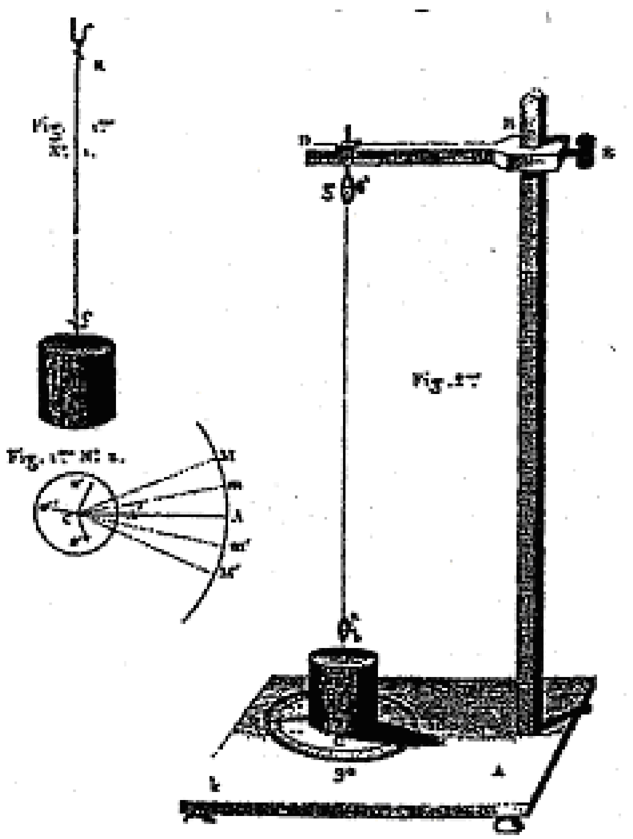
TAM 203 Lab Manual



Cornell University
Theoretical and Applied Mechanics

This manual has evolved over the years. Contributors in the past two decades include: Kenneth Bhalla, David Blocher, Jason Cortell, Drew Eisenberg, Jill Evensizer, Kwang Yul Kim, Richard Lance, Jamie Manos, Francis Moon, Dan Mittler, James Rice, Kevin Rompala, Andy Ruina, Bhaskar Viswanadham, and Alan Zehnder

Contents

TAM 203 Lab Introduction

Last Updated: January 22, 2008

PURPOSE

The laboratories in dynamics are designed to complement the lectures, text, and homework. They should help you gain a physical feel for some of the basic and derived concepts in dynamics: *force*, *velocity*, *acceleration*, *natural frequency*, *resonance*, *normal modes*, and *angular momentum*. You will also get exposure to equipment and computers which you may use in the future. Some mathematics from courses you have taken recently or are now taking will be used. We hope this will help you make the connection between mathematics and physical reality that is essential to much of engineering. The labs may come either before or after you cover the relevant material in lecture. Thus, they can be either a motivation for the lecture material or an application of what you have learned depending on the timing.

COURSE INFORMATION

There are four dynamics laboratories you will be performing during the semester:

1. One Degree-of-Freedom Oscillator
2. Two Degrees-of-Freedom Oscillator
3. Slider-Crank Mechanism
4. Gyroscopic Motion of a Rigid Body

Each of the four labs is taught for two or three weeks (depending on enrollment) in Thurston 101. You will be scheduled to attend lab during one of the weeks. The dates for your laboratory section will be posted outside Thurston 101 and on the course website. In general, you will have a lab once every two or three weeks, but be aware that this may vary due to exam and break schedules.

NOTE: See the Administrative Assistant in Kimball 212 if you have any problems with your lab schedule. You'll need to get his or her approval for any changes so that the lab sections do not become overly full. Turning in a course change form to the registrar is not enough.

LABORATORY ATTENDENCE

You are expected to attend the lab section you have signed up for. In the event of an **excused** absence you must make-up the lab. **All make-up labs must be arranged with your TA.** Your options for making-up labs are

1. Attend another of your lab TA's lab sections.

2. Attend another lab TA's lab section (requires permission from both lab TAs).
3. Attend the "Lab Make-Up Section" during the final week of the semester. Information regarding the date and time of this section will be given in lecture near the end of the semester.

If you show up for lab after it is under way, your lab instructor may ask you to leave and to perform the lab another time.

REQUIRED LABORATORY WORK

The laboratories will be done with physical equipment and some will also involve computer simulations. **It is essential that you read through the lab (especially the procedure section) before coming to lab.** It is not necessary that you understand all of the material perfectly before the lab period.

Prelab Questions

Each lab has prelab questions to be answered before you come to lab. These questions encourage you to review necessary theory and read through the laboratory procedure before attending the lab. Answers to prelab questions are due at the beginning of lab and will not be accepted for credit later.

Laboratory Notes

A rule of laboratory work is to keep a neat, complete record of what has been done, why it was done, how it was done, and what the result was.

The success or failure of an experiment in a research laboratory often depends critically upon the record made of the experiment. The outcome of a poorly documented experiment becomes a matter of personal recollection, which is not reliable enough to serve as a basis for further work (especially by someone else). You should take copious notes. If in doubt, **write it down.** One can ignore what is written, but one can not resurrect that which was never recorded. Similarly, **never** erase in your lab notes. If an erroneous reading was made, strike it out with a single line and record the new data. You may later decide that it was not in error.

All lab notes, signed by your lab TA and in their original form, must be stapled to the back of your final lab report.

Lab Report

Your laboratory report should be typed using a word processor. This report should communicate clearly and convincingly what was demonstrated or suggested by the lab work. Your TA is looking for evidence of thought and understanding on your part. Your logic and methods are as important as results or "correct" answers. It is essential that you provide information and calculations which indicate how you arrived at your conclusions. It is permissible (and a good idea if you want a very good grade) to discuss observations and material relevant to the lab which is not specifically asked about in the questions.

Each report must begin with a cover page containing the following (with appropriate substitutions for the words in quotes):

"NAME OF THE LAB"
TAM 203
By: "Your name and your signature (both partners if a joint report)"
Performed: "Date"
Performed with: "Name of person(s) with whom you performed the lab"
Discussed lab with: "Names of people with whom you discussed the lab, and nature of the discussions"
TA: "Lab TA's name"
TA signed the data on page: "Page #"

It is a good idea to include an introduction, abstract, or overview of the laboratory work you performed as this will help communicate that you successfully grasp the purposes and goals of the lab. It also gives you an opportunity to review your laboratory work before answering specific questions asked in the manual. If you deviate from the procuedure specified in the manual you should also state how and why you did so here.

You should concisely answer the questions that are asked and number them as they are numbered in the lab manual. Include any necessary plots, data, or calculations (make sure to include the correct dimensional units). Your answers should be self-contained and presented in an orderly fashion (i.e. the reader of the report should not have to refer back to the questions that are asked, nor should he or she have to hunt through the report to find your answers). While many questions require that you perform calculations, written explanations of what you are doing and diagrams can be very helpful. Show all calculations that you perform in arriving at your answers. If you are performing repetitive calculations you need show only one sample calculation.

Finally, at the end of your lab report you may want to include any observations, mistakes you made, or suggestions you have in a concluding section.

When answering questions, percentage difference calculations can be used to quantify how well experimental results agree with theoretical or expected values. Rather than writing "the experimental results agree very well with the theoretical calculations," this phrase can be changed to make a quantifiable statement; "the experimental results are within 5 percent of the theoretical calculations." Percentage difference is calculated as:

$100\% \times$ (Value being compared - Reference value) / (Reference value)

While formal error analysis can be used if it is necessary to make a point, your answers should include some discussion of the types and relative sizes of errors in your data.

All plots included with your lab report should be done on the computer using MATLAB (preferred) or Excel. Below are some guidelines for producing quality plots:

- All graphs should be titled and all axes labeled, with the appropriate units listed in parentheses.
- The independent variable should be placed on the horizontal axis.
- Numerical values on the axes should be set at reasonable intervals and scales chosen so that all of the data points can be displayed on the graphs.
- Curves should not be drawn between discrete data points unless the type of fitting used is explained and the equation of the curve given.
- On graphs with more than one curve a legend should be used to identify the curve. Data points can be enclosed by some symbol (i.e. circle, rectangle, etc.) to distinguish different data sets.

Figure 0.1 is an example of how your graphs should appear. The MATLAB code that produced the graph is given below:

```
t = linspace(0,10,1000);
x = 5*cos(2*t);
v = -10*sin(2*t);

figure(1); hold on;
plot(t,x,'b','LineWidth',2);
plot(t,v,'r--','LineWidth',2);
grid on;

plot_title = title('Plot of Position and Velocity vs. Time for Harmonic Oscillator');
x_axis_label = xlabel('Time (sec)');
plot_legend = legend('Position (m)','Velocity (m/s)');

hold off;

set(plot_title,'FontWeight','bold','FontSize',12);
set(x_axis_label,'FontWeight','bold','FontSize',12);
set(plot_legend,'FontWeight','bold','FontSize',12);
set(gca,'FontWeight','bold','FontSize',12);
```

For help with producing log-log and semi-log plots with MATLAB, type `help loglog`, `help semilogx`, or `help semilogy` in the main MATLAB window.

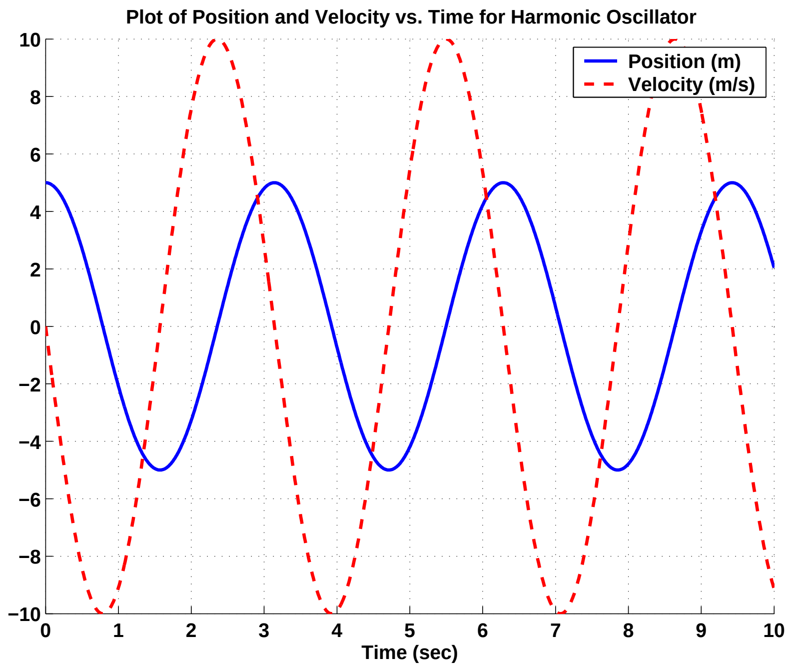


Figure 0.1: An example graph.

CREDIT AND GRADING

Lab reports are due at 10:00 AM one week from the day you performed the lab unless your TA specifies another time. Turn in reports in the boxes in the Don Conway room, Thurston 102. Put your report in the correct box corresponding to the TA in charge of your **laboratory** section. Reports placed in incorrect boxes might not be found.

Each laboratory is graded out of 15 points. The grade breakdown for each lab report will be determined by your lab TA. This grade will be given to your recitation TA.

ACADEMIC INTEGRITY

Your pre-lab answers and lab reports should be in your own words, based on your own understanding and your own calculations. You are encouraged to discuss the material with other students, friends, TAs, or even faculty. **Any help you receive from such discussions must be acknowledged on the cover of your lab report, including the name of the person or persons and the exact nature of the help.** Violations of this policy will be reported to the academic integrity board.

You may, however, do a joint report with your lab partners (turn in one report for your lab group). All partners get the same grade on the report but separate grades on pre-lab questions.

When you are done in the lab you must have your TA sign one of your data sheets. This sheet must include the name of your lab partners and the time and date the lab was performed.

The TA will not sign this sheet until your work station is clean and all equipment is accounted for. **No lab reports will be accepted without this signed sheet.**

Lab #1 - One Degree-of-Freedom Oscillator

Last Updated: March 4, 2009

INTRODUCTION

The mass-spring-dashpot is the prototype of all vibrating or oscillating systems. With varying degrees of approximation, car suspensions, violin strings, buildings responding to earthquakes, earthquake faults themselves, and vibrating machines are modeled as mass-spring-dashpot systems. This laboratory is aimed at demonstrating some of the basic concepts of the mass-spring-dashpot system. In this lab you will collect data on the motion of two different mass-spring-dashpot systems, and then use computer generated solutions of the equations of motion to determine system parameters. Phrases connected with some of the key ideas are: *natural frequency*, *resonance*, *forcing function*, and *frequency response*.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. Find the general solution to (1.4) if the forcing term is given by $F_s(t) = 0$ and there is no damping (c =0), i.e. solve $m\ddot{x} + kx = 0$. *Note: Use pencil and paper, not MATLAB.*

2. Repeat #1, this time numerically integrating the equation using *Matlab*. Choose $m = 1$, $k = 5$, and integrate over the time period $0 \leq t \leq 10$. Assume the mass starts from rest with an initial displacement of $x(0) = 1$. What is the period of the oscillation? Turn in a plot and an m-file of your code.

3. Define in your own words: *natural frequency*, *damped frequency*, *damping coefficient*, *underdamped*, *overdamped*, *resonance*, and *phase-shift*.

4. Suppose that you are measuring two sinusoidal waveforms of equal amplitude, $x_1(t)$ and $x_2(t)$, with phase difference of $\frac{\pi}{2}$. What would be the shape of the curve if you plotted $x_1(t)$ vs $x_2(t)$? What if the phase difference is zero? π? If you have trouble visualizing the situation, try calculating a few points and plotting them.

5. Find the period T, support amplitude of motion $A_{support}$, mass amplitude of motion $A_{response}$, and phase difference ϕ for the following two curves:

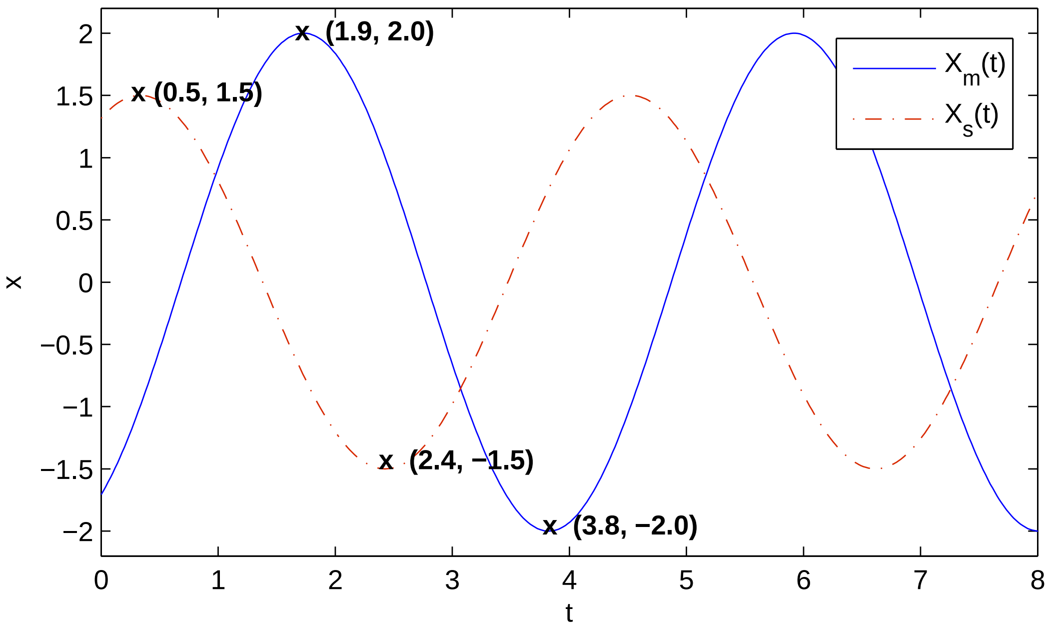


Figure 1.1: Sample lab data

THE MASS-SPRING-DASHPOT SYSTEM

The picture in Figure 1.2a shows a mathematical model of the laboratory mass-spring-dashpot, or one degree-of-freedom oscillator. A mass is supported by a spring and is constrained to move in the $\hat{\mathbf{e}}_\mathbf{x}$-direction. In this lab you will record the vertical motion of the mass both with a fixed support (*free vibration*) and with the support oscillating vertically (*forced vibration*). The spring is modeled as linear, i.e. the force it applies is proportional to its increase in length. The damping is also modeled as linear, i.e. the force transmitted by the dashpot is proportional to the rate at which it is being stretched. The vertical displacement of the mass is $x(t)$ and the vertical displacement of the support is $x_s(t)$ (*See Figure 1.2*).

