Lab #4 - Gyroscopic Motion of a Rigid Body

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INTRODUCTION

Gyroscope is a word used to describe a rigid body, that is spinning quickly about an axis about which it is symmetric. In other words it has a large angular velocity, ω, about a symmetry axis. Some examples are a flywheel, symmetric top, football, navigational gyroscopes, and the Earth. The gyroscope differs in some significant ways from the linear one and two degrees-of-freedom systems with which you have experimented so far. The governing equations are 3-dimensional equations of motion and thus mathematical analysis of the gyroscope involves use of 3-dimensional geometry. The governing equations for the general motion of a gyroscope are non-linear. Non-linear equations are in general hard (or impossible) to solve. In this laboratory you will experiment with some simple motions of a simple gyroscope. The purpose of the lab is for you to learn the relation between applied moment, angular momentum, and rate of change of angular momentum. You will learn this relation qualitatively by moving and feeling the gyroscope with your hands and quantitatively by experiments on the precession of the spin axis.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. What is a gyroscope?
2. Where is the fixed point of the lab gyroscope?
3. How will moments (torques) be applied to the lab gyroscope?
4. For a fixed applied moment, will increasing a gyroscope's spin rate, ω, increase or decrease its precession rate, $\dot{\phi}$.

THE GYROSCOPE

Our experiment uses a rotating sphere mounted on an air bearing (see Figure 4.3) so that the center of the sphere remains fixed in space (at least relative to the laboratory room). This is called a *gyroscope with one fixed point*.

As the gyroscope rotates about its spin axis it is basically stable. That is, the spin axis remains pointing in the same direction in space. As you should see in the experiment, the larger the spin rate the larger the applied moment needed to change the direction of the spin axis. When a moment is applied to a gyroscope, the spin axis will itself rotate about a new axis which is perpendicular to both the spin axis and to the axis of the applied moment. This motion of the spin axis is called *precession*, and comes from Angular Momentum Balance: $\sum \overrightarrow{M_{/o}} = \dot{\overrightarrow{H_{/o}}}$

DYNAMICS OF THE SYMMETRIC TOP

A common *gyroscope with one fixed point* which is analogous to our lab setup is a symmetric top acting under the influence of gravity. Imagine that the tip of the top can't move. We'll label that point on the gyroscope O and call it the fixed point of the gyroscope since it's location is fixed in space (see Figure (4.1)). Every other point on top will move as the top spins and wobbles (i.e. precesses), so there is only one fixed point. In this section, we will motivate an equation governing the precession rate. A detailed derivation requires the use of rotating coordinate frames.

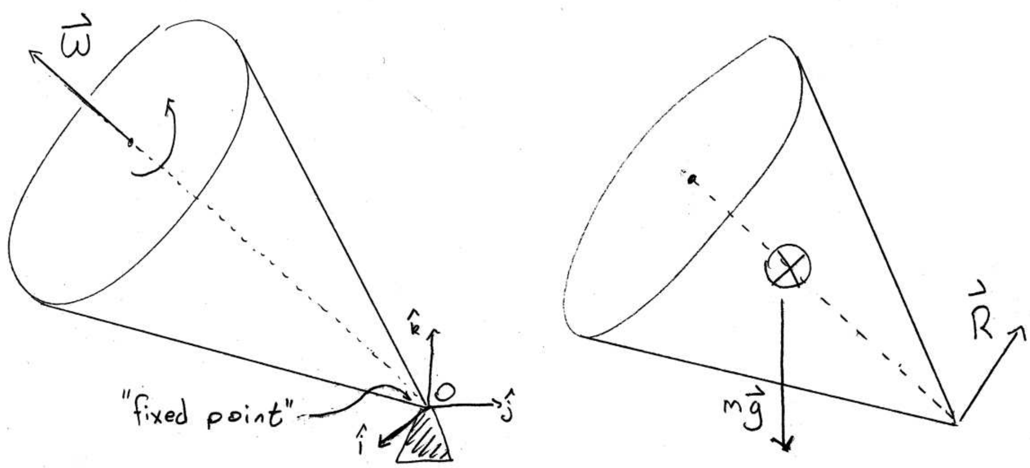

Figure 4.1: A symmetric top with one fixed point

Before we begin, let's remind ourselves how angular velocity vectors work - the direction of the vector gives the axis about which we are spinning, and the magnitude of the vector gives the spin rate. Note from Figure (4.1) that $\underline{\omega}$ points in the same direction as $\underline{\mathbf{r}}_{\mathbf{cm/o}}$, since the top is spinning around its symmetry axis. Now we're ready to 'solve' the equations of motion. Since we don't want to solve for the reaction force, $\underline{\mathbf{R}}$, at the point of contact O we'll use angular momentum balance about that point to find the equations of motion.

