How to get the highest score?

*Please* do these things:

- **Draw Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- Be (I) neat, (II) clear and (III) well organized.
- Tidily reduce and box in your answers (Don’t leave simplifyable algebraic expressions).
- Make appropriate Matlab code clear and correct.
  You can use shortcut notation like “$T_7 = 18$” instead of, say, “$T(7) = 18$”.
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($l, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($i, j, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems **poorly defined**, clearly state any reasonable assumptions (that do not oversimplify the problem).
- **Work for partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).

**Extra sheets.** Put your name on each extra sheet, fold it in, and refer to it at the relevant problem. Note the last page is blank for your use. Ask for more extra paper if you need it.

Problem 13: ___ /25

Problem 14: ___ /25

Problem 15: ___ /25

Problem 16: ___ /25

Problem 17: ___ /25
13) Making all the usual assumptions about masses and pulleys, find the acceleration of point C in terms of F and m. Neglect gravity.

\[
\begin{align*}
FBDs
\end{align*}
\]

\[
\begin{align*}
T & \leq \frac{F}{2} \\
& \Rightarrow T = \frac{F}{2}
\end{align*}
\]

\[
\begin{align*}
\frac{5F}{3T} &= \frac{1}{3}x''_c \\
3T &= m x'_d \\
\dot{x}_d &= \frac{3T}{m} = \frac{3F}{2m}
\end{align*}
\]

\[
\begin{align*}
\dot{x}_c &= \frac{3}{2} \dot{x}_d \\
\Rightarrow \dot{x}_c &= \frac{3}{2} \dot{x}_A \\
\Rightarrow x'_c &= \frac{9F}{4m}
\end{align*}
\]
14) A disk rolls down a ramp without slipping. How big does \( \mu \) have to be in order to prevent slip? (That is, if \( \mu \) is too small, slip would not successfully be prevented). Answer in terms of some or all of \( \theta, g, R, I^G \) and \( m \).

\[ \dot{\omega} = -\omega R \hat{R} \]

\[ \ddot{\omega} = (\dot{\omega} + \omega R) \hat{R} \]

\[ \theta = \omega R \]

\[ N = mg \cos \theta \]

\[ F = mg \hat{R} + m \dot{x}' \hat{R} \]

\[ F = m[\dot{x}' + g \sin \theta] \]

Unit circle
\[ \sum \vec{M}_c = \vec{H}_c \]

\[ \vec{F}_{bc} \times (-mg \hat{r}) = \vec{F}_{bc} \times m \vec{a}_c + Ic\omega \vec{a} \]

\[ \Rightarrow \vec{F}_{bc} = R_m \vec{a}_c + Ic\omega \vec{a} \]

\[ \{ -Rmg (-\sin \theta \hat{r}) = Rm \ddot{x'} (-\hat{k}) + Ic\omega \hat{k} \} \]

\[ \Rightarrow \sin \theta \cdot Rmg = -Rm \ddot{x'} + Ic\omega \]

\[ \Rightarrow \sin \theta \cdot Rmg = R^2m \omega + Ic\omega \]

\[ \Rightarrow \omega = \frac{Rmg}{Ic + mR^2} \sin \theta \]

\[ \ddot{x'} = -\frac{R^2mg \sin \theta}{Ic + mR^2} \sin \theta \]

\[ F = m \left[ -\frac{R^2mg \sin \theta + mg \sin \theta}{Ic + mR^2} \right] \]

\[ = mgsin\theta \left[ 1 - \frac{R^2m}{Ic + mR^2} \right] = mgsin\theta \frac{Ic}{Ic + mR^2} \]

\[ \mu \geq \frac{F}{N} = \frac{mgsin\theta \frac{Ic}{Ic + mR^2}}{mg \cos \theta} \]

\[ \mu \geq \frac{1 + mR^2/Ic}{\tan \theta} \]

Special cases:

- \[ Ic = \frac{mR^2}{\text{hoop}} \Rightarrow \mu \geq \frac{\tan \theta}{2} \]
- \[ Ic = \frac{mR^2/2}{\text{uniform disk}} \Rightarrow \mu = \frac{\tan \theta}{3} \]

Sanity checks:

A) \[ \theta = 0 \Rightarrow \mu \geq 0 \]

B) \[ Ic = 0 \Rightarrow \mu \geq 0 \]

\[ \uparrow F \rightarrow 0 \text{ to cause rotation} \]
15) A mass \( m \) hangs from a spring with constant \( k \) and rest length \( L_0 = 0 \) (the spring is a so-called zero-rest-length spring). The mass is released from rest at the position \( \vec{r}_0 = 0 \hat{i} + y_0 \hat{j} \).

a) Find the position of the mass at time \( t \) in terms of some or all of \( k, m, g \) and \( y_0 \).

b) Draw the trajectory (the path that the mass moves on).

c) In words, describe the shape of the trajectory.

\[
\sum \vec{F} = ma
\]

\[
m \vec{g} - k \vec{r} = m \vec{a}
\]

\[
m \ddot{\vec{r}} + k \vec{r} = m \vec{g}
\]

\[
x = A \cos(\omega t) + B \sin(\omega t) + \frac{mg}{k}
\]

\[
y = C \cos(\omega t) + D \sin(\omega t)
\]

\[
\begin{align*}
x(0) &= 0 & \Rightarrow B &= 0 \\
y(0) &= 0 & \Rightarrow D &= 0
\end{align*}
\]

\[
x_0 = 0 \Rightarrow A = -\frac{mg}{k} \Rightarrow x = \frac{mg}{k} \left( 1 - \cos(\omega t) \right)
\]

\[
y(0) = y_0 \Rightarrow C = y_0
\]

\[
y = y_0 \cos(\omega t)
\]
\[
\vec{r}(t) = \frac{mg}{k} (1 - \cos \omega t) \hat{i} + \frac{y_0 \cos \frac{\omega}{2} t}{g} \hat{j}
\]

\[a = \sqrt{\frac{mg}{k}}\]

\[
X = \frac{mg}{k} \left(1 - \frac{y}{y_0}\right)
\]

released from rest

\[y_0\]

\[a\]

\[2mg/k\]

\[X\]

trajectory is back and forth on this straight line

equilibrium position

\(\text{X and Y are both simple harmonic motion. In phase with each other.}\)

\(\text{X is offset by } \frac{\text{weight}}{\text{spring constant}}.\)
16) Write MATLAB commands to make a plot of $x_B(t)$. Pick any convenient non-zero values (in consistent units) for any variables or constants.

FBDs $
\begin{align*}
T_1 & \leftarrow \begin{array}{c} A \end{array} \rightarrow T_2 \\
T_2 & \leftarrow \begin{array}{c} B \end{array}
\end{align*}
$

\[
\frac{LMB}{\begin{align*}
m'X_A &= T_2 + T_c - T_1 \\
m'X_B &= -T_2 - T_c
\end{align*}}
\]

MATLAB Properties (Constitutive Laws)

\[
\begin{align*}
T_2 &= (X_B - X_A)k_2 \\
T_c &= (X_B - X_A)\xi \\
T_1 &= k_1X_A
\end{align*}
\]

Kinematics

\[
\begin{align*}
V_n &= X'_A \\
V_B &= X'_B
\end{align*}
\]

