‘Singularity’ functions
(Notes from office hours 12/1/2010 . . . and then some. Someone in office
hours asked me to write this up and send it out, so here you go.)

-Andy Ruina

Motivation. For loads $w(x)$ that are not smooth, singularity functions
make it easier to find the shear force $V$, bending moment $M$, slope $u'$ and dis-
placement $u$. To see singularity functions in use, see the solutions to Problem
15.27.

Context
In general if you know the downwards load per unit length $w(x)$ on a beam
you can integrate 4 times to find

$$
V(x) = -\int w(x) \, dx + C_1
$$

$$
E I u'' = M(x) = \int V(x) \, dx + C_2,
$$

$$
u' = \int u'' \, dx + C_3 \quad \text{and}
$$

$$
u = \int u' \, dx + C_4.
$$

These apply for any distributed loading $w(x)$.

But what kind of function $w(x)$ do you use for a concentrated load $P$;
or a reaction force $F$; or for a load that is constant in some regions and zero
in others; or when there is an applied couple? What are the functions $w(x)$
for such loads and how do you integrate them? The answers are: singularity
functions.

Without singularity functions you have to find separate expressions for
$V, M, \text{etc.}$ for the regions to the left and to the right of such discontinuities.
And then, as described in most books about beams, you have to pick inte-
gration constants in the two regions so that there is an appropriate jump, or
not of the $V, M, u'$ or $u$. Using singularity functions you can skip all this
matching. You just follow the integration rules and the right quantities jump
up and down the right ways.

Delta function and step function

Dirac delta function. The most famous\footnote{Actually the most famous singular-
ity function, the delta function, isn’t the most singular one, at least not in the
mathematical sense of the word ‘singular’.} singularity function is called the
Dirac delta function, or the ‘impulse’ function. If it applies at $a$ it is written:

$$
\delta(x - a) = \text{‘delta of x minus a’}.
$$

What function is it? In the classical mathematics sense the delta function
$\delta(x - a)$ isn’t a function. Mathematicians were upset about this for a while.
There are lots of ways to think about this function (See fig. 0.1). One way (that any real mathematician would hate) is to think of $\delta(x-a)$ as a function of $x$ that is zero except for very close to $x = a$. There, it is as tall as it is narrow. So the area underneath is one. Actually $\delta(x-a)$ is infinitely tall and infinitely narrow, but the area underneath is still one.

**Concentrated load $P$.** You can replace a downwards force $P$ with a very large force per unit length $w(x)$ acting on a small region near $x = a$ but that has total force $P$:

$$\int_{\text{left of } a}^{\text{right of } a} w(x) \, dx = P.$$ 

But if the concentrated load is very high in value and very narrow in spatial extent, its just like the delta function. Thus for a concentrated load $P$ at $a$ the associated distributed load $w$ is

$$w(x) = P \delta(x-a).$$

**Heaviside step function.** There are two primary ways to think of the Heaviside step function:

1. The step function is the integral of the delta function, informally:

$$\int \delta(x-a) \, dx = H(x-a)$$

or more formally

$$\int_{-\infty}^{x} \delta(x'-a) \, dx' = H(x-a).$$

That is, the Heaviside step function is the cumulative area under the delta function curve.

2. The Heaviside step function is that function of $x$ that is zero to the left of $a$ and one to the right of $a$,

$$H(x-a) \equiv \begin{cases} 
0 & \text{if } x < a, \\
1 & \text{if } x \geq a.
\end{cases}$$

This is pictured in the first of fig. 0.2.

**Shear $V$ for a concentrated load.** If there is various loading on a beam, part of which is a concentrated load $P$ at $a$ then the whole $w(x)$ function is several terms of which we only write out the key term $P\delta(x-a)$.

$$w(x) = \ldots P \delta(x-a) \ldots .$$
So, integrating,

\[ V(x) = - \int w(x) \, dx \]

\[ = - \int \ldots \frac{P \delta(x-a)}{\ldots} \, dx \]

\[ = \ldots -P \text{H}(x-a) \ldots. \]

\[ -P \text{ times the step function} \]

If there is a concentrated load \( P \) at \( a \) then the shear force function \( V(x) \) has a step down of size \( P \) at \( a \).

