Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

Final Exam

May 12, 2011

No calculators, books or notes allowed.

5 Problems, 150 minutes (no extra time)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**
- A+ Be (I) neat, (II) clear and (III) well organized.
- ◻ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
  Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 13: ____ /25

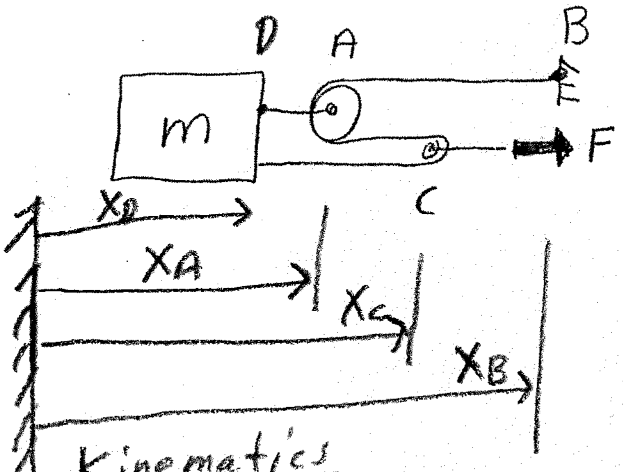
Problem 14: ____ /25

Problem 15: ____ /25

Problem 16: ____ /25

Problem 17: ____ /25

13) Making all the usual assumptions about masses and pulleys, find the acceleration of point C in terms of F and m. Neglect gravity.



Kinematics

L = Rope length = const

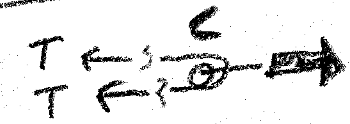
$$= (x_C - x_D) + \pi r_C + (x_C - x_A) + \pi r_A + (x_B - x_A)$$

$$0 = \dot{L} = (\underbrace{\dot{x}_C - \dot{x}_D}_{\dot{x}_D = \dot{x}_A}) + 0 + (\dot{x}_C - \dot{x}_A) + 0 + (\underbrace{\dot{x}_B}_{=0} - \dot{x}_A)$$

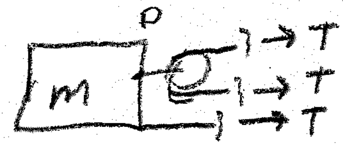
$$\Rightarrow \quad 2\dot{x}_C - 3\dot{x}_A = 0$$

$$\Rightarrow \boxed{\ddot{x}_C = \frac{3}{2}\underbrace{\ddot{x}_A}_{=\ddot{x}_D}} \quad (1)$$

FBDs



$\Rightarrow T = F/2$



$\{LMB\} \cdot \hat{\imath}$

$$\sum F = \dot{L}$$

$$3T = m\ddot{x}_D$$

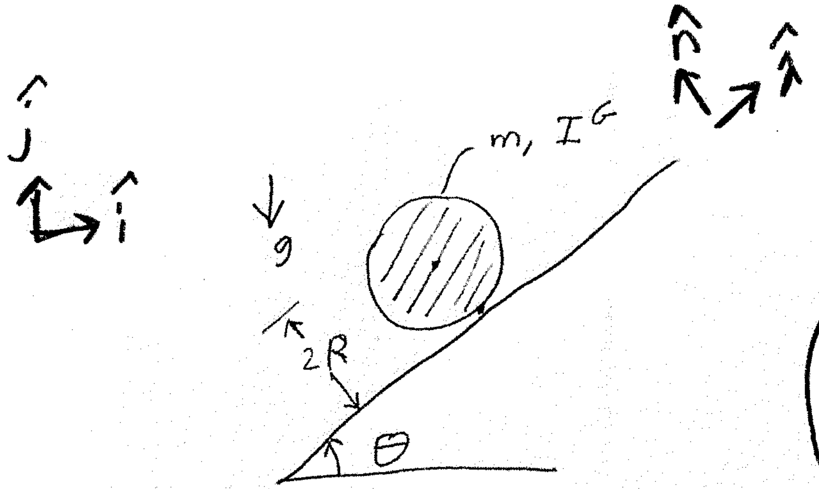
$$\boxed{\ddot{x}_A = 3T/m = 3F/2m} \quad (2)$$