Neglecting gravity (*Why can we neglect it?*), the mass has two forces acting on it in the $\hat{\mathbf{e}}_\mathbf{x}$-direction:

$$F_{sp}(t) = \text{ The spring force } = k\,(x_s - x) \tag{1.1a}$$

$$F_d(t) = \text{ The dashpot force } = c\dot{x} \tag{1.1b}$$

The system is a one degree-of-freedom system because a single coordinate is sufficient to describe the complete state of the system. (The support displacement $x_s(t)$ does not count as a degree of freedom since it is specified by the motor position and is thus considered to be a given.) From Newton's second law the equation of motion for this system is

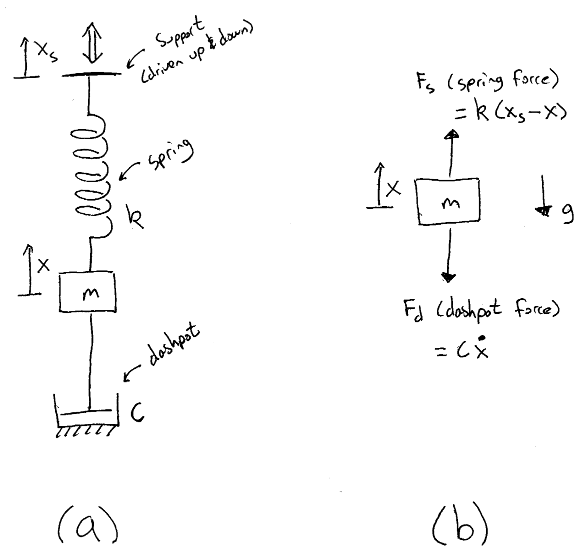


Figure 1.2: Model and free body diagram of the mass-spring-dashpot system.

$$\left\{\sum \underline{\mathbf{F}}\right\} \cdot \hat{\mathbf{e}}_\mathbf{x} \Rightarrow -F_d + F_{sp} = m\ddot{x} \tag{1.2}$$

and plugging in for spring and dashpot terms we get

$$m\ddot{x} = -c\dot{x} + kx_s - kx \tag{1.3}$$

and rearranging terms we get the equation of motion

$$m\ddot{x} + c\dot{x} + kx = F_s(t) \text{ with } F_s(t) = kx_s(t) \tag{1.4}$$

where $F_s(t)$ is the (presumably specified) "forcing function" due to the motion of the support. In this case the forcing function is the amount the spring is additionally stretched due the support motion multiplied by the spring stiffness.

In the first part of this experiment you will attempt to determine the value of the viscous damping constant c by measuring the rate at which oscillations decay towards zero, an experiment called a "ring-down test". In addition, the system response to both *free vibration* and *forced vibration* will be observed experimentally and through computer simulation.

A REAL-WORLD EXAMPLE: THE LOUDSPEAKER

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one degree-of-freedom mass-spring-dashpot system. The one you will observe in this lab is typical (see Figure 1.3). It has a plastic cone supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure and (not visible from outside) a coil of wire attached to the point of the cone which can slide up and down inside the magnet. When you turn on your stereo, it forces a current through the coil in time with the music, causing the coil to alternately repel and attract the magnet pushing the cone up and down in its housing. This results in the vibration of the cone which you hear as sound.

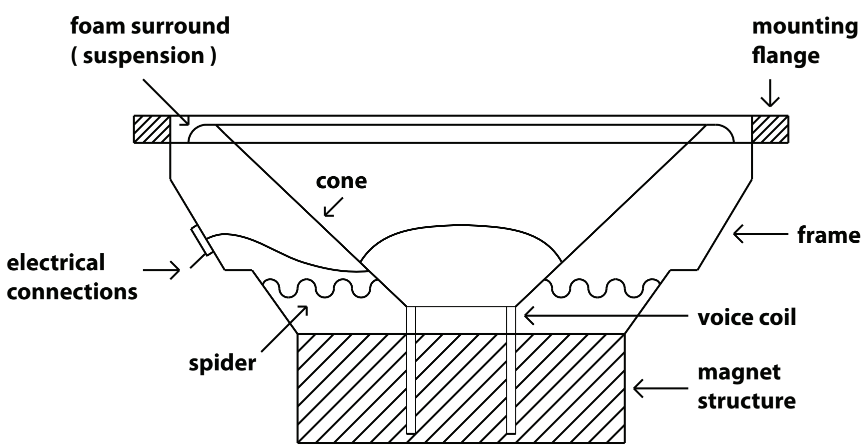


Figure 1.3: Cross-sectional view of a speaker.

A simplistic view is that the cone and coil provide inertia ($m\ddot{x}$), the foam surround and clothe bellows act as a spring (kx), viscous damping comes from the cone moving through the air ($c\dot{x}$), and the magnet provides external forcing $F_s(t)$. Putting it all together we get the familiar equation of motion of a driven mass-spring-dashpot system:

$$m\ddot{x} + c\dot{x} + kx = F_s(t) \tag{1.5}$$

In the second part of the lab, you will non-destructively measure the weight of the speaker coil and cone by examining the speakers dynamics.

SOLVING THE EQUATIONS OF MOTION

Recall that the equation of motion is given by:

$$m\ddot{x} + c\dot{x} + kx = F_s(t) \tag{1.6}$$

Our goal is to find the motion of the mass, $x(t)$, for a given forcing function $F_s(t)$. Two cases are of particular interest:

$$F_s(t) = 0 \text{ (unforced or 'free' vibration)} \tag{1.7}$$

$$F_s(t) = kx_s(t) = kA_{support}\cos\omega t \text{ (sinusoidal forcing)} \tag{1.8}$$

Equation (1.6) is a linear, second order ordinary differential equation with constant coefficients. The solution with $F_s(t)$ given either by (1.7) (homogeneous) or (1.8) (inhomogeneous) is discussed in every freshman or sophomore math text. Briefly, the solution can be found as follows:

From ordinary differential equation theory we can write the general solution to (1.6) as the sum of a complimentary (also referred to as the transient or homogeneous) solution $x_c(t)$ and a particular solution, $x_p(t)$.

$$x(t) = x_c(t) + x_p(t) \tag{1.9}$$

The homogeneous portion $x_c(t)$ is the solution to (1.8) with $F_s(t) = 0$ (*and appropriate initial conditions*). In this case, $x_c(t)$ goes to zero as $t \to \infty$ because any initial motion of the mass will eventual be damped out if there is no external forcing. Thus the particular solution $x_p(t)$ is what is left as $t \to \infty$ for any initial condition and includes the information about forcing.

In this section we are concerned with unforced vibrations, so we have $x(t) = x_c(t)$. We will deal with $x_p(t)$ later. As you may have seen in other courses, we posit the solution to be of the form $x_c(t) = Ae^{\lambda t}$ (*if this process seems unfamiliar to you, please review differential equations*). When we insert this into (1.5), we obtain the characteristic equation,

$$m\lambda^2 + c\lambda + k = 0 \tag{1.10}$$

which has roots given by the quadratic equation as,

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \tag{1.11}$$

Now, depending on the values of the parameters c,m, and k (specifically the *discriminant* $c^2 - 4mk$), there are three situations encountered, and thus three different behaviors of the displacement solution $x_c(t)$. These situations are:

- $c^2 - 4mk > 0$: This produces two distinct real roots λ_1 and λ_2, and the solution is :

$$x_c(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{1.12}$$

  This sytem is called *overdamped*– the system will slowly settle down to $x_c(t) = 0$ with **no oscillations**.

- $c^2 - 4mk = 0$: This produces a repeated real root $\lambda_1 = -c/2m$ and the solution is:

$$x_c(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} \tag{1.13}$$

  This system is called *critically damped* - the system will quickly settle down to $x_c(t) = 0$ with **no oscillations**. *Why is the decay more rapid than the overdamped case?*

- $c^2 - 4mk < 0$: This produces a complex conjugate pair $\alpha \pm i\beta$ with $\alpha < 0$ and the solution is:

$$x_c(t) = e^{\alpha t}\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right] \tag{1.14}$$

  This sytem is called *underdamped*–the mass will oscillate, but the oscillations will decay with time according to the exponential factor (see Figure 1.4). This is the most common situation - most real world systems are underdamped.

Naturally, the constants C_1 and C_2 will be determined from initial conditions for the speed and displacement of the mass.

A useful quantity (you will see why), termed the *natural frequency* ω_n is defined as,

$$\omega_n = \sqrt{\frac{k}{m}} \tag{1.15}$$

As you will show in prelab question (1), this is the system's frequency of free vibration when there is no damping ($c = 0$). Additionally, instead of employing the discriminant $c^2 - 4mk$ to describe the state of the system (over/under/critically damped) , it is convenient to define a *damping factor*, ζ, as

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{1.16}$$

ζ is defined in such a way that

- $\zeta > 1$ is an overdamped system

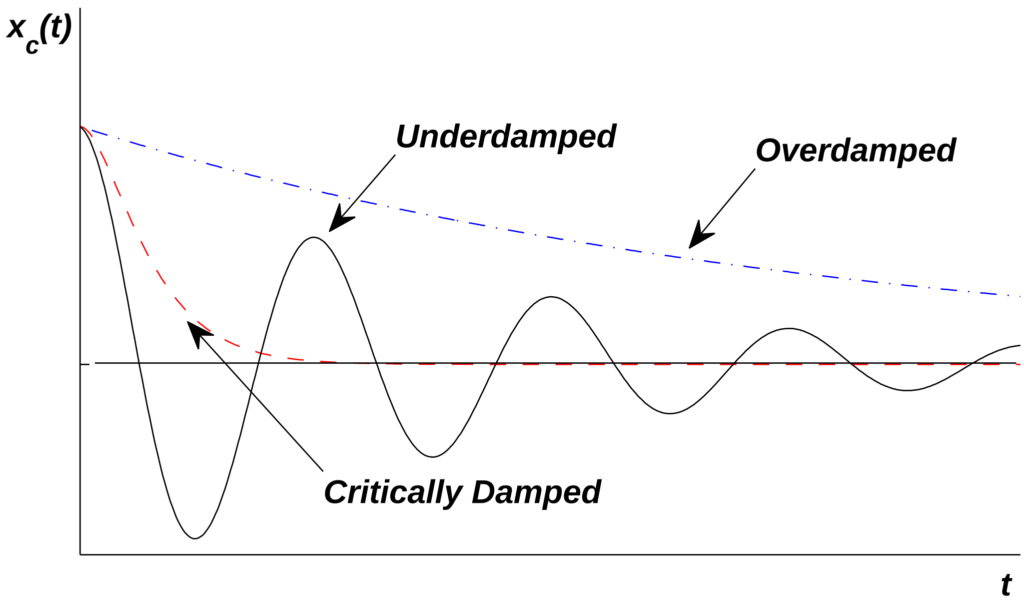


Figure 1.4: Typical solutions for *underdamped*, *overdamped*, and *critically damped* cases. Note that for *overdamped* and *critically damped* systems there are no oscillations.

- $\zeta = 1$ is a critically damped system
- $\zeta < 1$ is an underdamped system

Thus, ζ is a non-dimensional measure of the amount of damping in the system. **In this lab, we will assume that both the mass-spring-dashpot system and the speaker are underdamped.** ***In fact we will assume*** $\zeta \ll 1$***!***

Using these definitions, we can restate the quadratic equation we found above in terms of the new variables, which yields (after some algebra)

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{1.17}$$

Since we are studying the underdamped system in the lab, we take $\zeta < 1$ and find that the roots are

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d \tag{1.18}$$

where we defined the *damped natural frequency* (i.e. the frequency of oscillation *with* damping) as $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Thus, the solution for the underdamped system (1.14) is,

$$x_c(t) = e^{-\zeta\omega_n t}\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right] \tag{1.19}$$

which can be restated as,

$$x_c(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{1.20}$$

where $A = \sqrt{C_1^2 + C_2^2}$, and $\phi = tan^{-1}(-C_2/C_1)$ are two constants to be determined from the initial conditions.

THE LOGARITHMIC DECREMENT METHOD

It is often important to measure how much damping there is in an engineering system. The viscous damping coefficient, c, may be determined experimentally by measuring the rate of decay of unforced oscillations - this process is called a "ring down" test. We define the logarithmic decrement, D, as the natural logarithm of the ratio of any two successive amplitudes:

$$D = \ln\left(\frac{x_n}{x_{n+1}}\right) \tag{1.21}$$

where x_n and x_{n+1} are the heights of two successive peaks in the decaying oscillation (see Figure 1.5). **Note that "x" here refers to the mass displacement $x_c(t)$ with respect to equilibrium and not the x-axis.** The larger the damping, the greater will be the rate of decay of oscillations and the bigger the logarithmic decrement, D.

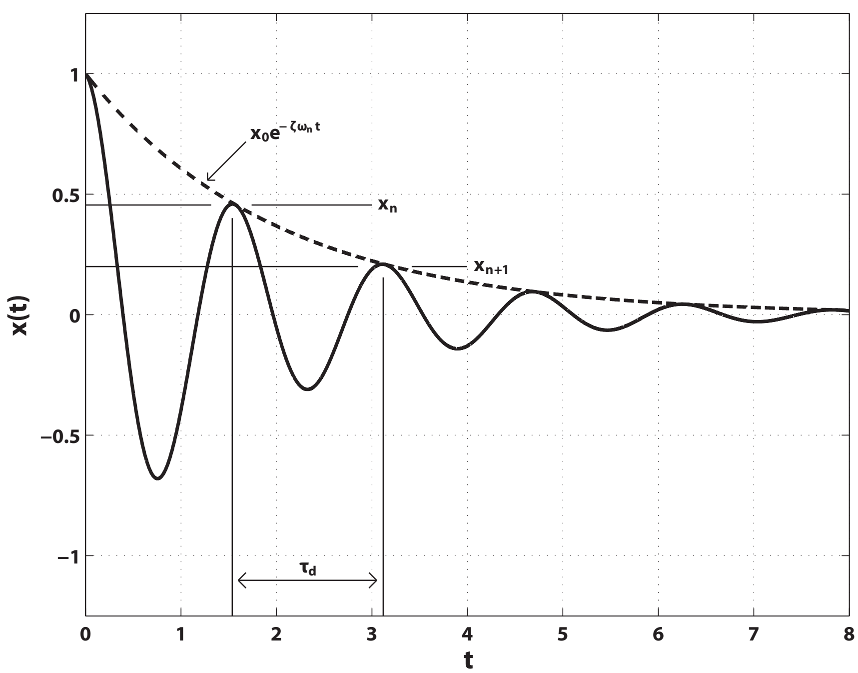


Figure 1.5: The logarithmic decrement method.