$Z = \begin{bmatrix} X_A \\ X_B \\ V_A \\ V_B \end{bmatrix}$
function final_exam_solution()
    % Nonzero things
    XAO = 1; XBO = 2; % Nonzero things
    VAO = 3; VBO = 4; % Nonzero things
    Z0 = [XAO XBO VAO VBO]';
    tspan = [0 10]; % 0 <= t <= 10
    [t zarray] = ode45(@myrhs, tspan, Z0)
    XB = zarray(:,2); % 2nd column
    plot(t,XB); % Skip the labels, it's an exam
end

function zdot = myrhs(t, z)
    K1 = 17; K2 = 39; C = pi'; % Randomish
    XA = Z(1); XB = Z(2); VA = Z(3); VB = Z(4);
    T1 = K1*XA'; T2 = K2*(XB-XA)';
    Tc = C*(VB-VA)';
    XA_dot = VA; % Kinematics
    XB_dot = VB; % "
    VA_dot = (1/m)*(T2+Tc-T1); % F=ma
    VB_dot = (1/m)*(-T2-Tc); % F=ma
    z_dot = [XA_dot XB_dot VA_dot VB_dot]';
end
17) A motor at O turns a rigid rod OA (mass \( M \), moment of inertia \( I^G \)) at constant angular rate \( \dot{\phi} \). A negligible-mass rod with length \( r \) is hinged at A and has mass \( m \) at its end. Neglect gravity.

a) Is angular momentum of the system OAB about O constant or not? (Explain your answer.)

b) Consider the special case that \( \phi = 0 \) and \( \dot{\phi} = 0 \) (for all time). Find \( \dot{\theta} \) in terms of as many of these terms are needed: \( \theta, \dot{\theta}, L, L_G, r, M, m \) and \( I^G \).

c) Now consider non-zero \( \dot{\phi} \). Find \( \dot{\theta} \) in terms of some or all of \( \phi, \phi, \dot{\theta}, L, L_G, r, M, m \) and \( I^G \).

\[ FBDs \]
\[ \text{System 1} \]

\[ \begin{aligned}
\vec{M}_{10} &= \vec{H}_{10} \\
\vec{N}_0 \cdot \vec{k} &= \vec{H}_{10} \neq \vec{0} \text{ in general} \\
\text{Some } \vec{N}_0 \neq 0 \text{ needed to keep } \phi = \text{constant} \\
\Rightarrow \quad \vec{H}_{10} \neq \vec{0} & \quad \text{Ang. Mom not conserved, } O
\end{aligned} \]
b) \( \hat{a}_A = \overrightarrow{0} \)

\[ \text{AMB/A system AB} \Rightarrow \Sigma \hat{r}_{IA} = \frac{\dot{\bar{r}}_{BA}}{m a_{BA}} \]

\[ \dot{\theta} = \frac{\bar{r}_{BA}}{m a_{BA}} \times \frac{m a_{BA}}{m} \]

\[ = m r^2 \hat{\theta} \hat{K} \]

\[ \Rightarrow \text{B goes in circles at const. rate} \]

\[ \odot \]

\[ \text{c) AMB/A system AB; } \Sigma \hat{m}_{IA} = \frac{\dot{\bar{r}}_{BA}}{m a_{BA}} \]

\[ \Rightarrow \dot{\theta} = \frac{\bar{r}_{BA}}{m a_{BA}} \times \frac{m a_{BA}}{m} \]

\[ \Rightarrow \dot{\theta} = -L \phi^2 \hat{r}_x \hat{\lambda} - r^2 \theta^2 \hat{r}_x \hat{e}_r + r^2 \dot{\theta} \hat{e}_r \hat{e}_\theta \]

\[ = \frac{-r \dot{L} \phi^2 \sin(\theta - \phi)}{r} \]

\[ \Rightarrow \hat{\theta} = \frac{-L \phi^2 \sin(\theta - \phi)}{\text{const.}} \]

\[ \hat{y} = -L \phi^2 \sin(\theta) \]

\[ \text{Because } \theta - \phi = \gamma \]

\[ \text{and } \dot{\theta} - \dot{\phi} = \dot{\gamma} \]

\[ \text{pendulum eqns w/ } g \text{ replaced by centripetal accel of pt. A: } a_A = L \phi^2 \]