**The ramp function.** See fig. 0.3. This one is not so famous\(^\circ\). Think of the ramp function as \((x-a)\) but for that its set to zero if \( x < a \). So we define

\[ R(x-a) \equiv < x-a > = \begin{cases} (x-a) & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases} \]

\[ \text{Unit ramp function} \]

The brackets <> mean that you should think of the whole expression as being zero if \( x < a \). For \( x > a \) the brackets are like ordinary parentheses (at least for \( n \geq 0 \)). Its easy to see that

\[ \int H(x-a) \, dx = R(x-a) \]

or more formally,

\[ \int_{-\infty}^{x} H(x' - a) \, dx' = R(x-a) \]

**The switched on parabola, etc.** We can keep defining new functions this way with higher and higher powers, e.g.,

\[ < x-a >^2 = \begin{cases} (x-a)^2 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases} \]

and

\[ < x-a >^3 = \begin{cases} (x-a)^3 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases} \]

In the same way that the step function integrates to the ramp we have

\[ \int < x-a >^1 \, dx = < x-a >^2 /2, \quad \text{and} \]

\[ \int < x-a >^2 \, dx = < x-a >^3 /3. \]

**Moment, slope and deflection due to a concentrated load.** A concentrated load \( P \) at \( a \) contributes \( P \delta(x-a) \) to \( w(x) \). We already integrated to

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get shear, now lets keep going,

\[ w(x) = \ldots P \delta(x-a) + \ldots \]
\[ V = - \int w \quad \Rightarrow \quad V(x) = \cdots - PH(x-a) + \cdots \]
\[ M = \int V \quad \Rightarrow \quad EIu''(x) = M(x) = \cdots - PR(x-a) + \cdots \]
\[ u' = \int u' \quad \Rightarrow \quad EIu'(x) = \cdots - P < x-a >^2 /2 + \cdots \]
\[ u = \int u' \quad \Rightarrow \quad EIu(x) = \cdots - P < x-a >^3 /6 + \cdots \]

(2)

Every concentrated load leads to a displacement term that is cubic in \( x \).

**More brackets.** So that we don’t have to remember all those names (Dirac, Heaviside and Macauley) nor the names of their functions (\( \delta, H, R \)) we invent a notation that covers all cases, the last of which we already defined.

\[ < x-a >^{-1} \equiv \delta(x-a), \]
\[ < x-a >^0 \equiv H(x-a), \quad \text{and} \]
\[ < x-a >^1 \equiv R(x-a). \]

We can now write eqn. (2) with this notation.

Given that \( w(x) = \ldots P < x-a >^{-1} + \ldots \)
\[ \Rightarrow \quad V(x) = \cdots - P < x-a >^{-0} + \cdots \]
\[ \Rightarrow \quad \underbrace{M(x)}_{EIu''} = \cdots - P < x-a >^1 + \cdots \]
\[ \Rightarrow \quad EIu'(x) = \cdots - P < x-a >^2 /2 + \cdots \]
\[ \Rightarrow \quad EIu(x) = \cdots - P < x-a >^3 /6 + \cdots \]

(3)

Note, if the superscript \( n \) is positive, think of it as an exponent. It it is negative it is not an exponent. Its just a label marking the degree of singularity: -1 is singular, -2 is more so, etc.

**Applied couples.** If a couple is applied to a beam it is like a two big equal and opposite forces next to each other. You can see this as the derivative of the delta function. For an applied counterclockwise couple \( M \) applied at \( a \) we write

\[ w(x) = M < x-a >^{-2} \]
and use the integration rule \( \int < x - a >^{-2} \, dx = < x - a >^{-1} \). Remember, for negative \( n \) you don’t think of the superscript \( n \) as a power, but just as a label.

**The general case**

Here is the full definition of the singularity functions and the general formulas for their integration.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n &lt; 0 )</td>
<td>( &lt; x - a &gt;^n = \begin{cases} 0 &amp; \text{if } x &lt; a, \ \text{undefined} &amp; \text{if } x = a, \ 0 &amp; \text{if } x &gt; a, \end{cases} )</td>
</tr>
<tr>
<td>( n \geq 0 )</td>
<td>( &lt; x - a &gt;^n = \begin{cases} 0 &amp; \text{if } x &lt; a, \ (x - a)^n &amp; \text{if } x \geq a, \end{cases} )</td>
</tr>
</tbody>
</table>

If \( n \leq 0 \): \( \int < x - a >^n \, dx = < x - a >^{n+1} \)

If \( n \geq 0 \): \( \int < x - a >^n \, dx = < x - a >^{n+1} / (n + 1) \)  

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Figure 0.5: Graphical representation of the singularity functions \( < x - a >^n \) for \(-2 \leq n \leq 3\). The most famous and most used of these, in all manner of scientific and engineering disciplines, are the **Dirac delta function** \( \delta(x-a) = < x - a >^{-1} \) and the **Heaviside step function** \( H(x-a) = < x - a >^0 \). The ramp, quadratic functions and so on are useful for beam problems, but not so useful in the rest of science and engineering.

The derivative of the delta function, \( < x - a >^{-2} \) is mathematically even more singular than the delta function. It has some uses outside of beam deflection problems, but is much more rarely seen than the delta impulse function or the Heaviside step function.