$$\sum \underline{\mathbf{M}}_{/\mathbf{o}} = \underline{\dot{\mathbf{H}}}_{/\mathbf{o}} \tag{4.1}$$

Calculating the left hand side of (4.1) is easy since there is just one moment, the one due to gravity:

$$\underline{\mathbf{M}}_{/\mathbf{o}} = \underline{\mathbf{r}}_{\mathbf{cm/o}} \times -mg\hat{\mathbf{k}} \tag{4.2}$$

As for the right hand side of (4.1), we'll start with just finding the angular momentum, $\underline{\mathbf{H}}_{/\mathbf{o}}$, and then take one derivative to get $\dot{\underline{\mathbf{H}}}_{/\mathbf{o}}$. As you know from class (see section 14.2 of your book), for a planar rigid object the angular momentum has two components: the first one due to the translation of the center of mass, and the second due to rotations about the center of mass.

$$\underline{\mathbf{H}}_{/\mathbf{o}} = \underline{\mathbf{r}}_{\mathbf{cm/o}} \times \mathbf{m_{tot}}\underline{\mathbf{v}}_{\mathbf{cm}} \quad + \quad I_{zz}\underline{\omega} \tag{4.3}$$

Recall that a gyroscope has a very large spin rate about a symmetry axis and so most of the angular momentum comes from rotation. This means that second term in (4.3) will be much larger than the first term. We'll assume that the spin rate $\underline{\omega}$ is so large that we can simply ignore the first term completely, giving the following simplification:

$$\underline{\mathbf{H}}_{/\mathbf{o}} \simeq I_{zz}\underline{\omega} \tag{4.4}$$

Thus $\underline{\mathbf{H}}_{/\mathbf{o}}$ is pointing in the same direction as $\underline{\omega}$. Now we simply take one derivative to find the rate of change of angular momentum. Note that I_{zz} is simply a geometric constant that tells us how hard it is to rotate our gyroscope, and is unaffected by taking a derivative:

$$\dot{\underline{\mathbf{H}}}_{/\mathbf{o}} \simeq I_{zz}\dot{\underline{\omega}} \tag{4.5}$$

Plugging $\dot{\underline{\mathbf{H}}}_{/\mathbf{o}}$ from (4.5) into the equation for angular momentum balance (4.1) we get an equation for $\dot{\underline{\omega}}$. Then we can can plug in for our applied moment (4.2):

$$\dot{\underline{\omega}} = \frac{\underline{\mathbf{M}}_{/\mathbf{o}}}{I_{zz}} \qquad \Rightarrow \qquad \dot{\underline{\omega}} = \frac{-mg}{I_{zz}}(\underline{\mathbf{r}}_{\mathbf{cm/o}} \times \hat{\mathbf{k}}) \tag{4.6}$$

Remember that $\underline{\omega}$ points along the the symmetry axis, so this equation tells us how the direction in which the gyroscope points ($\hat{\omega}$) changes in time, and will hopefully lead us to the precession rate. We know that the cross product of two vectors is perpendicular to both vectors. Since $\dot{\underline{\omega}}$ is perpendicular to $\hat{\mathbf{k}}$, it must lie in the xy-plane. Since $\dot{\underline{\omega}}$ is perpendicular to $\underline{\mathbf{r}}_{\mathbf{cm/o}}$ it must also be perpendicular to $\underline{\omega}$ (both $\underline{\mathbf{r}}_{\mathbf{cm/o}}$ and $\underline{\omega}$ point along the symmetry axis).