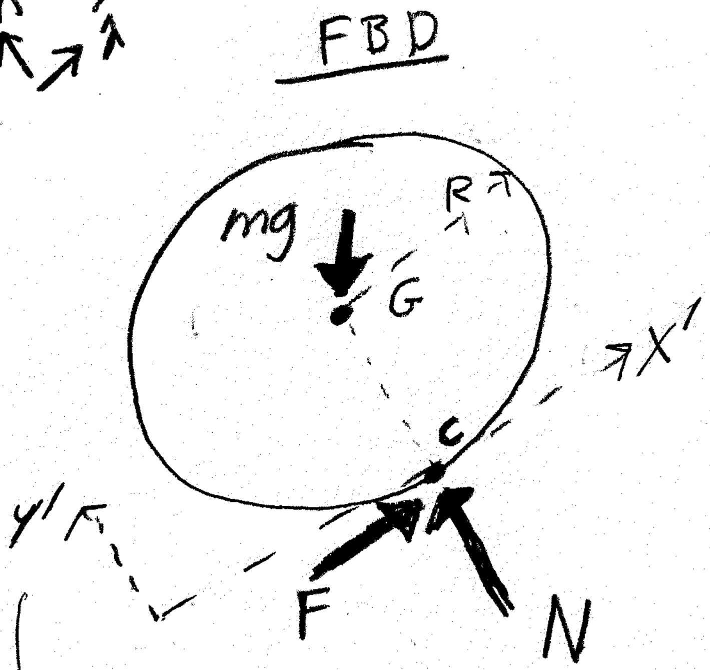
(1) + (2) $\Rightarrow$

$$\ddot{x}_C = \frac{3}{2}\ddot{x}_A$$

$$\Rightarrow \boxed{\ddot{x}_C = \frac{9}{4}\frac{F}{m}}$$

14) A disk rolls down a ramp without slipping. How big does μ have to be in order to prevent slip? (That is, if μ is too small, slip would not successfully be prevented). Answer in terms of some or all of θ, g, R, I^G and m.





Kinematics

$\vec{0} = \vec{V}_c = \dot{x}'\hat{\lambda} + \omega\hat{k}\times \underbrace{\vec{r}_{c/G}}_{-R\hat{n}}$

$\vec{0} = \dot{x}'\hat{\lambda} - \omega R(-\hat{\lambda})$

$\{\vec{0} = (\dot{x}' + \omega R)\hat{\lambda}\}$

$\{\}\cdot\hat{\lambda} \Rightarrow \{\dot{x}' = -\omega R\}$

$\frac{d}{dt}\{\} \Rightarrow \boxed{\ddot{x}' = -\dot{\omega}R}$ ①

LMB

$\sum \vec{F} = \dot{\vec{L}}$

$\{F\hat{\lambda} + N\hat{n} - mg\hat{j} = m\ddot{x}'\hat{\lambda}\}$

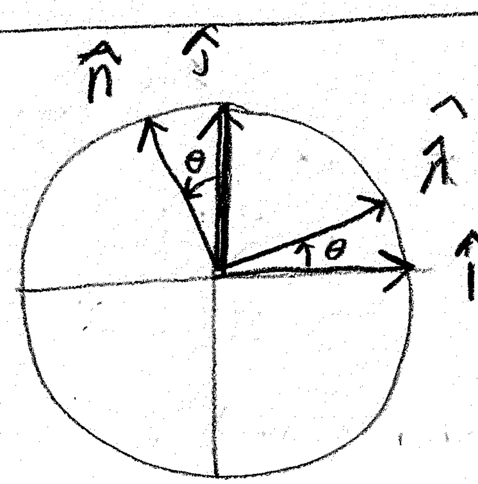
$\{\}\cdot\hat{n} \Rightarrow N - mg\underbrace{\hat{j}\cdot\hat{n}}_{\cos\theta} = 0$