Because of the exponential envelope that this curve has (refer to (1.20)), $x_n = (Const.) * e^{-\zeta\omega_n t}$ and $x_{n+1} = (Const.) * e^{-\zeta\omega_n(t+\tau_d)}$, where τ_d is the period of the damped oscillation, i.e. $\tau_d = \frac{2\pi}{\omega_d}$. Thus

$$D = \ln\left(\frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n(t+\tau_d)}}\right) = \zeta\omega_n\tau_d \tag{1.22}$$

We simplify this expression by substituting in (1.16) for ζ and then solve for the damping constant c, yielding *(algebra omitted)*

$$c = \frac{2mD}{\tau_d} \tag{1.23}$$

We can also obtain an equation for k from (1.22) , yielding

$$k = \frac{c^2\left(1 + \frac{4\pi^2}{D^2}\right)}{4m} = \frac{c^2}{4m\zeta^2} \tag{1.24}$$

Thus, by doing a "ring-down" test we can experimentally measure values of D and τ_d. Then using equations (1.23) and (1.24) and given the mass m, we can calculate the damping coefficient c and spring constant k for a one degree-of-freedom oscillator.

FORCED VIBRATIONS AND FREQUENCY RESPONSE

Often a system is periodically forced and we are interested in how it will respond, e.g. the tires on your car going over evenly spaced ruts in the road jostles the car. When the forcing function is sinusoidal with frequency ω, it can be shown that the steady state solution $x_p(t)$ is sinusoidal in time with the same frequency ω. *Note: here we use 'p' to denote the particular solution.* Furthermore, the amplitude of the system's response depends on the frequency and amplitude with which we drive it. When the frequency with which we force the system ω is close to the system's natural frequency of vibration ω_n, the response has a quite large amplitude. This phenomenon, called resonance, will be discussed in the next section.

Starting with our equation of motion (1.6):

$$m\ddot{x} + c\dot{x} + kx = F_s(t) \tag{1.25}$$

If we let the forcing term be given by:

$$F_s(t) = kx_s(t) = F_{drive}\cos\omega t \tag{1.26}$$

Then we are looking for a steady state solution of the form:

$$x_p(t) = A_{response}\cos(\omega t - \phi) \tag{1.27}$$

where $A_{response}$ is the amplitude of the system response and ϕ is the phase of the response $x_p(t)$ with respect to the exciting force $F_s(t)$. *Note: the phase of a curve is a shift of one graph to the left or right with respect to another graph and has units of radians. If the phase is 0 then the response is at a maximum when the forcing is at a maximum, and if the phase is π then the response is at a minimum when the forcing is at a maximum.* Using a trick from linear algebra we can solve for how the amplitude and phase of the response depend on the driving frequency. The results are given in section (9.10) of your book, and plotted above in Figure 1.6.

Note from the plot that:

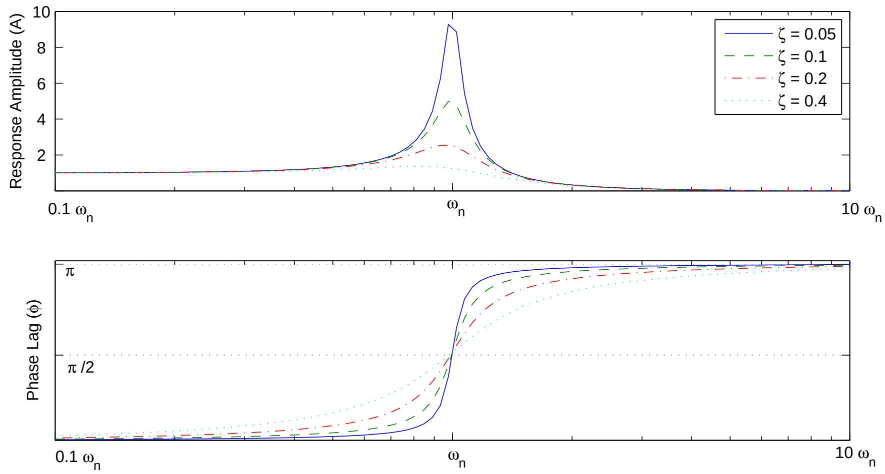


Figure 1.6: The system response $x(t) = A_{response} \cos(\omega t - \phi)$ as a function of forcing frequency ω, for various amounts of damping ζ. The forcing amplitude, F_{drive}, is fixed.

- for very low drive frequencies ($\omega \ll \omega_n$) the response is synchronized with the driving. The phase lag (ϕ) is 0, and the amplitude of vibration of the mass is the same as the amplitude of vibration of the support. *What is the physical argument for this?*

- for drive frequencies near ω_n the response amplitude is at a maximum and the phase lag is $\frac{\pi}{2}$.

- for very high drive frequencies ($\omega \gg \omega_n$) the response is completely out of phase with the driving ($\phi = \pi$) and the amplitude of vibration goes to zero. *What is the physical argument for why the amplitude vanishes?*

- the less damping there is, the sharper the change in phase is, and the greater the response near ω_n.

RESONANCE

Resonance as defined by *Merriam-Webster* is *a vibration of large amplitude in a mechanical or electrical system caused by a relatively small periodic stimulus of the same or nearly the same period as the natural vibration period of the system.* This definition confirms what we already noted in Figure (1.6), i.e. that the amplitude of response was a maximum when we drove the system at a frequency ω near the natural frequency ω_n. To find the exact resonant frequency, ω_r, we find the point on our graph of $A_{response}(\omega)$ with a slope of zero (see section (9.10) of your book for more details):

$$\left.\frac{dA_{response}}{d\omega}\right|_{\omega=\omega_r} = 0 \Rightarrow \omega_r = \omega_n\sqrt{1-2\zeta^2} \tag{1.28}$$

Note for small damping ($\zeta \ll 1$) we have $\sqrt{1-2\zeta^2} \sim 1$ and so the *resonant frequency* ω_r and the *natural frequency* ω_n are approximately equal $\omega_r \simeq \omega_n$. This supports what we observed in Figure 1.6 where the peak in the response seems to be very near to ω_n.

PHASE DIAGRAMS In our experiment we will need a way to tell if the system is near resonance. We could adjust the forcing frequency ω until the response is maximized. However, this is not a very precise method. A better way is to examine the response phase(ϕ). It can be shown that when we force the system at it's natural frequency (ω_n) that the phase is $\phi = \frac{\pi}{2}$ *(*Verify this by inspection (see Figure 1.6) or directly from the equation for ϕ in (9.10) of your book). The corresponding *phase diagram* will then be a circle *(If you are interested, further details on what a phase diagram is can be found in the appendix).* Thus when the phase diagram is a circle we are at (or very close to) resonance. Though it is more difficult to prove, we will see that when our forcing frequency ω is below resonance, the phase diagram will look like an ellipse tilted to the right, and when it is above resonance, the phase diagram will look like an ellipse tilted to the left.

LABORATORY SET-UP

- **Mass-Spring-Dashpot System**
  The apparatus consists of a laboratory-model mass-spring-dashpot system with displacement transducers (Linear Variable Differential Transformers or LVDTs) for measuring $x(t)$ and $x_s(t)$. The output from the LVDTs is communicated to the computer via the data acquisition board. An electric motor and controller, acting through a scotch yoke, enable a sinusoidal forcing function to be applied to the system. Note that the controller dial readings are arbitrary; frequency and period data must be obtained from your computer plots.

- **Loudspeaker**
  The apparatus consists of a speaker on a stand with one LVDT to measure cone displacement. Waveforms are generated by the computer, amplified, and sent through a resistor to drive the speaker. The computer is also used to measure current flow through the speaker and displacement of its cone (using the attached LVDT).

Please follow all safety precautions. Keep long hair and loose clothing well away from the electric motor, pulleys, and other moving parts.

- **Using the *LabView* Software**
  The four programs you will be using in the first part of the lab are: *FreeAcq* (Figure 1.7) for acquiring data on the unforced system; *FreeSim* (Figure 1.8) for measuring the data and simulation of the same; *ForcedAcq* (Figure 1.9) for acquiring data on the system with a sinusoidal forcing function; and *ForcedSim* (Figure 1.10) which may be used for measuring the data and simulation of the forced system. Although somewhat different

in appearance and function, the programs share many key features. The *SpeakerAcq* (Figure 1.11) program used in the second part of the lab is also similar.

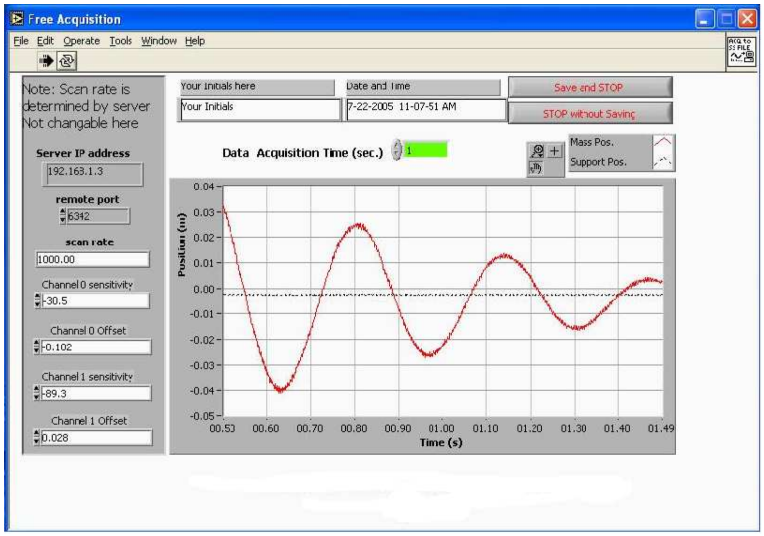


Figure 1.7: The *FreeAcq* program.

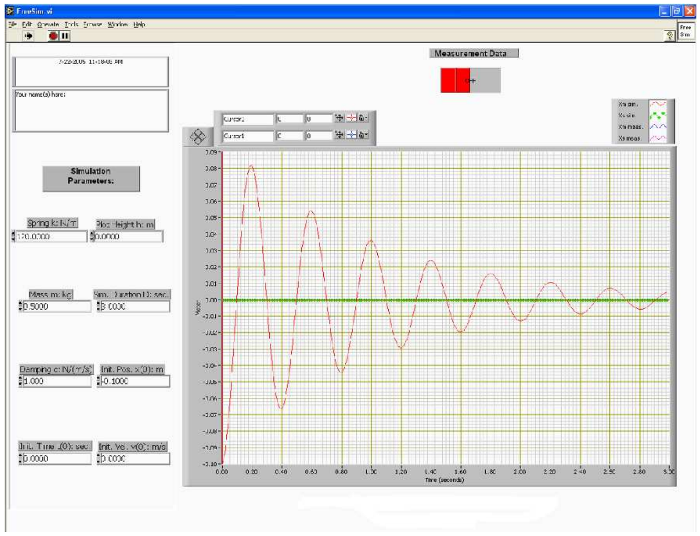


Figure 1.8: The *FreeSim* program.

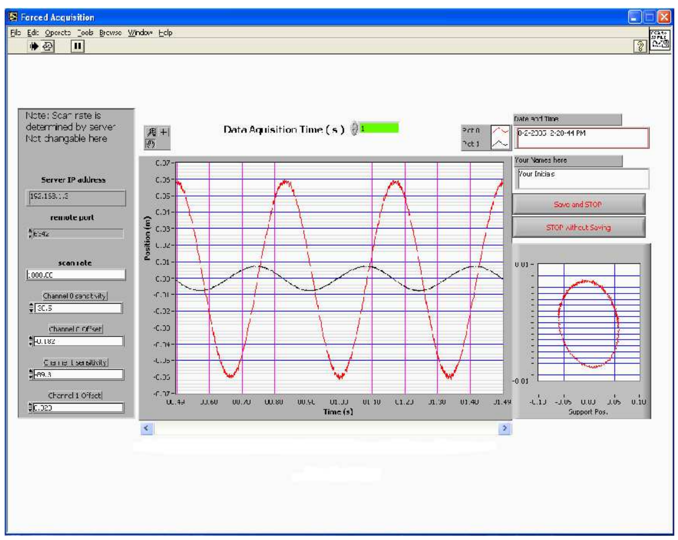


Figure 1.9: The *ForcedAcq* program.

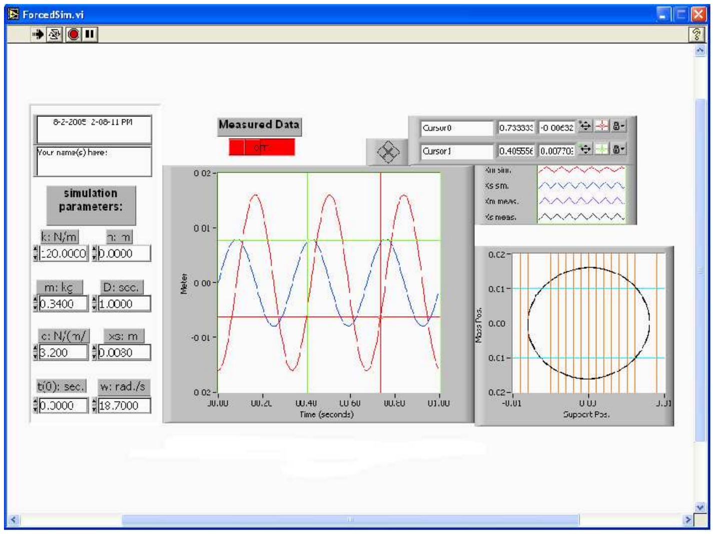


Figure 1.10: The *ForcedSim* program.

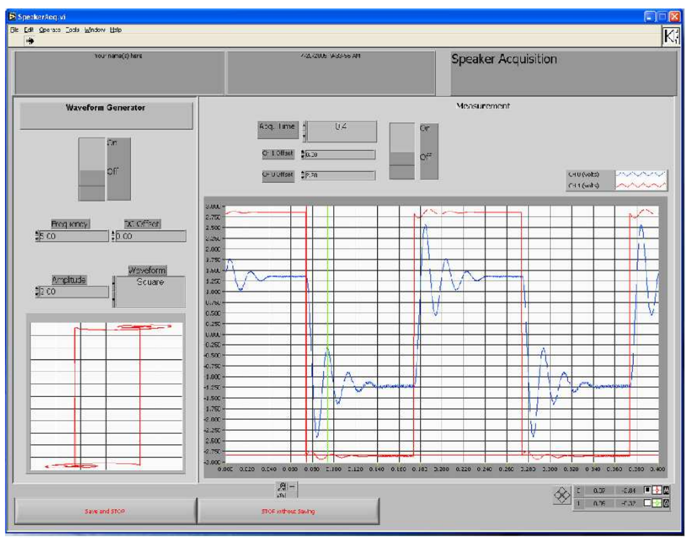


Figure 1.11: The *SpeakerAcq* program.

To run the program, you must hit the white arrow in the top left of the screen. If this arrow is black, that means that the program is already running. For the data acquisition programs, a green box on top will define the amount of time for which the program will record after hitting the arrow. To reset the data acquisition, press `STOP without Saving` and then press the white arrow to begin again.

After getting data, pressing the `Save and STOP` button stores your current data on disk. The data file is only used by the simulation programs *FreeSim* and *ForcedSim*- it is not available to the data acquisition programs.

You may find it convenient to obtain numerical data from your plots using the cursors, rather than using a ruler. Two cursors are available, one indicated by a circle and one by a square. To use a cursor, use the mouse to drag it to the point you want to measure. If your cursor has vanished off the screen, you can enter an on-screen position for it into the x and y display boxes, and it will reappear in the desired location. You can also lock the cursor to a curve by clicking the lock icon. Zoom and other features are available for the cursors and graphs; see the *LabView* manual for details.

PROCEDURE

- **Free Vibration, Mass-Spring-Dashpot**

  1. First you will measure the *free vibration* of the mass.
     - Start up the *FreeAcq* program. The data acquisition programs automatically convert the voltage output of the LVDTs to meters. To do this, they need a set of conversion factors, which are on a label on the mass-spring-dashpot base board. Make sure that the sensitivity and offset values on the left hand side of the window match the values listed on a small sheet of paper in front of the apparatus, and enter your name in the box provided. Set the data acquisition time to 6 seconds.
     - Pull down the mass and hold it still, then press the white run arrow in the top left of the toolbar, wait 1 second and then release the mass.
     - Repeat this procedure until you have a nice oscillation over the 6 seconds. Please note that the zero position is somewhat arbitrary. *You will need to take data long enough for the mass to stop oscillating in order to measure the equilibrium value.*
     - Save your best oscillation on disk by pressing the `Save and STOP` button. Save your data on the desktop with an appropriate title specific to your group.
  2. Next you will measure the logarithmic decrement D and estimate the spring stiffness k and damping coefficient c.
     - Close down the *FreeAcq* program and start the *FreeSim* program. Add the measured data to the graph by pressing the `Measurement Data` switch above

the graph. Set $k = 0$ to get the simulated data out of the way, and consult the legend to make sure you know what curve you are measuring.

