} = r \frac{\vec{r}_0}{|\vec{r}_0|}$

$$\therefore E_p(\vec{r}_0) = \int_0^{r_0} kr^2 \underbrace{\cancel{\frac{\vec{r}_0}{|\vec{r}_0|} \cdot \frac{\vec{r}_0}{|\vec{r}_0|}}}_{1} dr$$



$$= \int_0^{r_0} kr^2 dr$$

$$= \frac{k}{3} r_0^3$$

$\therefore E_p$ at a point $\vec{r} = x\hat{i} + y\hat{j}$ is

$$E_p = \frac{k}{3}\left(\sqrt{x^2+y^2}\right)^3$$

$$\therefore \boxed{E_K + E_p = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{k}{3}\left(\sqrt{x^2+y^2}\right)^3 \text{ is conserved.}}$$