$\boxed{N = mg\cos\theta}$ ②

LMB$\}\cdot\hat{\lambda} \Rightarrow$

$F - mg\underbrace{\hat{j}\cdot\hat{\lambda}}_{+\sin\theta} = m\ddot{x}'$

$\boxed{F = m[\ddot{x}' + g\sin\theta]}$ ③



unit circle

$\underline{AMB}_{/C}$: $\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

$$\underbrace{\vec{r}_{G/C}}_{R\hat{n}} \times (-mg\hat{j}) = \underbrace{\vec{r}_{G/C}}_{R\hat{n}} \times m\underbrace{\vec{a}_c}_{\ddot{x}'\hat{\lambda}} + I^G\dot{\omega}\hat{k}$$

$$\{-Rmg(-\sin\theta\hat{k}) = Rm\ddot{x}'(-\hat{k}) + I^G\dot{\omega}\hat{k}\}$$

$$\{\}\cdot\hat{k} \Rightarrow \sin\theta \cdot Rmg = -Rm\ddot{x}' + I^G\dot{\omega}$$

$$① \Rightarrow \quad \sin\theta \cdot Rmg = R^2 m\dot{\omega} + I^G\dot{\omega}$$

$$\Rightarrow \boxed{\dot{\omega} = \frac{Rmg}{I^G + mR^2}\sin\theta} \quad ④$$

$$\boxed{\ddot{x}' = \frac{-R^2 mg}{I^G + mR^2}\sin\theta} \quad ⑤$$

$$③⑤ \Rightarrow \quad F = m\left[\frac{-R^2 mg}{I^G + mR^2}\sin\theta + g\sin\theta\right]$$

$$= mg\sin\theta\left[1 - \frac{R^2 m}{I^G + mR^2}\right] = mg\sin\theta\frac{I^G}{I^G + mR^2}$$

$$\mu \geq \frac{F}{N} = \frac{\cancel{mg}\sin\theta\frac{I^G}{I^G + mR^2}}{\cancel{mg}\cos\theta}$$

$$\boxed{\mu \geq \frac{1}{1 + mR^2/I^G}\tan\theta}$$

Sanity checks

A) $\theta = 0 \Rightarrow \mu \geq 0$ ✓

B) $\underbrace{I^G = 0}_{F \to 0 \text{ to cause rotation}} \Rightarrow \mu \geq 0$ ✓

Special cases: $\underbrace{I^G = mR^2}_{\text{hoop}} \Rightarrow \mu \geq \frac{\tan\theta}{2}$

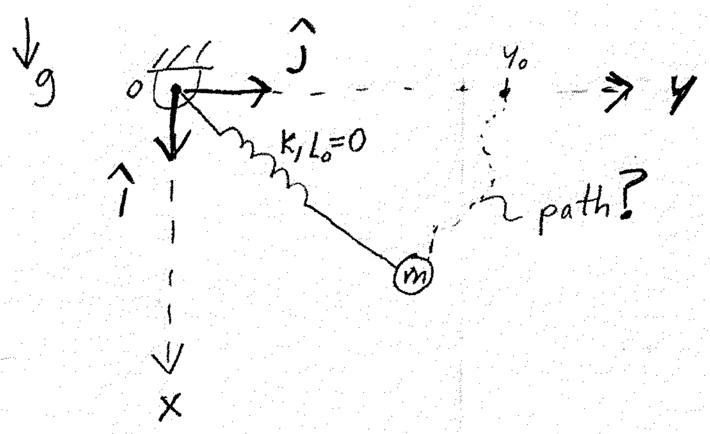
$\underbrace{I^G = mR^2/2}_{\text{uniform disk}} \Rightarrow \mu = \frac{\tan\theta}{3}$

15) A mass m hangs from a spring with constant k and rest length $L_0 = 0$ (the spring is a so-called zero-rest-length spring). The mass is released from rest at the position $\vec{r}_0 = 0\hat{\imath} + y_0\hat{\jmath}$.