- Using the cursors, calculate the logarithmic decrement D and the period of the damped oscillation τ_d for each set of successive peaks - **at least 3**. *Please note that x_n and x_{n+1} in (1.21) refer to the mass displacement from equilibrium and not the "x-axis". **You will need to measure the equilibrium value and take it into account in your calculations**.*
- Using these measured values, and the mass m, calculate the damping coefficient c and spring stiffness k *(The mass of the weight and spring are written at the base of the setup. For your 'm' use the total of the weight mass and the spring mass)*.
- Make a print-out of your curve.

3. Finally, you will simulate the *free vibration* of the mass-spring-dashpot system and verify your estimate of the system parameters k, c, and m which you just calculated.
   - Input the k, c and m which you just calculated and adjust the initial condition and viewing parameters $(t(0), h, x(0), D)$ to fit your data. *Don't change k or c.*
   - Make a print-out.
   - Now see if you can adjust k and c to get a better agreement. Take note of what aspects of the graph change when you change each of the parameters k and c *independently*.
   - Make another print-out.

- **Forced Vibration, Mass-Spring-Dashpot**

  1. Here you will be recording the motion of the mass as it undergoes sinusoidal forcing.
     - Close down any other open programs and start the *ForcedAcq* program.
     - Set the acquisition time to 10 seconds, start the data acquisition and turn on the motor. Two graphs will be displayed. The left one contains two plots. One is a plot of the mass position $x(t)$ vs. time and the second one is a plot of the spring support position $x_s(t)$ vs. time. The right graph plots the phase diagram.
     - For at least five different forcing frequencies get nice plots of several cycles of motion (see instructions below). Make sure to save each data set to disk in order to analyze them in the *ForcedSim* program. Print-outs are not necessary but may be helpful.
     - To acquire data, set the data acquisition time to 10 seconds and click the arrow to run the program. Now adjust the forcing frequency until you get the desired frequency. If the data acquisition stops before you are done adjusting the forcing frequecy, you will need to click `STOP without SAVING` and then

click the arrow to run it again. Once you've got the drive frequency where you want it, reduce the data acquisition time to ~ 1 second (or atleast long enough to get one whole cycle) and then run the program again. This time hit `SAVE and STOP`. Reducing the acquisition time will reduce demand on the server and save you time doing analysis. Forcing frequencies should include:

  * A low frequency for which the motor runs smoothly ($\sim \frac{1}{2}Hz$).
  * A frequency just lower than resonance.
  * Resonance. (*Hint: we can tell from the phase diagram that it is at resonance*)
  * A frequency just higher than resonance.
  * A high frequency ($\sim 3 * Resonance$).

2. Next we will measure our data in order to later calculate the response phase ϕ and amplitude $A_{response}$.
   - Close down the *ForcedAcq* program and open the *ForcedSim* program.
   - Turn on the measured data switch to view your saved data. To change the current measured data set you must close and then re-open the *ForcedSim* program. Once experimental data is loaded, make your necessary measurements (see pre-lab question) using the computer cursors.

You may also want to save the data to a USB storage device or write it to a CD for later analysis. To do this just copy the text files of the desired data onto your storage device.

- **Vibration of a Speaker**

- In the last part of the lab, you will non-destructively measure the mass of a speaker cone by measuring the shift in in its resonant frequency due to the addition of a known mass.

1. First you will find the resonant frequency of the loud speaker.
   - Set the `Waveform` control to `Sine` and the `Amplitude` control to 2. Leave the `DC Offset` control set to 0. Set the data acquisition time to 0.1 seconds. The `CH 0 Offset` and `CH 1 Offset` controls may be used to adjust the plots vertically if necessary.
   - Turn on the waveform generator and data acquisition switches and adjust the `Frequency` control value until you observe resonance of the speaker cone. **To change the frequency you must press `STOP without Saving`, enter the new desired frequency and then hit the start arrow.**
   - Make a print-out and record the resonant frequency *(note that the frequency here is given in Hz and not rad/sec)*.

Recall that the resonant frequency depends on both the mass m and spring stiffness k. By measuring the resonant frequency you cannot solve for both m and k

uniquely. However, if you also measure the resonant frequency when the mass is changed by a known amount then you will have 2 equations (i.e. (1.15), assume $\omega_r \sim \omega_n$) for 2 unknowns (m, k) in terms of measured data $(\omega 1_r, \omega 2_r, \triangle m)$. Now measure the mass of the rubber weight and then carefully press it onto the LVDT shaft. The best way is to spread the weight open, position it, and release it.

- Find the new resonant frequency, and record the mass of the rubber weight.
- Make a print-out and record the new resonant frequency.

LAB REPORT QUESTIONS

Please answer the following questions concerning the mass-spring-dashpot part of the lab within your lab report:

1. What is the spring constant k and damping coefficient c for your mass-spring-dashpot setup as calculated from your "ring-down" test? Indicate the measured data and formulas you used to calculate these values. Is the damping coefficient c really constant? How can you tell? What does this say about the air dashpot acting linearly?

2. Compare your experimental data to the simulated data for unforced motions. Comment on any similarities or differences of interest. How did changing c and k each change the simulation graph? Please attach print-outs.

3. Make a plot of the response amplitude $A(\omega)$ using your 5 data points. Make a plot of the phase-angle ϕ between $x(t)$ and $x_s(t)$ versus the forcing frequency ω. Indicate any formulas used. Do these plots match what you expect from Figure 1.6.

4. For your system, what is the theoretical percent difference between the natural frequency ω_n and the damped natural frequency ω_d (refer to 1.28)? Does the addition of a dashpot to a mass-spring system increase or decrease its oscillation frequency? Indicate any formulas used.

Please answer the following questions concerning the loudspeaker part of the lab within your lab report:

1. Calculate k and m for the speaker, using the resonant frequencies and mass you measured in lab.

2. Find another real-world vibrating system which could be reasonably modeled as a mass-spring-dashpot. Give the system a "push" and observe its response. Try applying a forcing function of various frequencies, and look for resonance.

   (a) Describe how you modeled your vibrating system as a mass-spring-dashpot. That is, what does the mass represent, what is the spring, and what is the dashpot? Be as specific as possible.

   (b) Is this system typically overdamped? Underdamped? If applicable, what was the resonant frequency (approximately)?

   (c) In what ways does the system you found most significantly differ from an ideal linear mass-spring-dashpot system?

CALCULATIONS & NOTES

Appendix: Phase Diagrams A phase diagram is a plot which contains the forcing function $F_s(t)$ on the y-axis and the response function $x(t)$ on the x-axis. The phase diagram is a graphical representation of the relative phase of the forcing and motion. Each point on the plot tells us both where we are in the drive cycle $y(t)$ and on the response cycle $x(t)$. Time is a parameter that moves us around on the diagram. Since we are only interested in the phase, we scale each term by its amplitude. Thus on our phase diagram we would plot the parametric function

$$y(t) = cos(\omega t) \tag{1.29}$$

$$x(t) = cos(\omega t - \phi) \tag{1.30}$$

When we force the system at $\omega_n \simeq$ and the phase is $\phi = \frac{\pi}{2}$ as shown before, we have:

$$y(t) = cos(\omega t) \tag{1.31}$$

$$x(t) = cos(\omega t - \frac{\pi}{2}) \tag{1.32}$$

Using the following trigonometric identities:

$$\cos^2 \omega t + \sin^2 \omega t = 1 \tag{1.33}$$

$$\cos(\omega t - \frac{\pi}{2}) = \sin \omega t \tag{1.34}$$

We can establish the following relationship for our phase plot:

$$x^2(t) + y^2(t) = 1 \tag{1.35}$$

Hopefully you will recognize this equation as the parametric form of the equation for a circle! Thus, when we force the system at its *natural frequency* which is very close to its *resonant frequency* the phase diagram is a circle. It is more difficult to show that when we forced the system below resonance the phase diagram will be an ellipse tilted to the right and above resonance an ellipse tilted to the left.

Lab #2 - Two Degrees-of-Freedom Oscillator

Last Updated: February 13, 2009

INTRODUCTION

The system illustrated in Figure (2.1) has two degrees-of-freedom. This means that two is the minimum number of coordinates necessary to uniquely specify the state of the system. The purpose of this laboratory is to introduce you to some of the properties of linear vibrating systems with two or more degrees-of-freedom. You have already seen a one degree-of-freedom vibrating system (the mass-spring-dashpot system) and should have some familiarity with the ideas of *natural frequency* and *resonance*. These ideas still apply to an undamped linear system with two or more degrees-of-freedom.

The new idea for many degrees-of-freedom systems is the concept of *modes* (also called *normal modes*). Each *normal mode* consists of a *mode shape* and corresponding *natural frequency*. The system will exhibit resonance if forced at one of its natural frequencies. The number of modes a system has is equal to the number of degrees-of-freedom. Thus the system below has two modes and two natural frequencies.

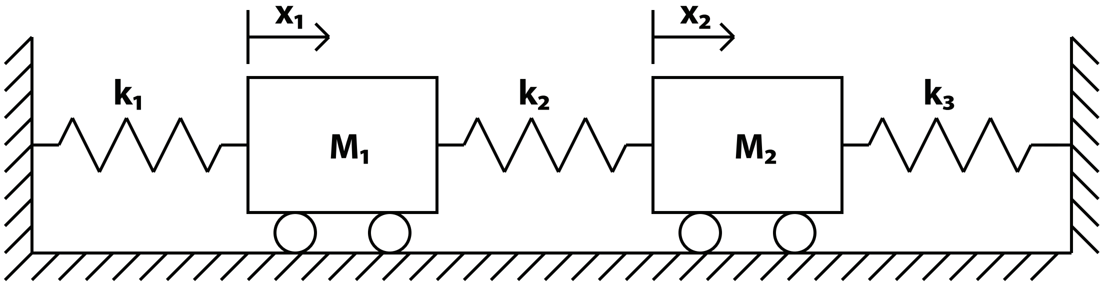


Figure 2.1: A simple two-degree-of-freedom system.

The primary goals of this laboratory are for you to learn the concept of normal modes in a two degrees-of-freedom system – the simplest system which exhibits such modes. You will learn this by experimentation and calculation.

PRE-LAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. Are the number of degrees of freedom of a system and the number of its *normal modes* related? Explain.
2. How can a normal mode be recognized physically?
3. What do you expect to happen when you drive a system at one of its *natural frequencies*?

4. Draw a free body diagram and derive the equations of motion for a three degrees-of-freedom system, with three different masses, four different springs, and no forcing. Put them in matrix form. *Your result should resemble equation (2.4) except your matrix will be* $3x3$ *and you will have no* $\underline{\mathbf{f}}(\mathbf{t})$ *term.*

5. Substitute the normal mode solution *(see (2.7))* into your matrix equation from (4) to get an eigenvalue problem *(see (2.5))*. How would the eigenvalues and eigenvectors of your matrix relate to the *mode shapes* and *natural frequencies*?

6. Using MATLAB, find the eigenvalues and eigenvectors of the following matrix and print the results (HINT: Type `help eig` for assistance).

$$[A] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{2.1}$$

NORMAL MODES

A *normal mode* is a special type of vibration what occurs when all of the points in the system are moving in simple harmonic motion. In addition, in a *normal mode* vibration all points move with the same angular frequency ω and are exactly in-phase or exactly out of phase. An example on the following page (See 2.2) illustrates a *normal mode* vibration for a two degrees-of-freedom-system. Note:

- Both masses are moving in simple harmonic motion. This is indicative of a normal mode vibration.
- The system has a period of $T = 4\pi$ (sec), and thus an angular frequency of $\omega = \frac{2\pi}{T} = \frac{1}{2}$.
- In this normal mode vibration, when one mass is at its maximum displacement, the other is at its minimum displacement - thus the masses are totally out of phase. There is another normal mode vibration for this system where the masses are moving in phase.

If we wanted to write out the equation of motion for this system, we would need a state vector $\underline{\mathbf{x}}(t)$ with two elements $x_1(t)$ and $x_2(t)$ - one to represent the position of each mass as a function of time. That equation might look something like this for our example normal mode vibration:

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \sin\left(\frac{1}{2}t\right) \tag{2.2}$$

Here, ω is the *natural frequency* of the *normal mode* (the same for all masses), and the vector $\underline{\mathbf{c}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is its *mode shape*. In this example, when x_1 is at it's maximum displacement to the left $c_1 = -2$, x_2 is at its maximum displacement to the right $c_2 = 2$. Here both masses have the same relative amplitude ($|c_1| = |c_2|$) *though in general that is not the case*, but are completely out of phase since c_1 has the opposite sign of c_2. Thus the *mode shape* $\underline{\mathbf{c}}$ tells you the relative amplitude of motion and phase of each mass by the relative magnitude and sign of its elements c_i.

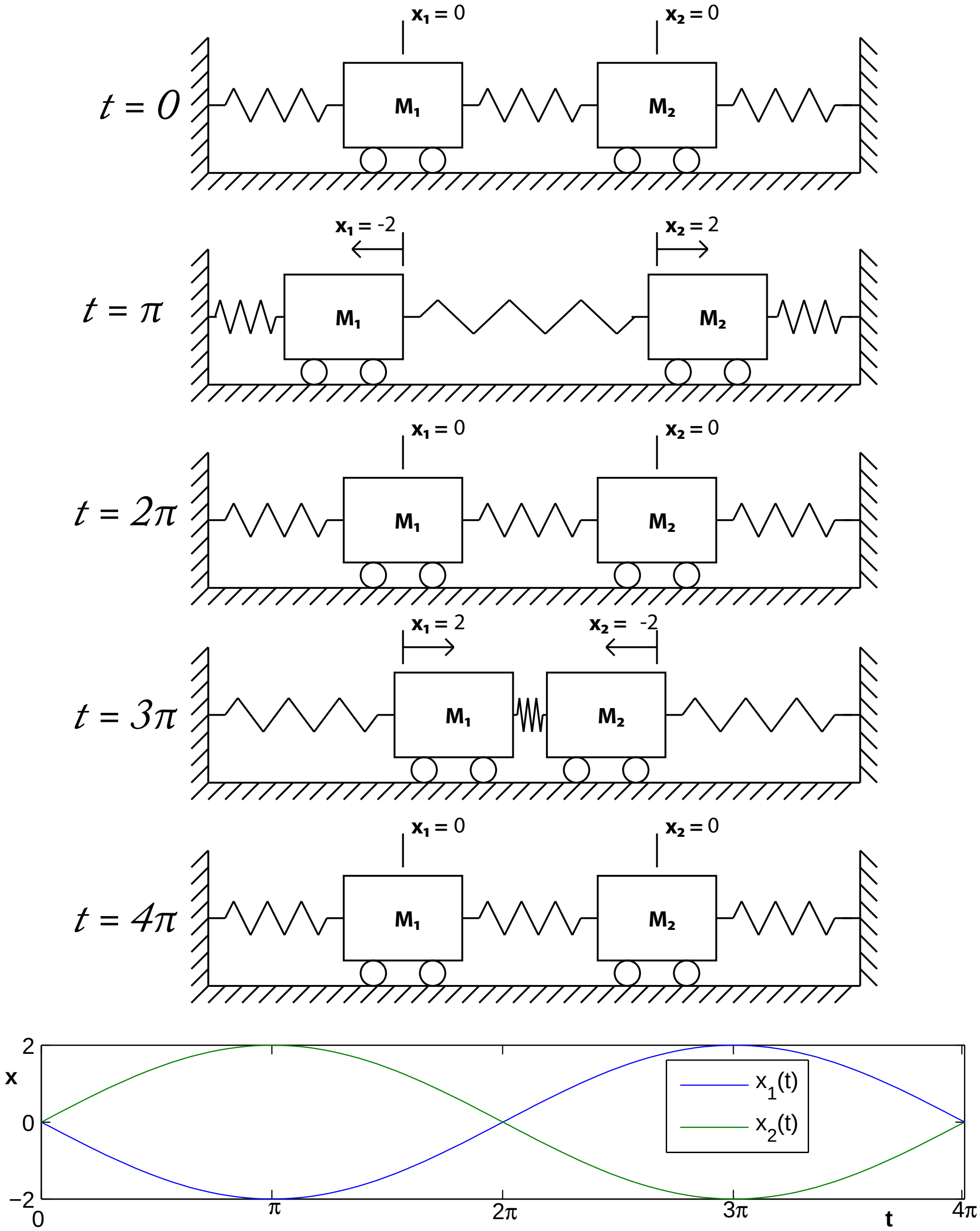


Figure 2.2: A normal mode vibration of a two-degree-of-freedom system.

