Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

Final exam

May 10, 2013

No calculators, books or notes allowed.

5 Problems, 150 minutes (+no extra time: University policy $\Rightarrow$ *budget your time!*)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 13: ____/25

Problem 14: ____/25

Problem 15: ____/25

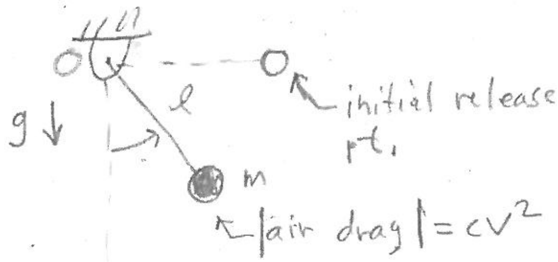
Problem 16: ____/25

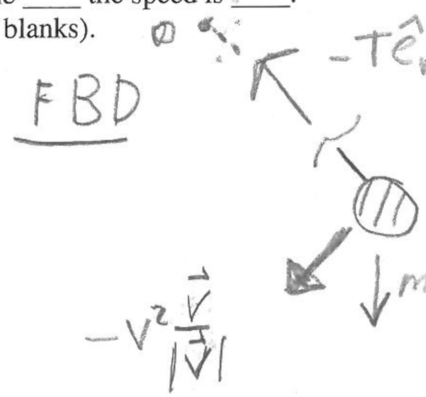
Problem 17: ____/25

13) 2D, with gravity g. A mass m is attached to the end of a negligible-mass rigid rod with a length ℓ. The other end of the rod is attached to a hinge with negligible friction. The mass is slowed by air friction which resists motion with a force with magnitude $|F| = cv^2$ where $v = |\vec{v}|$ is the speed of the mass. The pendulum is released from rest at time $t = 0$ with the rod horizontal and to the right of the hinge. Assume any non-zero positive values that please you for all parameters (*e.g.*, g, m, ℓ, c, and t_1). Do *not* attempt an analytic solution.

a) Using ODE23 or ODE45 write all the Matlab commands needed to find the speed of the mass at time t_1. Some partial credit if you never learned ODE23 or ODE45 and can write your own ODE solver.

b) The output to the command window should be 'At time ____ the speed is ____.' (with numbers, calculated by the computer, instead of blanks).





Drag Force

$$\vec{F}_D = -v^2 \frac{\vec{v}}{|\vec{v}|} c$$

$$\vec{F}_D = -c(\ell\dot{\theta})^2 \frac{\ell\dot{\theta}\hat{e}_\theta}{+\sqrt{(\ell\dot{\theta})^2}}$$

$$\boxed{\vec{F}_D = -c\ell^2\dot{\theta}|\dot{\theta}|\hat{e}_\theta}$$

AMB$_{/0}$:

$$\sum \vec{M}_{/0} = \dot{\vec{H}}_{/0}$$

$$\Rightarrow \ell\hat{e}_r \times (-mg\hat{j} - \ell^2\dot{\theta}|\dot{\theta}|\hat{e}_\theta) = \ell\hat{e}_r \times [-\ell\dot{\theta}^2\hat{e}_r + \ell\ddot{\theta}\hat{e}_\theta]m$$

$$\Rightarrow -\ell mg\sin\theta\hat{k} - c\ell^3\dot{\theta}|\dot{\theta}|\hat{k} = \ell^2 m\ddot{\theta}\hat{k}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \underbrace{\frac{-g}{\ell}\sin\theta - \frac{c\ell}{m}\dot{\theta}|\dot{\theta}|}_{*}}$$

$$\boxed{|\vec{v}| = |\dot{\theta}|\ell}$$

$$\omega \equiv \dot{\theta} \Rightarrow \boxed{\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= *\end{aligned}}$$

```
function givemeahighgrade( )
p.L=1; p.c=1; p.m=1; p.g=1; t1=10;
tspan= [0 t1];
theta0 = pi/2; omega0 = 0;
z0= [theta0 omega0]';

[t zarray]= ode45(@myrhs, tspan, z0,[], p);

vend= abs(p.L * zarray(end, 2));

disp(['At time ' num2str(t1) ...
      ' the speed is ' num2str(vend)])

end


function zdot = myrhs(t, z, p)
theta= z(1); omega= z(2)

thetadot= omega;

omegadot= -p.c *(p.L/p.m) * omega * abs(omega) ...
          - (p.g/p.L) sin(theta);
zdot = [thetadot ; omegadot];
end
```