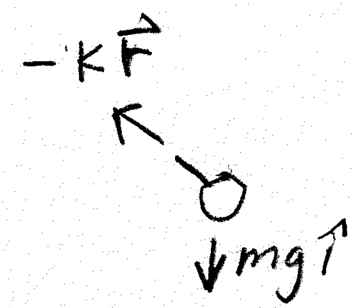
a) Find the position of the mass at time t in terms of some or all of k, m, g and y_0.

b) Draw the trajectory (the path that the mass moves on).

c) In words, describe the shape of the trajectory.



FBD



LMB

$$\sum \vec{F} = m\vec{a}$$

$$mg\hat{\imath} - k\vec{r} = m\ddot{\vec{r}}$$

$$\{ m\ddot{\vec{r}} + k\vec{r} = mg\hat{\imath} \}$$

$$\lambda = \sqrt{k/m}$$

$$\{\ \} \cdot \hat{\imath} \Rightarrow m\ddot{x} + kr = mg \Rightarrow x = A\cos(\lambda t) + B\sin\lambda t + mg/k$$

$$\{\ \} \cdot \hat{\jmath} = m\ddot{y} + ky = 0 \Rightarrow y = C\cos(\lambda t) + D\sin\lambda t$$

$\dot{x}(0) = 0 \Rightarrow B = 0$

"from rest"

$\dot{y}(0) = 0 \Rightarrow D = 0$

$$\Rightarrow x = A\cos\lambda t + mg/k$$

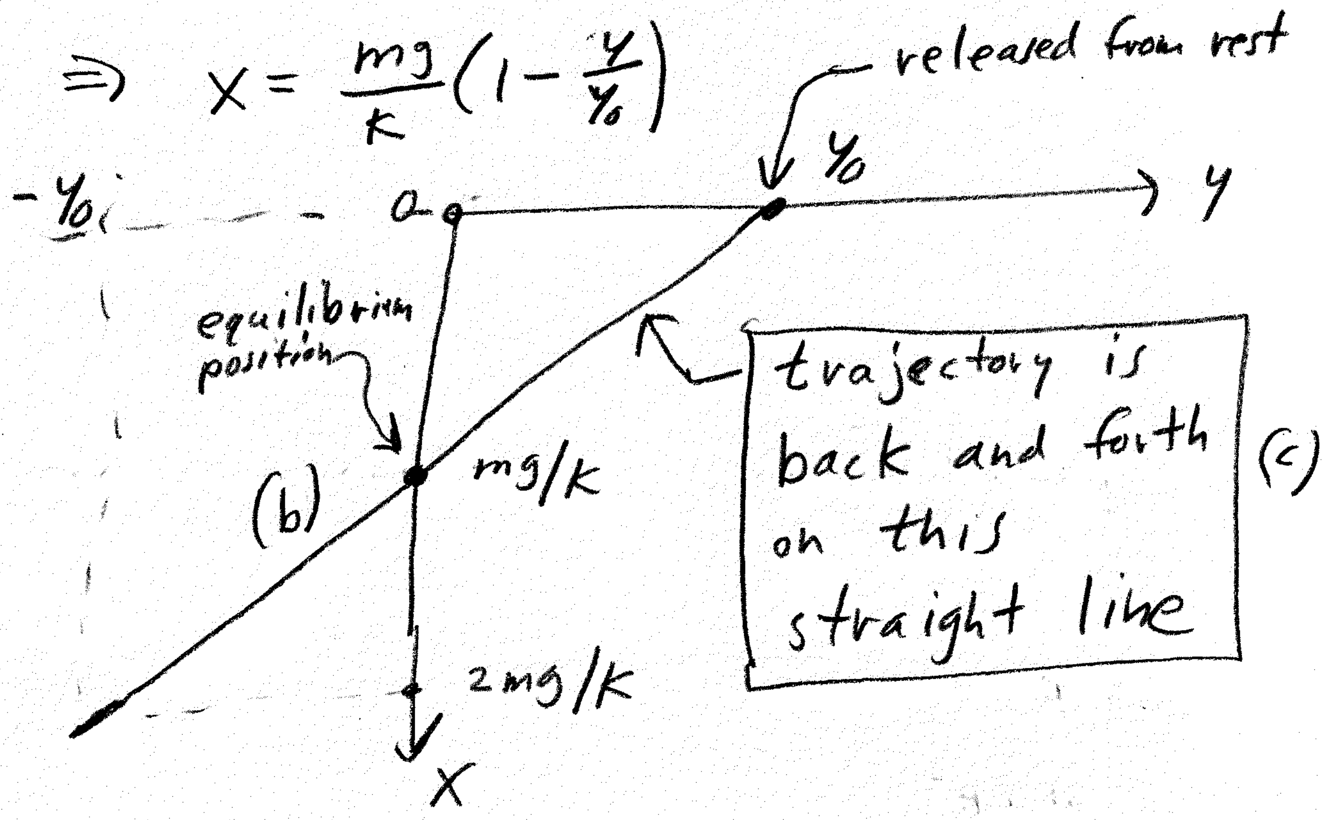
$$x_0 = 0 \Rightarrow A = -mg/k \Rightarrow x = \frac{mg}{k}(1 - \cos\lambda t)$$