DERIVING THE EQUATIONS OF MOTION

We will now derive the equations of motion for a driven two degrees-of-freedom system. The diagram and physical setup are shown in Figures 2.3 and 2.5.

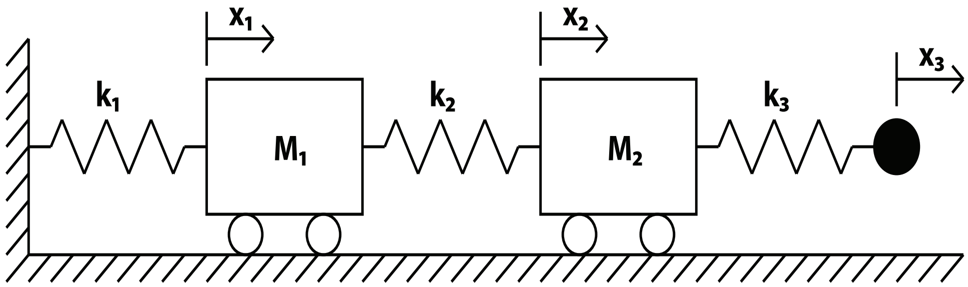


Figure 2.3: Illustration of a coupled mass-spring system.

Here, rather than having the rightmost spring attached to a fixed support, we have it attached to a sinusoidally driven support whose position is $x_3(t)$. Do not be fooled into thinking that x_3 counts as a degree of freedom - here we know how we are driving the system and so x_3 is a given. Look back over `Lab 1` if you are confused about this point - we use the same trick there to drive a one degree-of-freedom system. Now, we will draw the free-body diagram for each mass and work out its equation of motion. To help get the signs right, assume that

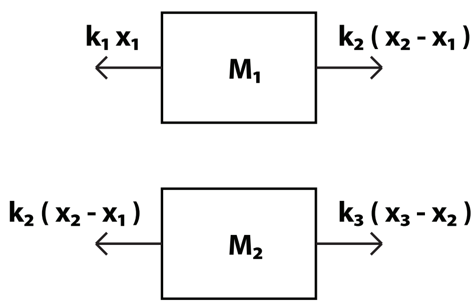


Figure 2.4: The free-body diagrams for masses m_1 and m_2.

the displacements are all positive (i.e. to the right) with $x_1 < x_2 < x_3$. This puts all of the springs into tension relative to their equilibrium condition. The equations of motion for each mass respectively are

$$k_2\left(x_2 - x_1\right) - k_1 x_1 = m_1 \ddot{x}_1 \tag{2.3a}$$

$$k_3\left(x_3 - x_2\right) - k_2\left(x_2 - x_1\right) = m_2 \ddot{x}_2 \tag{2.3b}$$

We can rewrite this in matrix form as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_3 x_3(t)}{m_2} \end{bmatrix} \tag{2.4}$$

or as

$$\underline{\ddot{\mathbf{x}}} = [A]\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t) \tag{2.5}$$

Where the matrix $[A]$ contains information about the system response to forcing and the vector $\underline{\mathbf{f}}(\mathrm{t})$ contains information about the external forcing.

SOLVING THE EQUATIONS OF MOTION USING NORMAL MODES

To make matters easier, let's consider the case where there is no external forcing, thus $\underline{\mathbf{f}}(\mathbf{t}) = 0$ and our equation of motion (2.5) reduces to:

$$\ddot{\underline{\mathbf{x}}} = [A]\,\underline{\mathbf{x}} \tag{2.6}$$

Now we'll look for the normal mode solutions of the system. Remember - a *normal mode* vibration is when both masses are moving in simple harmonic motion with the same angular frequency ω, but potentially different relative amplitudes of motion c_i. Before we gave an example of a normal mode solution. Here is the general form of a normal mode solution for a two degrees-of-freedom system:

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} (A\cos(\omega t) + B\sin(\omega t)) \tag{2.7}$$

Once again, ω is the *natural frequency* of the mode which tells you the angular frequency with which every mass vibrates, and $\underline{\mathbf{c}}$ is the *mode shape* which tells you the phase and relative amplitude of motion of each mass. If we plug in our *ansatz* for the solution (2.7) into the equation of motion (2.6), we can solve for the natural frequency and mode shape that will make the equation hold. Substituting (2.7) into (2.6) and canceling out cosine and sin terms yields

$$-\,\omega^2\underline{\mathbf{c}} = [A]\,\underline{\mathbf{c}} \tag{2.8}$$

It turns out that we have non-trivial solutions to (2.8) only for certain values of $-\omega^2$ and then only when $\underline{\mathbf{c}}$ is a multiple of a specific vector. Equation (2.8) is in the form of an "eigenvalue problem" from linear algebra, and these sets of solutions are called the "eigenvalues" (λ_i) and corresponding "eigenvectors" ($\hat{\lambda}_{\mathbf{i}}$) of the matrix $[A]$. These can easily be solved for by hand, or by using a computer algebra program such as `MATLAB` or `SciLab`. For an n degrees-of-freedom system there will be n such sets of "eigenvalues" and "eigenvectors".

Thus, a n degrees-of-freedom system has n *natural frequencies* and n *mode shapes* given by

$$\omega_i = \sqrt{-\lambda_i}; \qquad \underline{\mathbf{c}} = \hat{\lambda}_{\mathbf{i}} \tag{2.9}$$

where (λ_i) and ($\hat{\lambda}_{\mathbf{i}}$) are the eigenvalues and eigenvectors (respectively) of the matrix $[A]$.

The general motion of the system can then some combination of the normal modes. For a 2-degrees-of-freedom system, it would look like this:

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{c}}_{\mathbf{1}}(A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)) + \underline{\mathbf{c}}_{\mathbf{2}}(A_2\cos(\omega_2 t + \phi_2) + B_2\sin(\omega_2 t + \phi_2)) \tag{2.10}$$

Here the coefficients A_i and B_i would depend on the initial conditions and tell the amplitude of vibration - note that the mode shape $\underline{\mathbf{c}}$ only tells the *relative* amplitude of vibration of each mass, not the overall magnitude of the system vibration. In the language of linear algebra, we say that the *normal modes* span the space of possible solutions.

LABORATORY SET-UP

- **Air Track**
  The lab set-up consist of an air-track hooked up to the lab's air system, four or more air track gliders, four plug-in springs, a mechanical oscillator (for external forcing), a photogate timer, and a digital stopwatch. Please note that there are two somewhat incompatible styles of glider which should only be used on the appropriate air tracks. Each glider has a label listing its approximate mass (including spring) and the air tracks on which it will work. **You should remeasure the masses of the gliders and springs at the start of your lab.**

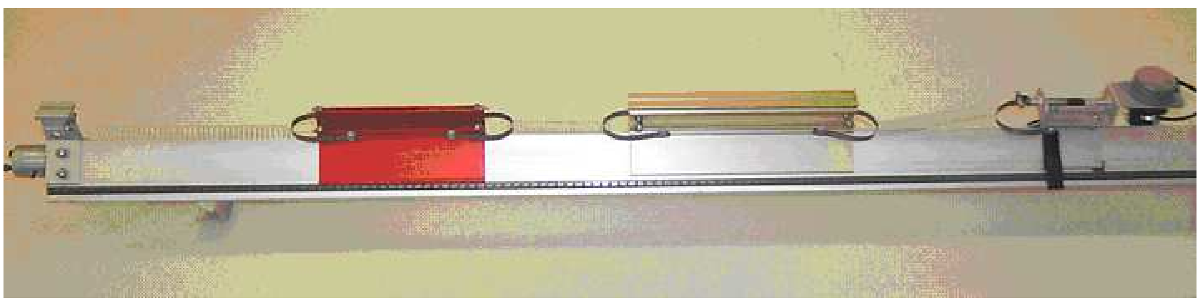


Figure 2.5: The laboratory set-up you will be working with.

- **Using the *SciLab* Software**

  To open *SciLab* click on its icon located on the desktop of your computer. This program is a freeware program similar to *MatLab* and should look quite similar.

  To find the eigenvalues and eigenvectors of a matrix you must use the function `spec()` as shown below. Input the matrix $[A]$, call `spec()` on it, and the program will return the eigenvalues along the diagonal of a square matrix and the eigenvectors as the columns of the second returned matrix (just like `eig()` in MATLAB).

PROCEDURE

1. Play with the air track, gliders, and timer. Adjust the mechanical driver left or right so that each spring, at equilibrium, has a total length of about 20 cm (the driver is attached with a Velcro strap). Have the TA turn on the main air supply, if it is not already on, and turn on the valve at the end of the air track.

2. Measure the mass of each of your carts, and one of your springs (you may assume that all springs have the same mass).

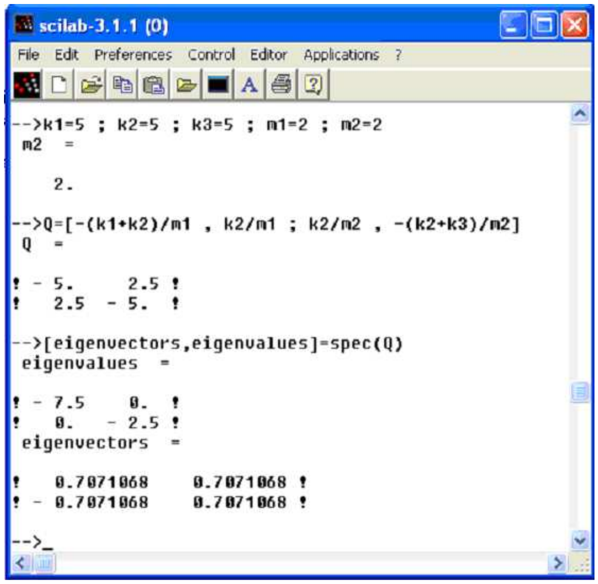


Figure 2.6: Screenshot of *Scilab* in use.

3. Next you will find the spring constant for your springs.
   - Attach a small weight (40 to 50 grams) to one end of a spring and hold the other end solidly against the tabletop.
   - Pull the weight down a few centimeters and release it, and then measure the period of oscillation (*average over 10 periods*). Use $\omega = \sqrt{\frac{k}{m}}$ to find k. Remember to include part of the spring as well as the plug mass in "m" – half is a good approximation in this case. Also remember to convert your period T to angular frequency ω.
   - Repeat this calculation for each spring. *Note: there will be variability in k from spring to spring.*
4. Choose two gliders of different sizes to work with, and set them up on the track.
   - Using the measured masses and spring constants, calculate the *mode shapes* and *natural frequencies* of the system - *see (2.9)*. You may do the calculations by hand or use *SciLab* on the computer. Remember to add the the mass of a plug-in spring when calculating the cart mass.

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5. The system is set into a normal mode oscillation by applying the appropriate initial conditions. Choose one of your normal modes to use.
   - First, place the system in equilibrium. One simple method is to turn the air track on and off repeatedly until the gliders stop moving.

- With the air off, displace m_1 by an amount $B*c_1$cm, and displace m_2 by $B*c_2$cm, where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is the appropriate "eigenvector" or *mode shape*, and B is an arbitrary constant that sets the scale of the vibrations. Pick B large enough that neither mass is displaced by less than 1 cm, but small enough that neither mass slides on the track with the air off.
- Turn on the air track valve abruptly. The system should oscillate in the normal mode which your chose.
- Measure the angular frequency (ω) of oscillation, and verify that it is approximately equal to the natural frequency calculated in `SciLab`. The angular frequency of the masses is found by timing a number of oscillations (i.e. 10) and then converting the resulting period to ω (which has units of rad/sec). Digital stopwatches are available at the air track.
- Note the phase difference between the two masses - are they in phase or out of phase?
- Repeat the above procedure for the other normal mode.

6. Use some arbitrary initial conditions and set the system into a non-normal mode oscillation. Observe the motion. (It should be difficult to see that it is the sum of normal mode vibrations.)

7. Next we will attempt to obtain normal mode vibrations by driving the system at each natural frequency - the driving frequency of oscillation is obtained by timing the motion of the driving rod connected to the motor, using either a stopwatch or a photogate timer.

   - Choose one of the normal modes you have calculated. With the air off, set the driving frequency to the corresponding natural frequency which you have calculated. Now turn on the air.
   - Does the system resonate when you turn the air on? Be patient, it may take a try or two to get resonance. Start the system from rest every time you change the motor speed.
   - When you get resonance, measure the frequency of oscillation and compare it to the natural frequency which you calculated in SciLab. Note, as best you can, the relative phase between the scotch yoke (driving) and the masses (response) at resonance.
   - Repeat this procedure for the other normal mode.

8. Drive the system at a frequency close to (but not equal to) one of the natural frequencies. Is the amplitude of motion of either mass constant? Now drive the system at a frequency much higher than either natural frequency. Note the amplitude of motion of each mass.

9. Set up the air track with three (approximately) equal masses and four (approximately) equal springs. Adjust the mechanical oscillator to give an equilibrium spring length of about 20 cm. Verify by observation that $[1 \quad -1.414 \quad 1]^T$ is approximately a normal mode for this system.

10. Find another normal mode for this system by observation/experimentation. Find another still. If you are having difficulty finding modes experimentally, ask your TA for help. Are there any more? Now use *SciLab* to find the normal modes and natural frequencies.

LAB REPORT QUESTIONS

1. List your values of k for the springs and a sample calculation. What is the average value of k, and what was the largest variation from the average (in percent)?

2. List your mode shapes and natural frequencies calculated in (4). Did you obtain normal mode oscillations using initial conditions based on your eigenvectors? How could you tell? How close were the natural frequencies what you measured in (5) to the ones you calculated in (4)? Was the phase correct?

3. Describe what you observed when you forced the system at the following 3 frequencies:

   (a) forcing frequency = a natural frequency

   (b) forcing frequency close to a natural frequency (Was the amplitude of the oscillations constant in this case? If not, how did it vary?)

   (c) forcing frequency far from a natural frequency.

4. How many normal modes are there in the three equal mass system (*theoretically*) and what are they? How many were you able to find experimentally and how did you recognize them as normal modes? How do the modes you found compare with those you calculated?