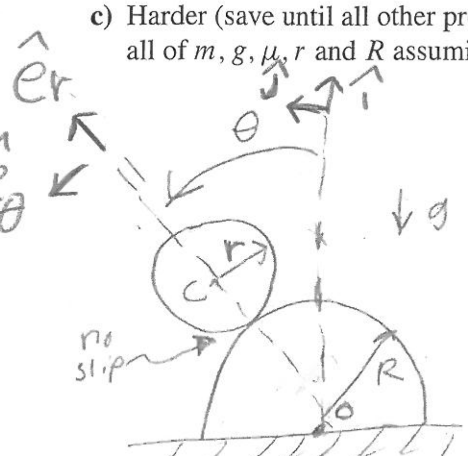
the heart of the matter

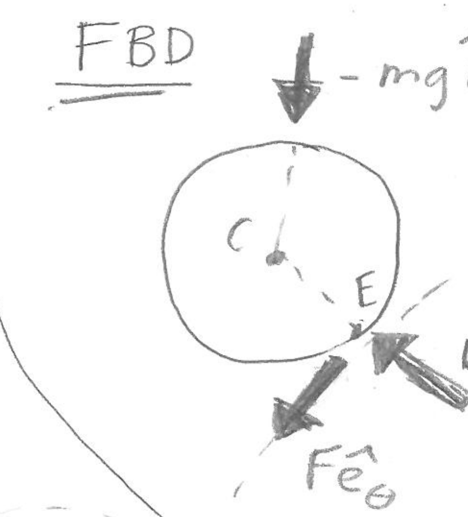
14) 2D, with gravity g. A solid uniform disk with radius r and mass m rolls on the top of a rigid unmoving hollow pipe with radius R. Line OC, between the center of the pipe and the center of the disk makes an angle of θ CCW (counterclockwise) from straight up. Assume θ and $\dot\theta$ are small enough, and μ big enough, so there is no separation or slip.

a) Find the equations of motion (That is, find $\ddot\theta$ in terms of some or all of $\theta, \dot\theta, m, g, r, \mu$ and R).

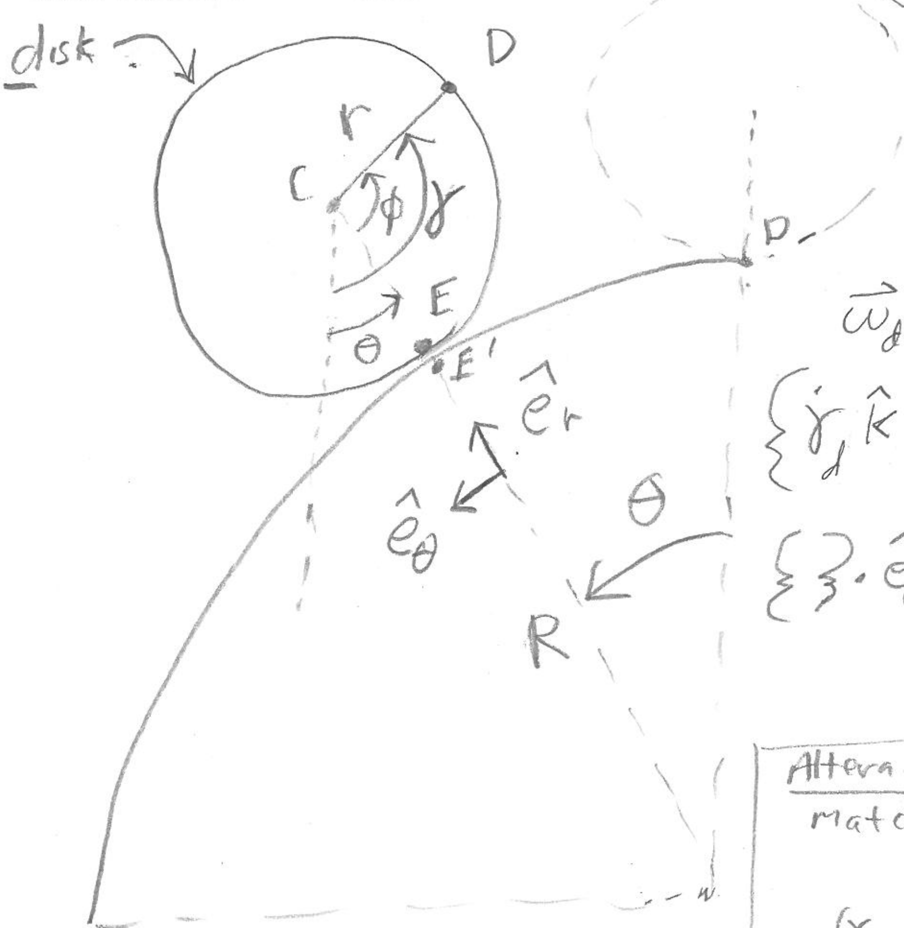
b) Find a function f so that the equation $0 = f(\theta, \dot\theta, m, g, \mu, r, R)$ describes the condition when the wheel would first lose contact.