$$y(0) = y_0 \Rightarrow C = y_0$$

$$\Rightarrow \qquad y = y_0\cos\lambda t$$

$$\Rightarrow \boxed{\vec{r}(t) = \frac{mg}{k}(1-\cos\lambda t)\,\hat{i} + y_0 \cos \lambda t \qquad \lambda = \sqrt{k/m}} \quad (a)$$

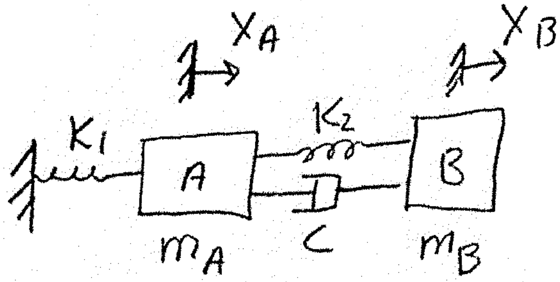
$$\Rightarrow \quad x = \frac{mg}{k}\left(1 - \frac{y}{y_0}\right)$$



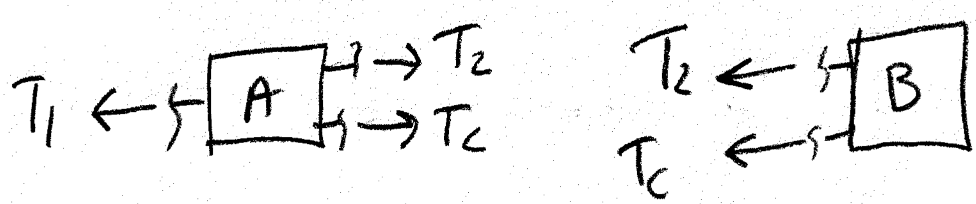
x & y are both simple harmonic motion. In phase with each other.

x is offset by $\frac{\text{weight}}{\text{spring constant}}$.

16) Write MATLAB commands to make a plot of $x_B(t)$. Pick any convenient non-zero values (in consistent units) for any variables or constants.



FBDs



LMB

$m\ddot{x}_A = T_2 + T_c - T_1$ $\qquad$ $m\ddot{x}_B = -T_2 - T_c$

Mat. Properties (Constitutive Laws)