CALCULATIONS & NOTES

APPENDIX: SOLVING THE EQUATIONS OF MOTION VIA A CHANGE OF BASIS

So far we have discussed how normal modes are the simplest oscillatory functions from which *all* motions of the two degrees-of-freedom system can be thought to be comprised of. Mathematically, the normal modes y_1 and y_2 satisfy the equations of motion for simple harmonic oscillators with natural frequencies ω_1 and ω_2 respectively.

$$\ddot{y}_1 + \omega_1^2 y_1 = 0 \tag{2.11a}$$

$$\ddot{y}_2 + \omega_2^2 y_2 = 0 \tag{2.11b}$$

Since the equations of motion for the normal modes are simple in terms of the y_1, y_2 coordinates, it would be nice if we could find some transformation between the physical coordinates x_1, x_2 and these new variables, i.e. $\underline{\mathbf{x}} = f(\underline{\mathbf{y}})$, so that we can solve the problem in terms of the easier coordinates and then transform back into the original ones. We can accomplish this mathematically by performing a change-of-basis from the original basis into the *eigenbasis* of $[A]$. We define our new normal mode coordinates by

$$\underline{\mathbf{x}} = [S]\,\underline{\mathbf{y}} \tag{2.12}$$

where the change-of-basis matrix $[S]$ is defined as

$$[S] = \begin{bmatrix} \underline{\mathbf{v_1}} & \underline{\mathbf{v_2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.13}$$

Plugging this change of variables into (2.5) we get the new equation

$$[S]\,\ddot{\underline{\mathbf{y}}} = [A]\,[S]\,\underline{\mathbf{y}} + \underline{\mathbf{f}}\,(t) \tag{2.14}$$

Left-multiplying both sides by $[S^{-1}]$ gives us

$$\ddot{\underline{\mathbf{y}}} = \left[S^{-1}\right][A]\,[S]\,\underline{\mathbf{y}} + \left[S^{-1}\right]\underline{\mathbf{f}}\,(t) = [\Lambda]\,\underline{\mathbf{y}} + \underline{\tilde{\mathbf{f}}}(t) \tag{2.15}$$

where

$$[\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & -3k \end{bmatrix} \tag{2.16}$$

Looking at the unforced case, $\underline{\tilde{\mathbf{f}}}(t) = \underline{\mathbf{0}}$, we see from (2.15) that in the new normal mode coordinates we now have two uncoupled second-order ODEs,

$$\ddot{y}_1 + k y_1 = 0 \tag{2.17a}$$

$$\ddot{y}_2 + 3k y_2 = 0 \tag{2.17b}$$

the solutions of which are

$$y_1 = A_1 \cos\sqrt{k}t + B_1 \sin\sqrt{k}t \tag{2.18a}$$

$$y_2 = A_2 \cos\sqrt{3k}t + B_2 \sin\sqrt{3k}t \tag{2.18b}$$

Using (2.12) we can now transform back into the original x_1, x_2 coordinates giving

$$\begin{aligned} \underline{\mathbf{x}} &= [S]\,\underline{\mathbf{y}} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} = \\ & \underline{\mathbf{v_1}}\left(A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)\right) + \underline{\mathbf{v_2}}\left(A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t)\right) \end{aligned} \tag{2.19}$$

where we have substituted $\omega_1 = \sqrt{k}$ and $\omega_2 = \sqrt{3k}$. This is the same result we found before in (2.10), so you might not think much was gained by performing this change-of-basis. However, the real advantage of this method appears when we consider the forced case.

APPENDIX: FORCED TWO-DEGREE-OF-FREEDOM SYSTEM

We now reconsider equation (2.15) when $\underline{\tilde{\mathbf{f}}}(t) \neq \underline{\mathbf{0}}$.

$$\underline{\ddot{\mathbf{y}}} = [\Lambda]\,\underline{\mathbf{y}} + \underline{\tilde{\mathbf{f}}}(t) \tag{2.20}$$

The two resulting equations are

$$\ddot{y}_1 + \omega_1^2 y_1 = \frac{k x_3}{2m_1} \tag{2.21a}$$

$$\ddot{y}_2 + \omega_2^2 y_2 = -\frac{k x_3}{2m_1} \tag{2.21b}$$

where $x_3(t) = F\cos\omega t$ and ω is the forcing frequency. Solving both of these non-homogeneous second-order ODEs yields

$$y_1(t) = A_1\cos\omega_1 t + B_1\sin\omega_1 t - \frac{Fk}{2m_1}\left(\frac{1}{\omega^2 - \omega_1^2}\right)\cos\omega t \tag{2.22a}$$

$$y_2(t) = A_2\cos\omega_2 t + B_2\sin\omega_2 t + \frac{Fk}{2m_1}\left(\frac{1}{\omega^2 - \omega_2^2}\right)\cos\omega t \tag{2.22b}$$

Once again we use (2.12) to transform back into the original coordinates to get

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x_c}}(t) + \frac{Fk}{2m_1}\begin{bmatrix} \frac{1}{\omega^2-\omega_2^2} - \frac{1}{\omega^2-\omega_1^2} \\ -\frac{1}{\omega^2-\omega_2^2} - \frac{1}{\omega^2-\omega_1^2} \end{bmatrix}\cos\omega t \tag{2.23}$$

where we have suppressed the homogeneous (or complementary) part of the solution. We note that the particular solution becomes unbounded as the forcing frequency approaches either $\omega = \omega_1$ or $\omega = \omega_2$. In other words, *resonance* occurs when we force the two degrees-of-freedom system at one of the normal modes' natural frequencies. (Obviously the oscillations you will observe in the lab will not be unbounded as the lab set-up is not entirely frictionless.)

We now rewrite the particular solution as

$$\underline{\mathbf{x_p}}(t) = \frac{F}{2}\begin{bmatrix} \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2-3} - \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2-1} \\ -\frac{1}{\left(\frac{\omega}{\omega_1}\right)^2-3} - \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2-1} \end{bmatrix}\cos\omega t \tag{2.24}$$

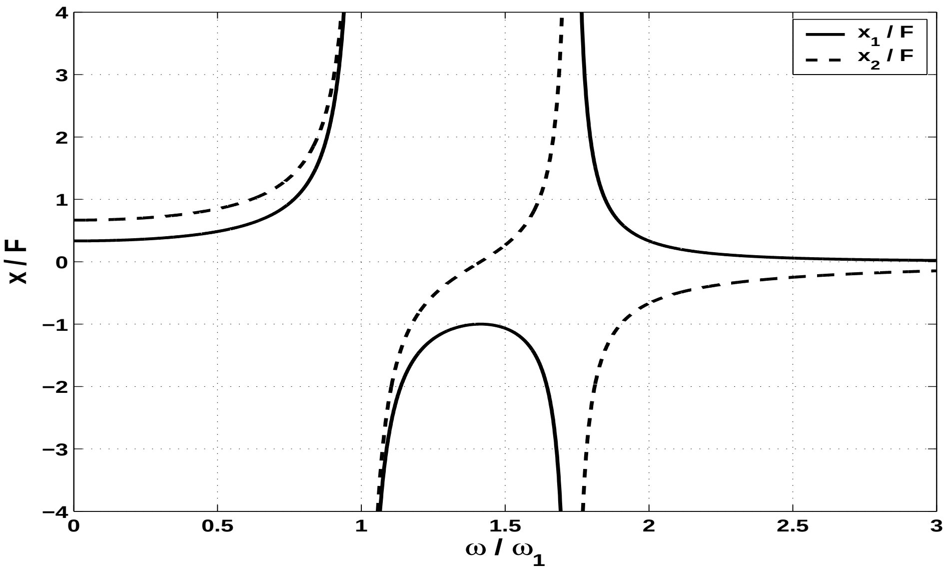

Figure 2.7: Plot of the response amplitude to forcing amplitude ratio for the forced two degrees-of-freedom system.

where we have written it in terms of the ratio of the forcing frequency to the smaller normal mode frequency ω_1. Figure 2.7 graphically shows how the amplitudes of the particular (or steady-state) solutions change as the forcing frequency ω is varied.

The plot graphically illustrates what we found earlier – that when the forcing frequency is near the natural frequency of a normal mode, that mode resonates. As $\omega \to \omega_1$ the two masses move in-phase and when $\omega \to \omega_2$ the masses move out-of-phase.

Lab #3 - Slider-Crank Lab

Last Updated: March 4, 2009

INTRODUCTION

In this laboratory we will investigate the kinematics of some simple mechanisms used to convert rotary motion into oscillating linear motion and vice-versa. In *kinematics* we use geometry and calculus to study the motion of objects without consideration of the forces which produce those motions. The first machine we will study is the slider-crank - a mechanism widely used in engines to convert the linear thrust of the pistons into the useful rotary motion of the drive-shaft. In this lab you will find that the linear acceleration of the piston of a lawn mower engine is a function of the rotation rate of the drive shaft and the engine's geometry. This result exemplifies the simple relation between speed and acceleration for kinematically restricted motions. An adjustable slider-crank apparatus and a computer simulation will show you some effects of changing the proportions of the slider-crank mechanism on piston velocity and acceleration. Other linkages and cam mechanisms may also be used for linear-rotary motion conversion and some of these will be included in the lab.

Once an object's acceleration, $\underline{\mathbf{a}}$, has been determined through kinematics, linear momentum balance allows us to find the net force acting on the object ($\underline{\mathbf{F}} = m\underline{\mathbf{a}}$). Knowledge of these forces is crucial if one is to choose the right material, proportions, and operating conditions for a new design.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. What data will you collect from the lawn-mower engine and what will you simulate on the computer?
2. Which parameter(s) can be adjusted on the adjustable slider-crank? Which are fixed?
3. Derive an equation relating the piston displacement x to the crankshaft speed, ω, time, t, connecting rod length, L, and crank radius R. *Do not assume* $L >> R$.

SLIDER-CRANK KINEMATICS & INTERNAL COMBUSTION ENGINES

Figure 3.1 shows a sketch of the slider-crank mechanism. The point A is on the piston, line AB (with length L) is the *connecting rod*, line BC (with length R) is the crank, and point C is on the *crankshaft*. In an engine, a mixture of gasoline and air in the cylinder is ignited in an exothermic (heat producing) reaction. As a result, the pressure in the cylinder rises, forcing the piston out. The force transmitted through the connecting rod has a moment about the center of the crankshaft, causing the shaft to rotate. An exhaust valve releases the gas pressure once the piston is extended. Inertia of machinery (often a flywheel) connected to the crankshaft (as well as forcing from other pistons in multi-cylinder engines) forces the piston back up the cylinder. In a standard "four-cycle" engine the crankshaft makes another full revolution before another ignition (to bring in fresh air and compress it before ignition).

The *kinematic constraint* comes about from the geometry of the engine. If we know the angle through which the crank arm has rotated, $\theta = \omega t$, then we can determine the piston displacement, x. The crank arm R, connecting rod L, and piston displacement x, form a triangle containing the angle $\theta = \omega t$. Using this triangle we can find our kinematic constraint $x = f(\theta(t); R, L)$ with basic geometry. We can then take derivatives to find a relationship between the piston's velocity or acceleration and the crankshaft's spin rate ω and engine geometry R, L.

Notice that if the connecting rod L is much longer then the crank arm R then it will remain close to parallel to the x-direction throughout the stroke. The x displacement of the piston will then be the x-displacement of the end of the crank arm plus a fixed length L. The former we know from circular motion, giving us a simple kinematic constraint:

$$x(t) \simeq Rcos(\omega t) + L; \qquad (L >> R) \tag{3.1a}$$

$$v(t) \simeq -R\omega sin(\omega t); \qquad (L >> R) \tag{3.1b}$$

$$a(t) \simeq -R\omega^2 cos(\omega t); \qquad (L >> R) \tag{3.1c}$$

The kinematic constraint is a little bit more complicated if we do not assume $L >> R$. You will work it out in the Pre-Lab.

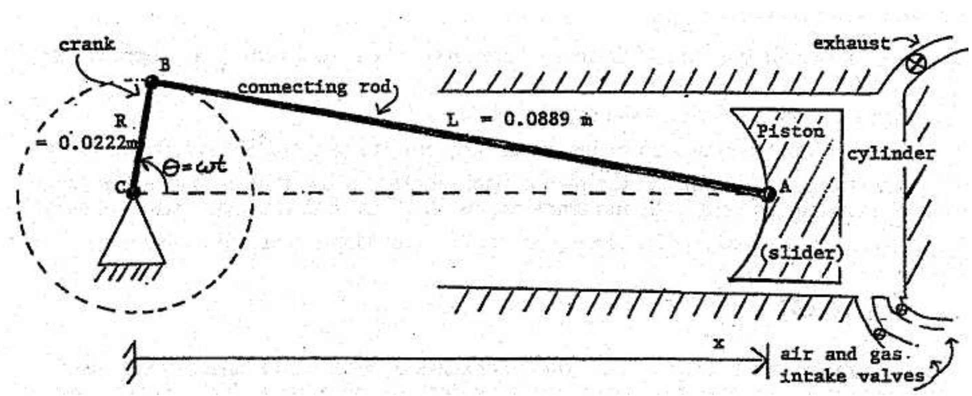

Figure 3.1: A diagram of the slider-crank system.

In this experiment the crankshaft is driven by an electric motor at a more or less constant rate, ω. This in turn, drives the piston. The same motion results as when the combustion process takes place in the piston, driving the crankshaft at a more or less constant rate. As the crankshaft rotates the piston moves in the positive and negative x direction. The basic measurements in this lab are the position and velocity of the piston in the x direction (which happens to be vertical in the laboratory). These measurements can be compared to those calculated by hand (if you are energetic) or to the results of a computer simulation. The simulation and the adjustable crank will allow you to see some of the effects of varying the ratio of connecting rod length L to crank length R.

LABORATORY SET-UP

A stripped-down lawn mower engine is driven by a variable-speed electric motor. Sensors are installed on the engine's piston to measure displacement and velocity. A data acquisition program is used to measure, analyze, and record the piston data. Look at the engine and see how its various parts fit together. It may help to look at Figure 3.1 and at the various demonstration slider-cranks present in the dynamics laboratory. Identify the piston, connecting rod, and crankshaft (the connecting rod won't be visible at your lab set-up, but you can see it in the demonstration slider-cranks). The cylinder head has been removed, exposing the top of the piston and allowing sensors to be attached.

The speed and direction of the electric motor are controlled by a knob and switch on the motor controller. The numbers on the speed controller are arbitrary; do not write them down as r.p.m. or radians per second (instead obtain angular velocity information from the data acquisition program). *Does the direction of motor rotation affect the slider-crank kinematics?*

The displacement and velocity data are measured using a LVDT and velocity transducer. Acceleration is calculated by the computer through numerical differentiation of the velocity data. This process magnifies any noise in the data. The computer also measures and displays the angular frequency by timing successive crossings of the zero line and converting to radians per second. The displacement, velocity, and acceleration are all plotted in *LabView* along with their minimum and maximum values (see Figure 3.2). A simple simulation program lets you compare your data to theoretical values and look at the effects of different slider-crank geometries.

Please follow safety precautions. The electric motor driving the lawn mower engine is powerful enough to cause serious injury if you get in its way. Keep long hair and loose clothing well away from the belt and pulleys at the back of the engine. If you need to touch the pulley, piston, or LVDT for some reason, check first that the electric motor power is off and that the speed control is set to zero. Make sure your lab partner knows what you are doing.

Using the LabView software

1. To run the software, open up the *Engrd203Lab* account and then open the folder *Crank* on the desktop. Open the program *Crank*. As soon as the program is running, it will ask you to move the piston to the top of its travel. Press `Ready` after you have done this and wait until the next pop-up comes before moving the piston again. Then once prompted move the piston to the bottom of its travel and press `Ready` again and

allow the computer a few seconds to calibrate. This calibration procedure allows the computer to convert the output of the LVDT (in volts) into displacement (in meters). Do this carefully. It may help to rock the pulley back and forth slightly as you try to home in on the highest (or lowest) piston position. If you make a mistake, you can redo the procedure by clicking on the `SET-UP` button. The *Crank* program has a box for the initials of your lab group. Click on the box with the mouse, type your initials, and then press the `Enter` key, not the `Return` key. Your initials will then appear on your plots, making it easier to identify them as they emerge from the laser printer.

2. When the data acquisition "switch" on the screen is turned on, the computer acquires and displays a new set of data every ten seconds or so. Allow ten or twenty seconds for the data plot to stabilize after changing the motor speed. If you have a plot that you want to keep, turn the data acquisition off. Also turn the motor off promptly when you are not acquiring data to save wear and tear on the lab set-ups and on the nerves of other students.