c) Harder (save until all other problems are done). Find the angle θ when contact is first lost in terms of some or all of m, g, μ, r and R assuming that the rolling starts from rest at $\theta_0 = 0^+$ and all rolling is without slip.





Kinematics



$$\boxed{\vec\omega_d = \dot\gamma \hat k}$$

$$\vec V_E = \vec V_{E'}$$

$$\vec\omega_d \times \vec r_{E/C} + \vec V_C = \vec 0$$

$$\left\{ \dot\gamma_d \hat k \times (-r\hat e_r) + (R+r)\dot\theta \hat e_\theta = \vec 0 \right\}$$

$$\{\}\cdot \hat e_\theta \Rightarrow -\dot\gamma r + (R+r)\dot\theta = 0$$

$$\Rightarrow \boxed{\dot\gamma = \frac{R+r}{r}\dot\theta} *$$

Alternative Kinematics

match arc lengths:

$$\phi r = \theta R$$
$$(\gamma - \theta) r = \theta R$$
$$\gamma r - \theta r = \theta R$$
$$\gamma r = (r+R)\theta$$
$$\boxed{\gamma = \frac{R+r}{r}\theta} *$$

(again)

(14 cont'd)

AMB$_{/E}$:

$$\sum \vec{M}_{/E} = \dot{\vec{H}}_{/E}$$

$$\underbrace{\vec{r}_{c/E}}_{r\hat{e}_r} \times (-mg\hat{\imath}) = \vec{r}_{c/E} \times m\vec{a}_c + I\dot{\omega}_d\hat{k}$$

$\dot{\omega}_d = \frac{R+r}{r}\ddot{\theta}$ from *

$\vec{a}_c = -\dot{\theta}^2(R+r)\hat{e}_r + \ddot{\theta}(R+r)\hat{e}_\theta$

$$\left\{ mgr\sin\theta\,\hat{k} = \left[mr(R+r)\ddot{\theta} + I\left(\frac{R+r}{r}\right)\ddot{\theta}\right]\hat{k} \right\}$$

$\{\}\cdot\hat{k} \Rightarrow$

$$\ddot{\theta} = \frac{mgr\sin\theta}{mr(R+r) + I\frac{R+r}{r}}$$

$I = mr^2/2$ uniform disk

$$= \frac{g\sin\theta}{(R+r) + (R+r)/2}$$

$$\boxed{\ddot{\theta} = \frac{2g}{3(R+r)}\sin\theta} \quad \boxed{a}$$

LMB $\left\{\sum \vec{F} = \dot{\vec{L}}\right\}\cdot\hat{e}_r$

$\Rightarrow N = -(R+r)\dot{\theta}^2 m + mg\cos\theta$

Liftoff $\Rightarrow N=0$

$\Rightarrow \boxed{(b)} \quad \boxed{\cos\theta = \frac{(R+r)\dot{\theta}^2}{g}}$

C) Cons. of Energy

$E_0 = E_1$

$$(R+r)\cancel{m}g = (R+r)\cancel{m}g\cos\theta + m(R+r)^2\dot{\theta}^2/2 + I\left(\frac{R+r}{r}\dot{\theta}\right)^2/2$$