$T_2 = (x_B - x_A) k_2$ $\qquad$ $T_1 = k_1 x_A$

$T_c = (\dot{x}_B - \dot{x}_A) c$

$$Z = \begin{bmatrix} x_A \\ x_B \\ V_A \\ V_B \end{bmatrix}$$

Kinematics

$V_A = \dot{x}_A$

$V_B = \dot{x}_B$

MATLAB CODE

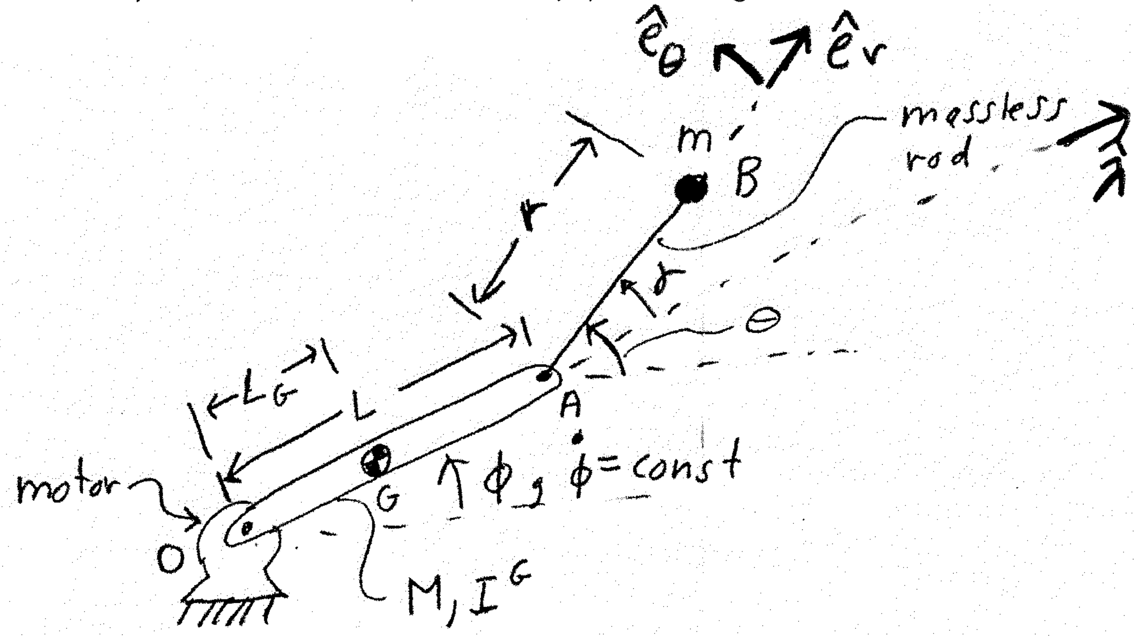
```
function finalexamfun()
XA0=1;  XB0=2;  % nonzero things
VA0=3;  VB0=4;  %      "
z0 = [XA0 XB0 VA0 VB0]';
tspan=[0 10]; % 0≤t≤10
[t zarray] = ode45(@myrhs, tspan, z0)
XB= zarray(:,2); % 2nd column
plot (t,XB); % skip the labels, its an exam
end

function zdot = myrhs(t,z)   m=12;
K1=17; K2 =39; C=pi; % randomish
XA= z(1); XB= z(2); VA= z(3); VB=z(4);
T1= K1*XA; T2= K2*(XB-XA);
 TC = C*(VB-VA);
 XADOT=VA; % kinematics
 XBDOT=VB; %     "
 VADOT=(1/m)*(T2+TC-T1); % F=ma
 VBDOT=(1/m)*(-T2-TC); % F=ma
 zdot=[XADOT XBDOT VADOT VBDOT]'
 end
```

17) A motor at O turns a rigid rod OA (mass M, moment of inertia I^G) at constant angular rate $\dot{\phi}$. A negligible-mass rod with length r is hinged at A and has mass m at its end. Neglect gravity.