Let's think about that: the derivative of a vector $\frac{\mathbf{d}}{\mathbf{dt}}\overrightarrow{\omega}$ is always perpendicular to the original vector $\overrightarrow{\omega}$. This should remind us of circular motion, where the velocity $\frac{\mathbf{d}}{\mathbf{dt}}\overrightarrow{r}$ is always

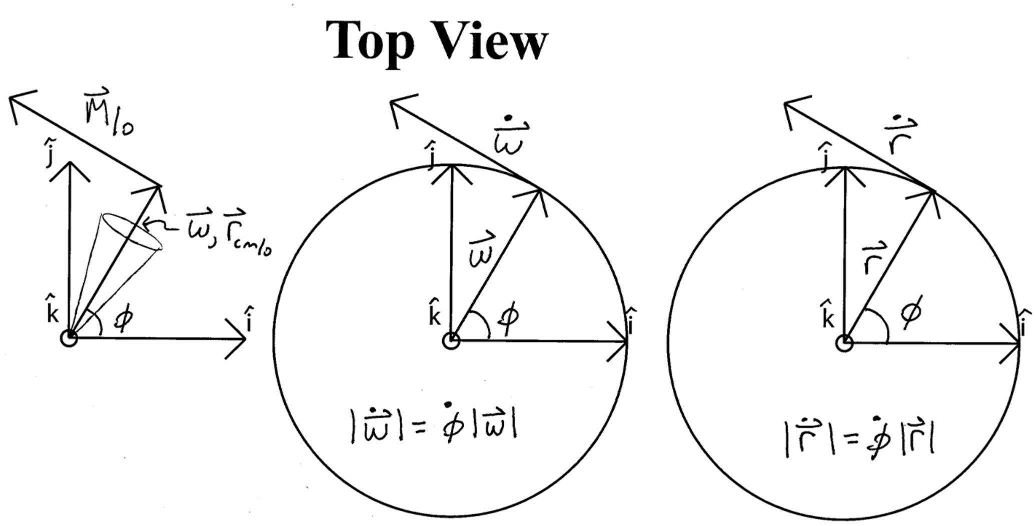

Figure 4.2: A top view of our gyroscope and an anlogy to circular motion

perpendicular to the position $\overrightarrow{r}$! By using what we know from circular motion, we'll posit a solution for how the spin axis $\underline{\omega}$ changes in time. *Be careful about notation!* In circular motion θ and ω are typically used to denote the angle that the position vector makes w.r.t. the positive x-axis, and the rate of change of that angle, respectively. However, here ω is the spin rate of the top about it's symmetry axis, so we'll use ϕ to denote the angle that ω makes with the positive x-axis (see Figure 4.2), i.e. the angle through which the top has precessed:

$$'v = \dot{\theta} r' \qquad \Rightarrow \qquad |\dot{\underline{\mathbf{r}}}| = \dot{\phi}|\underline{\mathbf{r}}| \qquad \Rightarrow \qquad |\dot{\underline{\omega}}| = \dot{\phi}|\underline{\omega}| \tag{4.7}$$

Let's assume that the gyroscope is tilted down so far that it's axis lies in the xy-plane. Then we can easily find the magnitude of the applied moment. Plugging in (4.6)for $\dot{\underline{\omega}}$ and $r_{cm}mg$ for $|\underline{\mathbf{M}}_{/\mathbf{o}}|$ we get:

$$\frac{r_{cm}mg}{I_{zz}} = \dot{\phi}\omega \tag{4.8}$$

Again, note that $\dot{\phi}$ is the precession rate - the rate at which the spin axis is rotating, and ω is the spin rate - the rate at which the top is spinning about it's symmetry axis.

Thus for a gyroscope (or rotor) whose spin axis is orthogonal to the applied torque we find that the product of the moment of inertia, spin rate, and precession rate is equal to the applied torque. In your lab report you will verify this fact.

LABORATORY SET-UP

Our lab gyroscope is a 4" diameter steel ball on an air bearing (see Figure 4.3). On one side of the ball a rod is mounted in order to spin the top and apply moments to it. This side of the ball has also been bored out so that the rod side is lighter and the center of mass can be adjusted to either side of the center of the sphere by sliding a balance weight in or out. The balance weight is black, with reflective tape, to make rotation rate measurements easier. The sphere is supported in a spherical cup into which high pressure air is supplied so that the sphere is actually supported by a thin layer of air (similar to the air track).

To experimentally measure the spin rate ω of the gyroscope you will use a tachometer (measures in rotations per minute, or rpm). To measure the precession rate $\dot{\phi}$ you will use a stop-watch. We will apply a net moment about the fixed point by adding a weight of a known mass to the rod. We can use a diagram of our gyroscope (see Figure 4.3) to estimate the torque-arm and thus calculate the net applied moment.