$I = \frac{1}{2}\cancel{m}r^2$

$$\Rightarrow (R+r)g(1-\cos\theta) = \left[\frac{(R+r)^2}{2} + \frac{(R+r)^2}{4}\right]\dot{\theta}^2$$

$$\Rightarrow \dot{\theta}^2 = \frac{4}{3}\frac{(1-\cos\theta)g}{R+r} \quad (**)$$

apply ** to b $\Rightarrow$

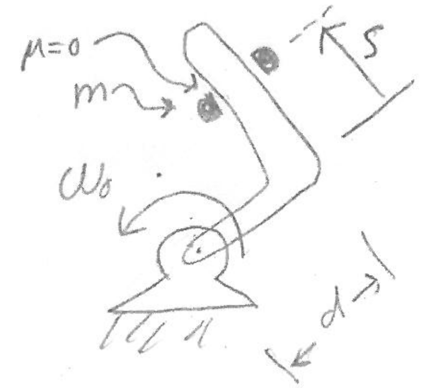
$\cos\theta = \frac{4}{3}(1-\cos\theta)$

$\Rightarrow \frac{7}{3}\cos\theta = \frac{4}{3} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{4}{7}\right)} \quad \boxed{c}$

15) 2D, no gravity. A bead with mass m slides on an L-shaped (right angle at bend) frictionless rigid rod which is turned by a motor at constant angular velocity $\vec{\omega}_0 = \omega_0 \hat{k}$. The bead only moves on the straight part of the bar marked by s. If s passes through zero in your solution, only consider until $s = 0$.

a) Find the equations of motion of the bead. That is, find $\ddot{s}$ in terms of some or all of $\omega_0, s, \dot{s}, m$ and d.

b) Given that $s(0) = s_0 > 0$ and $\dot{s}(0) = 0$, find s in terms of some or all of t, ω_0, m, s_0 and d. (No Matlab).



FBD

Kinematics: $\vec{a}_P = \vec{a}_{O'} + \vec{a}_{P/O'}$

$$= -\omega_0^2 d\hat{e}_r + (\ddot{s} - s\omega_0^2)\hat{e}_\theta + (0 + 2\dot{s}\omega_0)\hat{e}_r$$

LMB: $\sum \vec{F} = m\vec{a}$

$$\{N\hat{e}_r = m[\quad]\}$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow \quad 0 = \ddot{s} - s\omega_0^2 \Rightarrow \boxed{\ddot{s} = s\omega_0^2} \text{ (a)}$$

ODE soln is: $s = A\cosh(\omega_0 t) + B\sinh(\omega_0 t)$

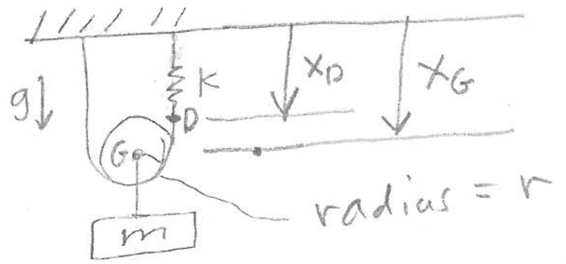
$\dot{s}(0) = 0$

$s(0) = s_0 \Rightarrow \boxed{s = s_0\cosh(\omega_0 t)}$ (b) o

16) 1D with gravity g. A mass m hangs from an ideal round negligible-mass frictionless pulley, an inextinsible strings, and spring k as shown. Give all answers in terms of some or all of m, g and k. As for all problems, clearly define any other variables you may use in your solution.

a) At equilibrium how much lower is the pulley than when there is no mass (but the string and pulley are not slack)?

b) What is the frequency of small oscillation (so small that the strings do not go slack)? You can find ω or f, as you please.



String length $= L_S$
Spring rest length $= L_0$
Spring stretched length $= X_D$
Spring stretch $= X_D - L_0 = \Delta\ell$

Before stretch: $2X_{G0} + \pi r = L_S + L_0$ (1)

After stretch: $2X_G + \pi r = L_S + X_D$ (2)