   The legend and scale factors for the plots are displayed in the top left corner. Multiply the y-axis reading (between -1 and 1) by the appropriate scale factor to obtain the actual measured value, in the units given in the legend. For example, if the velocity plot has a y-value of 0.5 at a particular time, and a scale factor of 4 m/s, the measured velocity at that time would then be 0.5*4 m/s = 2 m/s.

3. The `SAVE` button stores your data on the hard disk. The file created in this way can only be used by the simulation program (*CrankSim2*).

4. To exit from the program, click the "close" box in the top right corner of the window. To leave *LabView* completely, at any time, pull down the `File` menu and select `Quit`. If the program tells you that "Quitting now will stop all active VIs" select `OK`.

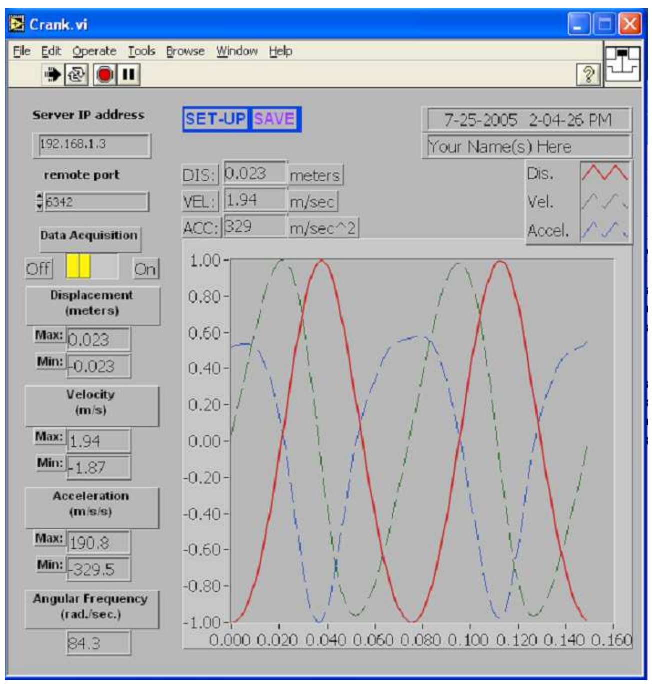

Figure 3.2: Using the *LabView Crank* program.

PROCEDURE

You will record and analyze $x(t)$, $v(t)$, and $a(t)$ while spinning the lawn mower engine at various speeds.

1. Check that the electric motor power switch is off, the speed control knob is at zero, and the data acquisition is on. Twist the pulley back and forth by hand and look at the resulting plot of piston position, velocity, and acceleration. If the piston moves upwards, in what direction does the plotted curve move? *(i.e. how is the coordinate system defined for our system?)* You will need to wait several seconds for the data to be displayed.

2. Put a penny on top of the piston, turn on the motor, and adjust the motor speed so that the penny just barely starts to bounce on top of the piston. You should be able to hear a faint clinking sound. Wait until you have a good graph of the data and then turn off first the data acquisition and then the motor. Record the angular velocity and the minimum and maximum values for the displacement, the velocity, and the acceleration in a table. There is no need to print up your data, you will be able to examine it in `CrankSim2`. Check that the displacement plot makes sense, given that the crank length is known to be 0.0222 m. Check that the acceleration data makes sense - *what should the acceleration be when the penny begins to leave the top of the piston? Given the coordinate system for our engine should that be the maximum or minimum value of the acceleration for this data?*

3. Remove the penny and repeat the procedure above for at least four additional speeds. Try to get as wide a variety of speeds as possible. At very slow speeds the motor does not turn smoothly and the data is drowned out by noise. When using very high speeds, try to acquire data quickly, turn off the data acquisition "switch", and shut the motor off immediately. Record your data in a table (including the penny data).

You will now simulate the slider-crank mechanism on the computer. The *CrankSim2* program (Figure 3.3) will be used to compare the theoretical values for displacement, velocity, and acceleration with the values measured above. The effects of changing the crank length R, connecting rod length L, and angular velocity ω of the crankshaft may also be observed.

1. To start up the simulation program double-click on *CrankSim2* in the *Crank Lab* folder. If you want to compare your simulation to your most recently saved data, turn the measured-data "switch" on; otherwise, turn it off to eliminate the clutter of all the extra graphs. Described below are the parameters you can change in the simulation:

   - R is the crank length in meters.
   - L is the connecting rod length in meters.
   - ω is the angular velocity of the drive shaft in radians per second.

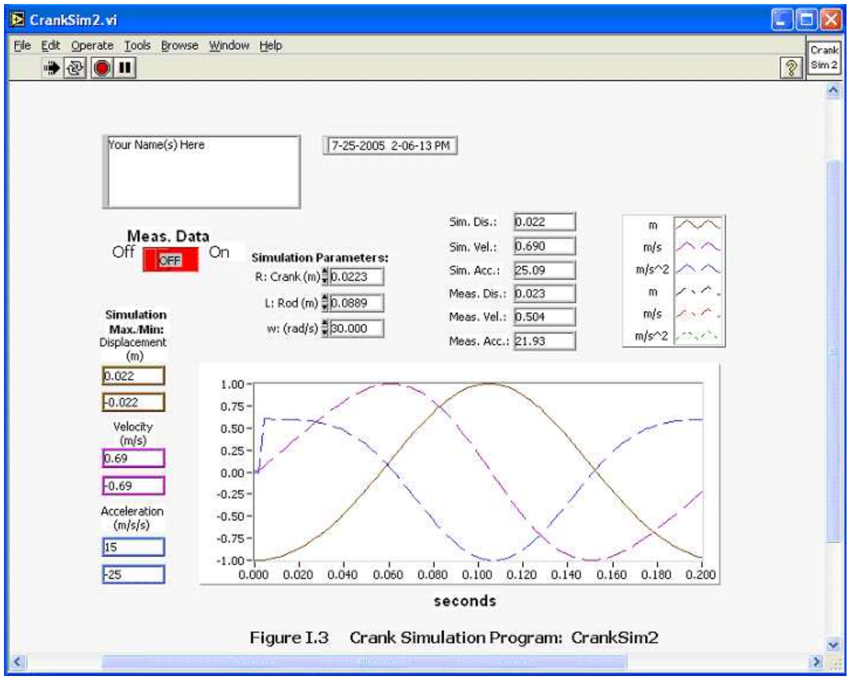

Figure 3.3: Using the *LabView CrankSim2* program.

As with the data acquisition program, the maximum and minimum values are displayed. These are the simulation maxima and minima. Note that the displacement shown is the value x in Figure 3.1 minus the connecting rod length L. This makes it more easily comparable to the measured data. The $x = 0$ point is thus defined to be halfway between the piston's top and bottom positions instead of at the center of the crankshaft.

2. Set up the simulation with the crank length and connecting rod length of the lawn mower engine. Enter the angular velocity from one of your previously saved data sets (the penny data) and turn the `Measured Data` switch on. **Make a printout of your data.**

3. Switch off the measured-data curve. Now simulate slider-cranks with different geometries by varying the crank length R and the connecting rod length L. Observe and record velocities and accelerations for the following cases. **Please make print-outs to support your observations and conclusions.**

   - L is much greater than R - i.e. L of 10 m, and R of 0.0223 m
   - R is increased, but still much smaller than L - i.e. L of 10 m and R of 0.223 m
   - L is decreased, but still much larger than R - i.e. L of 1 m and R of 0.0223 m
   - R and L equal the values for the lawn mower engine - i.e. L of 0.089 m and R of 0.0223
   - L is only slightly greater than R - i.e. L of 0.0224 m, and R of 0.0223 m *What happens physically when R is greater than L?*

Next you will work with the adjustable slider-crank. This device allows you to adjust the crank length to connecting rod ratio $\frac{R}{L}$ from zero to slightly more than one, using an adjustment knob which changes the effective crank length. A handle is located underneath to rotate the apparatus by hand. **Please be gentle with it!** Large forces can be generated with even a small input torque when the ratio is close to 1. If you see things bending, back off. When turning the hand crank, do it slowly. You can also push and pull on the masses at the end of the "piston" to look at the way it converts linear to rotary motion. Be sure you can identify the crank, connecting rod, and piston on the adjustable crank apparatus as first appearances may be misleading. Here is a hint: the long thin rod with a weight on each end is the piston. Compare the shapes of the curves you saw in the simulation above to what you observe and feel with the adjustable crank.

The slider-crank is just one of many devices that have been invented to convert linear to rotational motion or vice-versa. The scotch yoke, the cam, and the four-bar linkage are some others.

1. Look over the scotch yoke mechanism, which is driven by an electric motor and gearbox. Try it at different speeds and (with the motor off) push and pull on its various parts. Rotate the pulley by hand while watching the motion of the rod. Take measurements or make a drawing if you wish. Be prepared to find a kinematical equation relating disk rotation to yoke displacement and think about the advantages and disadvantages of the scotch yoke relative to the slider-crank.

2. Cam-and-follower mechanisms are a particularly versatile way to convert rotary to linear motion because you can select the type of motion you want by changing the shape of the cam. For example, cams are used in an internal combustion engine to open and close the intake and exhaust valves. Cam shapes are chosen to optimize fuel economy, power, and emission control. The cam in this lab is a simple eccentric disk - i.e. a circle rotating about a point other than its center. Try out the cam mechanism by turning it with your hand. Feel the output from the follower as the cam is rotated and then try rotating the cam by pushing and pulling on the follower.

LAB REPORT QUESTIONS

1. Plot peak piston acceleration vs. crankshaft angular velocity on linear and log-log paper. From these graphs find an appropriate equation relating the two variables. Does this equation make sense in terms of the theory? Explain.

2. How does the peak piston velocity depend on the angular velocity of the crankshaft? Plot your experimental data and find an appropriate formula relating the two variables from your graph. Does this equation make sense in terms of the theory? Explain.

3. Examine the plot of your penny data from `Part 2` of the procedure which compares the measured data and the corresponding simulation data. What explanations can you give of the similarities or differences in the graphs? *Note: kinematic constraints arise from geometry not forces and so "friction is not accounted for" is not an adequate explanation.*

4. From your experimental data, what is the crankshaft angular velocity for which an ant standing on the top of the piston would start to need sticky feet in order to not lose contact with the piston? Explain.

5. Using your simulation data, how does the length of the connecting rod, relative to the crank length, affect the shape of the displacement, velocity, and acceleration curves?

6. The lawn mower engine piston weighs 0.175 kg. Suppose that the net force on the piston should not to exceed 10 kN. Using the equation you found in `Part 1` of the lab report, what is the maximum crankshaft angular velocity for which the engine can safely run.

7. Argue for or against the following point: for all slider-cranks the peak velocity occurs exactly at the midpoint of the stroke. Back up your arguments with either your print-outs for `Part 3` of the procedure, or any other appropriate analysis and logic. Make sure you consider the case when ($L \simeq R$).

8. For the scotch yoke, work out the equation relating rotation of the pulley to linear motion of the rod. You will need to draw a picture and label your variables.

9. Why is the slider-crank, and not a scotch yoke, used in an engine? Also, what special advantages does the scotch yoke have in some applications?

10. How does the cam-follower mechanism you saw in lab compare kinematically to the scotch yoke? What reasons might a designer have for choosing one over the other?

CALCULATIONS & NOTES

Lab #4 - Gyroscopic Motion of a Rigid Body

Last Updated: May 25, 2009

INTRODUCTION

Gyroscope is a word used to describe a rigid body, that is spinning quickly about an axis about which it is symmetric. In other words it has a large angular velocity, ω, about a symmetry axis. Some examples are a flywheel, symmetric top, football, navigational gyroscopes, and the Earth. The gyroscope differs in some significant ways from the linear one and two degrees-of-freedom systems with which you have experimented so far. The governing equations are 3-dimensional equations of motion and thus mathematical analysis of the gyroscope involves use of 3-dimensional geometry. The governing equations for the general motion of a gyroscope are non-linear. Non-linear equations are in general hard (or impossible) to solve. In this laboratory you will experiment with some simple motions of a simple gyroscope. The purpose of the lab is for you to learn the relation between applied moment, angular momentum, and rate of change of angular momentum. You will learn this relation qualitatively by moving and feeling the gyroscope with your hands and quantitatively by experiments on the precession of the spin axis.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. What is a gyroscope?
2. Where is the fixed point of the lab gyroscope?
3. How will moments (torques) be applied to the lab gyroscope?
4. For a fixed applied moment, will increasing a gyroscope's spin rate, ω, increase or decrease its precession rate, $\dot{\phi}$.

THE GYROSCOPE

Our experiment uses a rotating sphere mounted on an air bearing (see Figure 4.3) so that the center of the sphere remains fixed in space (at least relative to the laboratory room). This is called a *gyroscope with one fixed point*.

As the gyroscope rotates about its spin axis it is basically stable. That is, the spin axis remains pointing in the same direction in space. As you should see in the experiment, the larger the spin rate the larger the applied moment needed to change the direction of the spin axis. When a moment is applied to a gyroscope, the spin axis will itself rotate about a new axis which is perpendicular to both the spin axis and to the axis of the applied moment. This motion of the spin axis is called *precession*, and comes from Angular Momentum Balance: $\sum \overrightarrow{M_{/o}} = \dot{\overrightarrow{H_{/o}}}$

DYNAMICS OF THE SYMMETRIC TOP

A common *gyroscope with one fixed point* which is analogous to our lab setup is a symmetric top acting under the influence of gravity. Imagine that the tip of the top can't move. We'll label that point on the gyroscope O and call it the fixed point of the gyroscope since it's location is fixed in space (see Figure (4.1)). Every other point on top will move as the top spins and wobbles (i.e. precesses), so there is only one fixed point. In this section, we will motivate an equation governing the precession rate. A detailed derivation requires the use of rotating coordinate frames.