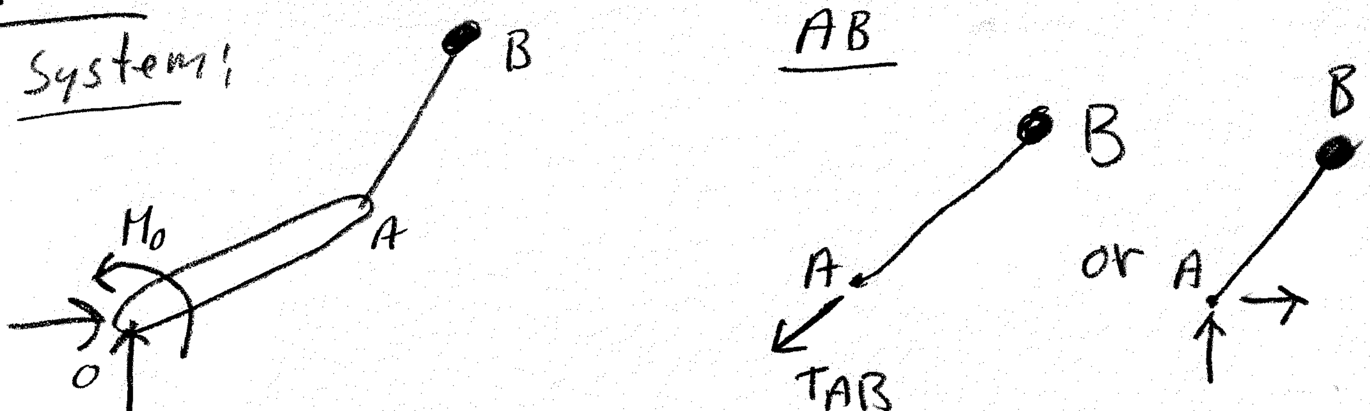
a) Is angular momentum of the system OAB about O constant or not? (Explain your answer.)

b) Consider the special case that $\phi = 0$ and $\dot{\phi} = 0$ (for all time). Find $\ddot{\theta}$ in terms of as many of these terms are needed: $\theta, \dot{\theta}, L, L_G, r, M, m$ and I^G.

c) Now consider non-zero $\dot{\phi}$. Find $\ddot{\theta}$ in terms of some or all of $\phi, \dot{\phi}, \theta, \dot{\theta}, L, L_G, r, M, m$ and I^G.



FBDs



$\Sigma \vec{M}_{/O} = \dot{\vec{H}}_{/O}$

$M_O \hat{k} = \dot{\vec{H}}_{/O} \neq \vec{0}$ in general

Some $M_O \neq 0$ needed to keep $\dot{\phi}$ = const

$\Rightarrow$ $\boxed{\dot{\vec{H}}_{/O} \neq \vec{0}}$ Ang. Mom not conserved, (a)

b) $\vec{a}_A = \vec{0}$

$\underline{AMB_{/A}}$ system $AB \Rightarrow \quad \sum \vec{M}_{/A} = \dot{\vec{H}}_{/A}$

$$\vec{0} = \vec{r}_{B/A} \times m\vec{a}_B$$

$$= m r^2 \ddot{\theta} \hat{k}$$

$\Rightarrow \boxed{\ddot{\theta} = 0}$ B goes in circles at const. rate

c) $\underline{AMB}_{/A}$ system AB; $\sum \vec{M}_{/A} = \dot{\vec{H}}_{/A}$

$$\Rightarrow \quad \vec{0} = \underbrace{\vec{r}_{B/A}}_{r\hat{e}_r} \times \cancel{m}\vec{a}_B$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$\vec{a}_{B/A} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$$

$$\vec{a}_A = -L\dot{\phi}^2\hat{\lambda}$$

$$\Rightarrow \quad \vec{0} = -rL\dot{\phi}^2 \underbrace{\hat{e}_r \times \hat{\lambda}}_{-\sin(\theta-\phi)\hat{k} = -\sin\gamma\,\hat{k}} - r^2\dot{\theta}^2 \underbrace{\hat{e}_r \times \hat{e}_r}_{\vec{0}} + r^2\ddot{\theta} \underbrace{\hat{e}_r \times \hat{e}_\theta}_{\hat{k}}$$

$$\left\{ \quad = \left[\cancel{r}L\dot{\phi}^2 \sin\theta + r^{\cancel{2}}\ddot{\theta}\right]\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad \boxed{\ddot{\theta} = \underbrace{\frac{-L}{r}}_{\text{const.}}\dot{\phi}^2 \sin(\theta-\phi)}$$

$$\boxed{\ddot{\gamma} = \frac{-L}{\gamma}\dot{\phi}^2 \sin(\gamma)} \rightarrow$$ pendulum eqn. w/ g replaced by centripal accel. of pt. A: $a_A = L\dot{\phi}^2$

Because $\theta - \phi = \gamma$

and $\ddot{\theta} - \ddot{\phi} = \ddot{\gamma}$

$\ddot{\phi} = 0$ {given}