Subtract (2) − (1) ⇒ $\boxed{2\Delta X_G = \underbrace{X_D - L_0}_{\Delta\ell}}$

FBD: $T \uparrow \ \uparrow T = k(\Delta\ell)$, $\downarrow mg$

LMB: $\sum \vec{F} = m\vec{a}$

$$mg - 2T = m\ddot{X}_G$$

$$T = k\Delta\ell = 2k\,\Delta X_G$$

$$\ddot{X}_G = g - \frac{4k(\Delta X_G)}{m} \quad (1)$$

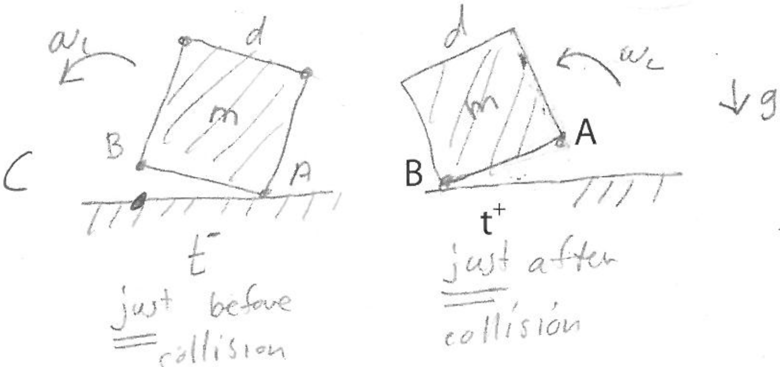
equilib stretch ⇒ $\ddot{X}_G = 0$ ⇒ $\boxed{\Delta X_G = \frac{mg}{4k}}$ (a)

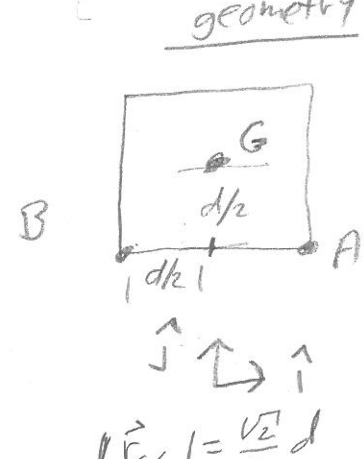
(1) ⇒ $m\ddot{X}_G + 4kX_G = 4kX_{G0}$

$\Rightarrow (X_G - X_{G0}) = A\sin\sqrt{\frac{4k}{m}}\,t + B\sin\sqrt{4k/m}\,t$

$\Rightarrow \boxed{\omega = 2\sqrt{\frac{k}{m}} \qquad f = \frac{1}{\pi}\sqrt{\frac{k}{m}}}$ b

17) 2D, with gravity g. A uniform cube with mass m and side d rocks on edge A and tips until it has a collision with edge B. Then edge A breaks free, and then the cube rocks about edge/hinge B. *Just* before the collision, at $t = t^-$, the angular velocity of the cube is known to be $\omega_1\hat{k}$. Just before and after the collision the tip angles are negligibly small. What is the angular velocity $\omega_2\hat{k}$ *just* after the collision at $t = t^+$? Answer in terms of some or all of m, g, d and ω_1.





AMB$/_C$

$$\vec{H}_C^- = \vec{H}_C^+$$

$$\underbrace{\vec{r}_{G/C}\times m\vec{v}^-}_{=\vec{0}} + I\omega^-\hat{k} = \vec{r}_{G/C}\times m\vec{v}^+ + I\omega^+\hat{k}$$

$\vec{v}^- = \vec{\omega}^-\times\vec{r}_{G/A}$ $\qquad$ $\vec{v}^+ = \vec{\omega}^+\times\vec{r}_{G/C}$

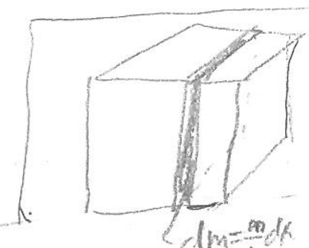
$= \vec{0}$ because $\vec{v}^-$ is $\parallel$ to $\vec{r}_{G/C}$

$$\Rightarrow \left\{ I\omega^-\hat{k} = |\vec{r}_{G/C}|^2 m\,\omega^+\hat{k} + I\omega^+\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad \omega^+ = \frac{I}{I + md^2/2}\,\omega^-$$

$$\omega^+ = \frac{(1/6)md^2}{(1/6)md^2 + \frac{1}{2}md^2}\,\omega^-$$

$$\boxed{\omega^+ = \frac{1}{4}\omega^-}$$



for cube

$I^G = \frac{1}{6}md^2$

Check: $\int r^2 dm = \int x^2 + y^2\, dm$

$= 2\int x^2 dm \qquad dm$

$= 2\int_{-d/2}^{d/2} x^2 \left(\frac{m}{d}\right)dx$

$= 2\,\frac{x^3}{3}\Big|_{-d/2}^{d/2}\,\frac{m}{d}$

$= md^2/6$