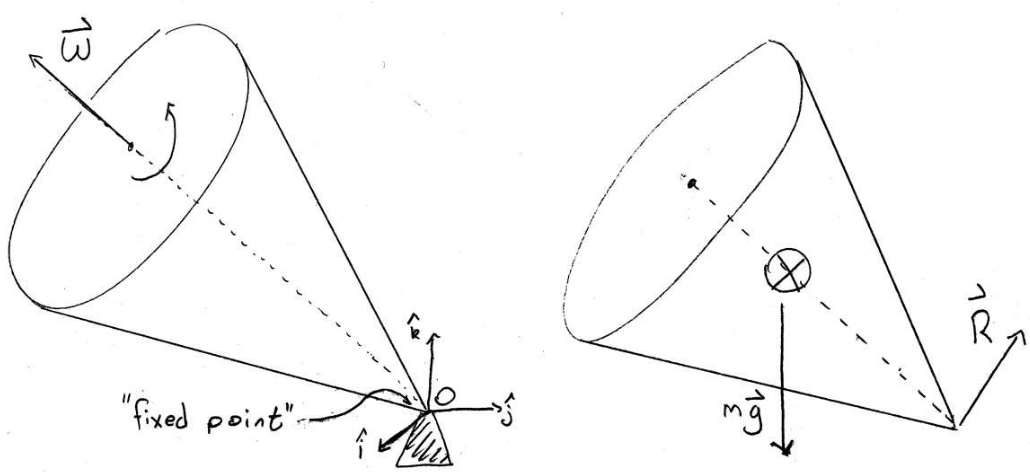

Figure 4.1: A symmetric top with one fixed point

Before we begin, let's remind ourselves how angular velocity vectors work - the direction of the vector gives the axis about which we are spinning, and the magnitude of the vector gives the spin rate. Note from Figure (4.1) that $\underline{\omega}$ points in the same direction as $\underline{\mathbf{r}}_{\mathbf{cm/o}}$, since the top is spinning around its symmetry axis. Now we're ready to 'solve' the equations of motion. Since we don't want to solve for the reaction force, $\underline{\mathbf{R}}$, at the point of contact O we'll use angular momentum balance about that point to find the equations of motion.

$$\sum \underline{\mathbf{M}}_{/\mathbf{o}} = \underline{\dot{\mathbf{H}}}_{/\mathbf{o}} \tag{4.1}$$

Calculating the left hand side of (4.1) is easy since there is just one moment, the one due to gravity:

$$\underline{\mathbf{M}}_{/\mathbf{o}} = \underline{\mathbf{r}}_{\mathbf{cm/o}} \times -mg\hat{\mathbf{k}} \tag{4.2}$$

As for the right hand side of (4.1), we'll start with just finding the angular momentum, $\underline{\mathbf{H}}_{/\mathbf{o}}$, and then take one derivative to get $\dot{\underline{\mathbf{H}}}_{/\mathbf{o}}$. As you know from class (see section 14.2 of your book), for a planar rigid object the angular momentum has two components: the first one due to the translation of the center of mass, and the second due to rotations about the center of mass.

$$\underline{\mathbf{H}}_{/\mathbf{o}} = \underline{\mathbf{r}}_{\mathbf{cm/o}} \times \mathbf{m_{tot}}\underline{\mathbf{v}}_{\mathbf{cm}} \quad + \quad I_{zz}\underline{\omega} \tag{4.3}$$

Recall that a gyroscope has a very large spin rate about a symmetry axis and so most of the angular momentum comes from rotation. This means that second term in (4.3) will be much larger than the first term. We'll assume that the spin rate $\underline{\omega}$ is so large that we can simply ignore the first term completely, giving the following simplification:

$$\underline{\mathbf{H}}_{/\mathbf{o}} \simeq I_{zz}\underline{\omega} \tag{4.4}$$

Thus $\underline{\mathbf{H}}_{/\mathbf{o}}$ is pointing in the same direction as $\underline{\omega}$. Now we simply take one derivative to find the rate of change of angular momentum. Note that I_{zz} is simply a geometric constant that tells us how hard it is to rotate our gyroscope, and is unaffected by taking a derivative:

$$\dot{\underline{\mathbf{H}}}_{/\mathbf{o}} \simeq I_{zz}\dot{\underline{\omega}} \tag{4.5}$$

Plugging $\dot{\underline{\mathbf{H}}}_{/\mathbf{o}}$ from (4.5) into the equation for angular momentum balance (4.1) we get an equation for $\dot{\underline{\omega}}$. Then we can can plug in for our applied moment (4.2):

$$\dot{\underline{\omega}} = \frac{\underline{\mathbf{M}}_{/\mathbf{o}}}{I_{zz}} \qquad \Rightarrow \qquad \dot{\underline{\omega}} = \frac{-mg}{I_{zz}}(\underline{\mathbf{r}}_{\mathbf{cm/o}} \times \hat{\mathbf{k}}) \tag{4.6}$$

Remember that $\underline{\omega}$ points along the the symmetry axis, so this equation tells us how the direction in which the gyroscope points ($\hat{\omega}$) changes in time, and will hopefully lead us to the precession rate. We know that the cross product of two vectors is perpendicular to both vectors. Since $\dot{\underline{\omega}}$ is perpendicular to $\hat{\mathbf{k}}$, it must lie in the xy-plane. Since $\dot{\underline{\omega}}$ is perpendicular to $\underline{\mathbf{r}}_{\mathbf{cm/o}}$ it must also be perpendicular to $\underline{\omega}$ (both $\underline{\mathbf{r}}_{\mathbf{cm/o}}$ and $\underline{\omega}$ point along the symmetry axis).

Let's think about that: the derivative of a vector $\frac{\mathbf{d}}{\mathbf{dt}}\overrightarrow{\omega}$ is always perpendicular to the original vector $\overrightarrow{\omega}$. This should remind us of circular motion, where the velocity $\frac{\mathbf{d}}{\mathbf{dt}}\overrightarrow{r}$ is always

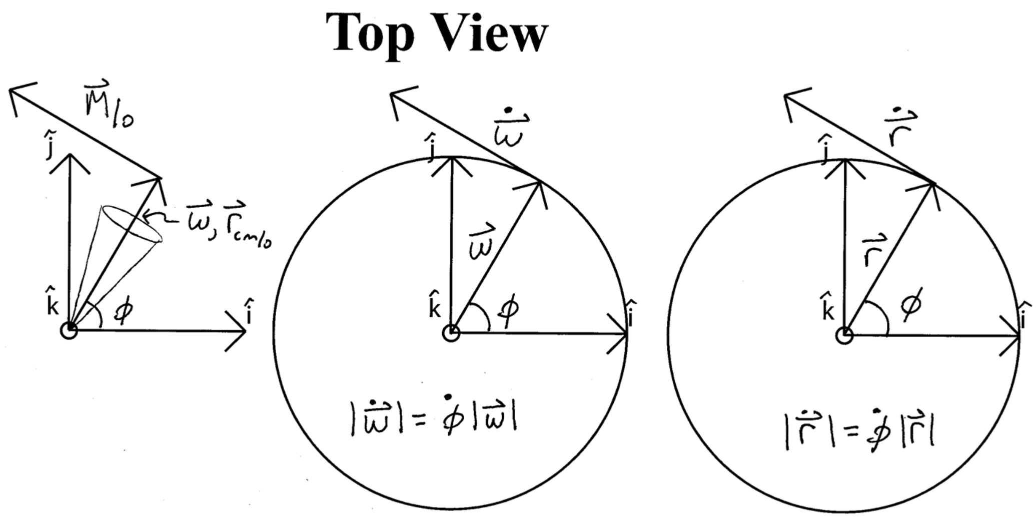

Figure 4.2: A top view of our gyroscope and an anlogy to circular motion

perpendicular to the position $\overrightarrow{r}$! By using what we know from circular motion, we'll posit a solution for how the spin axis $\underline{\omega}$ changes in time. *Be careful about notation!* In circular motion θ and ω are typically used to denote the angle that the position vector makes w.r.t. the positive x-axis, and the rate of change of that angle, respectively. However, here ω is the spin rate of the top about it's symmetry axis, so we'll use ϕ to denote the angle that ω makes with the positive x-axis (see Figure 4.2), i.e. the angle through which the top has precessed:

$$'v = \dot{\theta} r' \qquad \Rightarrow \qquad |\dot{\underline{\mathbf{r}}}| = \dot{\phi}|\underline{\mathbf{r}}| \qquad \Rightarrow \qquad |\underline{\dot{\omega}}| = \dot{\phi}|\underline{\omega}| \tag{4.7}$$

Let's assume that the gyroscope is tilted down so far that it's axis lies in the xy-plane. Then we can easily find the magnitude of the applied moment. Plugging in (4.6)for $\underline{\dot{\omega}}$ and $r_{cm}mg$ for $|\underline{\mathbf{M}}_{/\mathbf{o}}|$ we get:

$$\frac{r_{cm}mg}{I_{zz}} = \dot{\phi}\omega \tag{4.8}$$

Again, note that $\dot{\phi}$ is the precession rate - the rate at which the spin axis is rotating, and ω is the spin rate - the rate at which the top is spinning about it's symmetry axis.

Thus for a gyroscope (or rotor) whose spin axis is orthogonal to the applied torque we find that the product of the moment of inertia, spin rate, and precession rate is equal to the applied torque. In your lab report you will verify this fact.

LABORATORY SET-UP

Our lab gyroscope is a 4" diameter steel ball on an air bearing (see Figure 4.3). On one side of the ball a rod is mounted in order to spin the top and apply moments to it. This side of the ball has also been bored out so that the rod side is lighter and the center of mass can be adjusted to either side of the center of the sphere by sliding a balance weight in or out. The balance weight is black, with reflective tape, to make rotation rate measurements easier. The sphere is supported in a spherical cup into which high pressure air is supplied so that the sphere is actually supported by a thin layer of air (similar to the air track).

To experimentally measure the spin rate ω of the gyroscope you will use a tachometer (measures in rotations per minute, or rpm). To measure the precession rate $\dot{\phi}$ you will use a stop-watch. We will apply a net moment about the fixed point by adding a weight of a known mass to the rod. We can use a diagram of our gyroscope (see Figure 4.3) to estimate the torque-arm and thus calculate the net applied moment.

As a final example of the gyroscopic effect you will play around with a bicycle wheel and rotating platform for hands-on experience and a demonstration of the conservation of angular momentum.

PROCEDURE

1. Turn on the air source.

2. Place the black balance weight on the rod so that if the sphere is released with no spin the rod does not tend to fall down or pop upright from a horizontal position. Note that this is easier said than done, so try to get it as close to motionless as possible. *Where is the center of mass of the system (sphere, rod, and disk) after the gyroscope is balanced? What effect does gravity have on the motion of the balanced gyroscope? If you don't perfectly balance the gyroscope it will result in an error in the calculation of what quantity?*

3. Without spinning the ball, point the rod in some particular direction (up, or towards the door, for example). Carefully release the rod and watch it for several seconds. *Does it keep pointing in the same direction?* Touch the rod lightly with a small strip of paper. *How much force is required to change the orientation of the rod? In which direction does the rod move?* Rotate the table underneath the air bearing. *Does the rod move?*

4. Get the ball spinning and repeat step #3. One good way to do this is to roll the rod between your hands. Stop any wobbling motion by holding the tip lightly and briefly. Avoid touching the ball itself. **Do not allow the rod to touch the base and do not jar the ball while it is spinning.** *What is the effect of spin on the gyroscope motion? Why are navigation gyroscopes set spinning?*

5. While the ball is spinning, apply forces to the end of the rod using one of the pieces of Teflon on a string. The ball should continue to rotate freely as you apply the force because of the low friction of the Teflon. Gently move the end of the rod (keep the rod from touching the bearing cup, or the rod may spin wildly).

6. For a more quantitative look at the motion of a gyroscope:

   (a) Add another weight to the rod so that the gyroscope is no longer balanced. Record its mass and position on the rod for use in calculations later (see Figure 4.3).

   (b) Get the ball spinning, but not wobbling, with the rod in the plane of the table. Now measure the procession rate of the top with a stopwatch and spin rate with a tachometer. You can use the 3 support screws on the air bearing to measure the angle through which the top processes, each being seperated by one-third of a revolution. For the spin rate, measure it at the middle of your period of observation, or measure it at the beginning and end and then average.

   (c) Repeat the procedure for at least two additional spin rates. Try to use a wide range of spin rates; e.g., 200, 400, and 600 r.p.m.

7. Remove the weight and repeat step #6 with at least two more weights for a total of at least three different weights and three different spin rates per weight. The spin rates need not be the same as the ones you used before, but they should cover a similarly wide range of r.p.m.

8. Turn off the air source and clean up your lab station.

9. Hold the bicycle while someone else gets it spinning. Twist it different ways. Hold your hands level and turn your body in a circle. *How do the forces you apply depend on the direction you twist the axle and on the rotation speed and sense?*

10. Now stand on the rotatable platform. Hold the bike wheel so that it's axis is vertical, and get the wheel spinning by yourself - this is a bit tricky, particularly with the larger wheel. *Note the speed and direction of your rotation.*

LAB REPORT QUESTIONS

1. Suppose that the rod on one spinning air gyroscope is pointed north along the earth's axis of rotation. In Ithaca, that would mean at an angle of 42.5 degrees from the horizontal. A second air gyroscope is pointed due east, with its rod horizontal. Assume that the gyroscope is perfectly balanced and that air friction is negligible. How does the orientation of each spinning gyroscope change over a period of several hours?

2. Use your recorded data from parts 6 and 7 of the lab procedure for the following questions.

   (a) Plot the precessional period τ vs. the spin rate ω for your different applied torques. Make sure to use a different color and/or symbol for each data point.

   (b) **From your plot** what is the relationship between the precessional period τ and the spin rate ω?

   (c) **Using your data**, show that for a fixed torque, the product of the precessional rate $\dot{\phi}$ and the spin rate ω is a constant.

   (d) The torque should be proportional to the product of the spin rate and the precession rate. Find the constant of proportionality and plot the relationship between torque and the product of spin rate and precession rate (i.e. M_o vs. $\dot{\phi}\omega$).

   (e) In `(d)` you found a simple formula relating torque, spin rate and precession rate. What is the meaning of and common name for the numerical constant in the formula? *You might want to consider what units it has.*

3. Explain why when you stand on the platform with a spinning bicycle wheel and rotate the wheel, the platform begins to rotate.

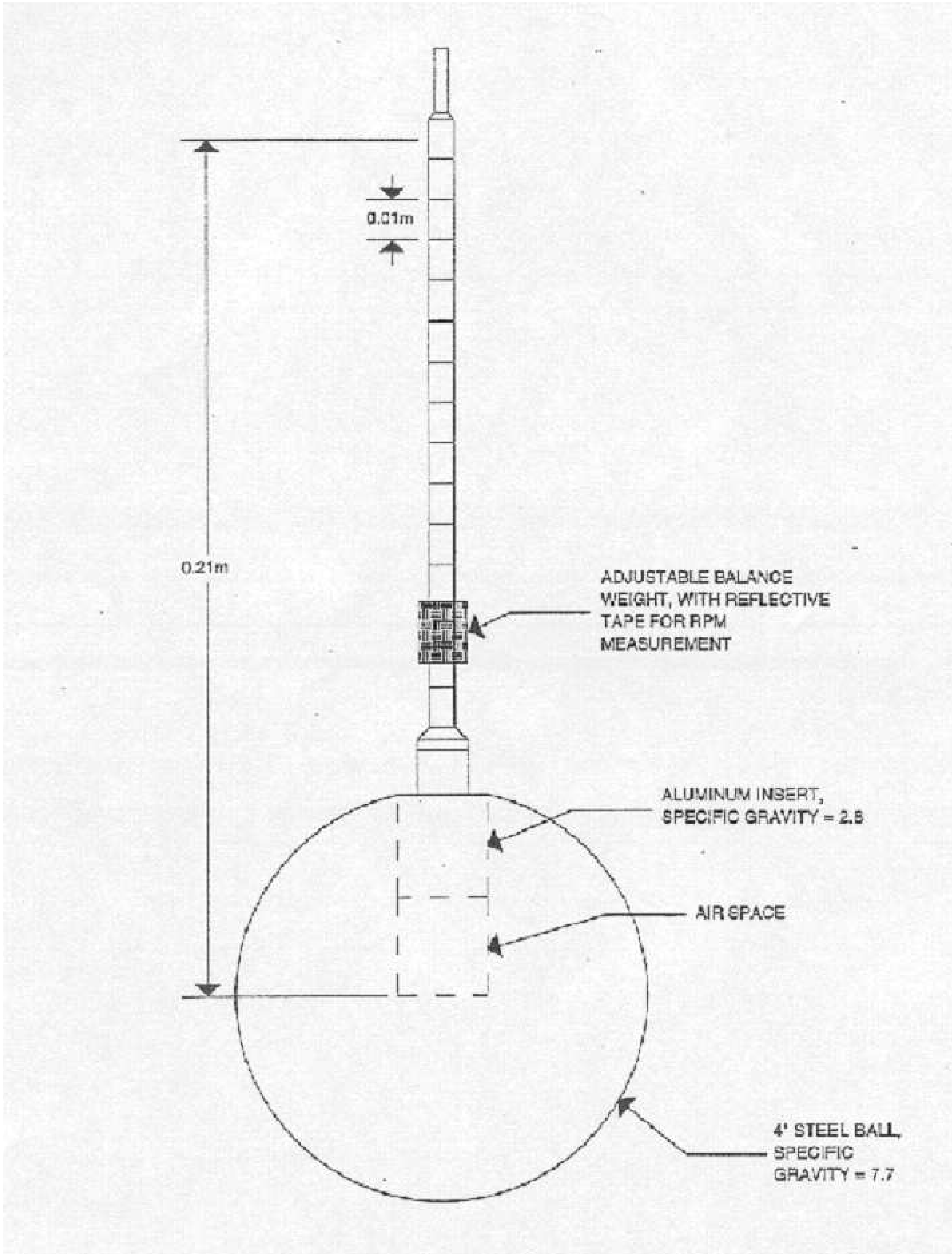

Figure 4.3: A diagram of the lab gyroscope.

CALCULATIONS & NOTES