As a final example of the gyroscopic effect you will play around with a bicycle wheel and rotating platform for hands-on experience and a demonstration of the conservation of angular momentum.

PROCEDURE

1. Turn on the air source.

2. Place the black balance weight on the rod so that if the sphere is released with no spin the rod does not tend to fall down or pop upright from a horizontal position. Note that this is easier said than done, so try to get it as close to motionless as possible. *Where is the center of mass of the system (sphere, rod, and disk) after the gyroscope is balanced? What effect does gravity have on the motion of the balanced gyroscope? If you don't perfectly balance the gyroscope it will result in an error in the calculation of what quantity?*

3. Without spinning the ball, point the rod in some particular direction (up, or towards the door, for example). Carefully release the rod and watch it for several seconds. *Does it keep pointing in the same direction?* Touch the rod lightly with a small strip of paper. *How much force is required to change the orientation of the rod? In which direction does the rod move?* Rotate the table underneath the air bearing. *Does the rod move?*

4. Get the ball spinning and repeat step #3. One good way to do this is to roll the rod between your hands. Stop any wobbling motion by holding the tip lightly and briefly. Avoid touching the ball itself. **Do not allow the rod to touch the base and do not jar the ball while it is spinning.** *What is the effect of spin on the gyroscope motion? Why are navigation gyroscopes set spinning?*

5. While the ball is spinning, apply forces to the end of the rod using one of the pieces of Teflon on a string. The ball should continue to rotate freely as you apply the force because of the low friction of the Teflon. Gently move the end of the rod (keep the rod from touching the bearing cup, or the rod may spin wildly).

6. For a more quantitative look at the motion of a gyroscope:

   (a) Add another weight to the rod so that the gyroscope is no longer balanced. Record its mass and position on the rod for use in calculations later (see Figure 4.3).

   (b) Get the ball spinning, but not wobbling, with the rod in the plane of the table. Now measure the procession rate of the top with a stopwatch and spin rate with a tachometer. You can use the 3 support screws on the air bearing to measure the angle through which the top processes, each being seperated by one-third of a revolution. For the spin rate, measure it at the middle of your period of observation, or measure it at the beginning and end and then average.

   (c) Repeat the procedure for at least two additional spin rates. Try to use a wide range of spin rates; e.g., 200, 400, and 600 r.p.m.

7. Remove the weight and repeat step #6 with at least two more weights for a total of at least three different weights and three different spin rates per weight. The spin rates need not be the same as the ones you used before, but they should cover a similarly wide range of r.p.m.

8. Turn off the air source and clean up your lab station.

9. Hold the bicycle while someone else gets it spinning. Twist it different ways. Hold your hands level and turn your body in a circle. *How do the forces you apply depend on the direction you twist the axle and on the rotation speed and sense?*

10. Now stand on the rotatable platform. Hold the bike wheel so that it's axis is vertical, and get the wheel spinning by yourself - this is a bit tricky, particularly with the larger wheel. *Note the speed and direction of your rotation.*

LAB REPORT QUESTIONS

1. Suppose that the rod on one spinning air gyroscope is pointed north along the earth's axis of rotation. In Ithaca, that would mean at an angle of 42.5 degrees from the horizontal. A second air gyroscope is pointed due east, with its rod horizontal. Assume that the gyroscope is perfectly balanced and that air friction is negligible. How does the orientation of each spinning gyroscope change over a period of several hours?

2. Use your recorded data from parts 6 and 7 of the lab procedure for the following questions.

   (a) Plot the precessional period τ vs. the spin rate ω for your different applied torques. Make sure to use a different color and/or symbol for each data point.

   (b) **From your plot** what is the relationship between the precessional period τ and the spin rate ω?

   (c) **Using your data**, show that for a fixed torque, the product of the precessional rate $\dot{\phi}$ and the spin rate ω is a constant.

   (d) The torque should be proportional to the product of the spin rate and the precession rate. Find the constant of proportionality and plot the relationship between torque and the product of spin rate and precession rate (i.e. M_o vs. $\dot{\phi}\omega$).

   (e) In `(d)` you found a simple formula relating torque, spin rate and precession rate. What is the meaning of and common name for the numerical constant in the formula? *You might want to consider what units it has.*

3. Explain why when you stand on the platform with a spinning bicycle wheel and rotate the wheel, the platform begins to rotate.

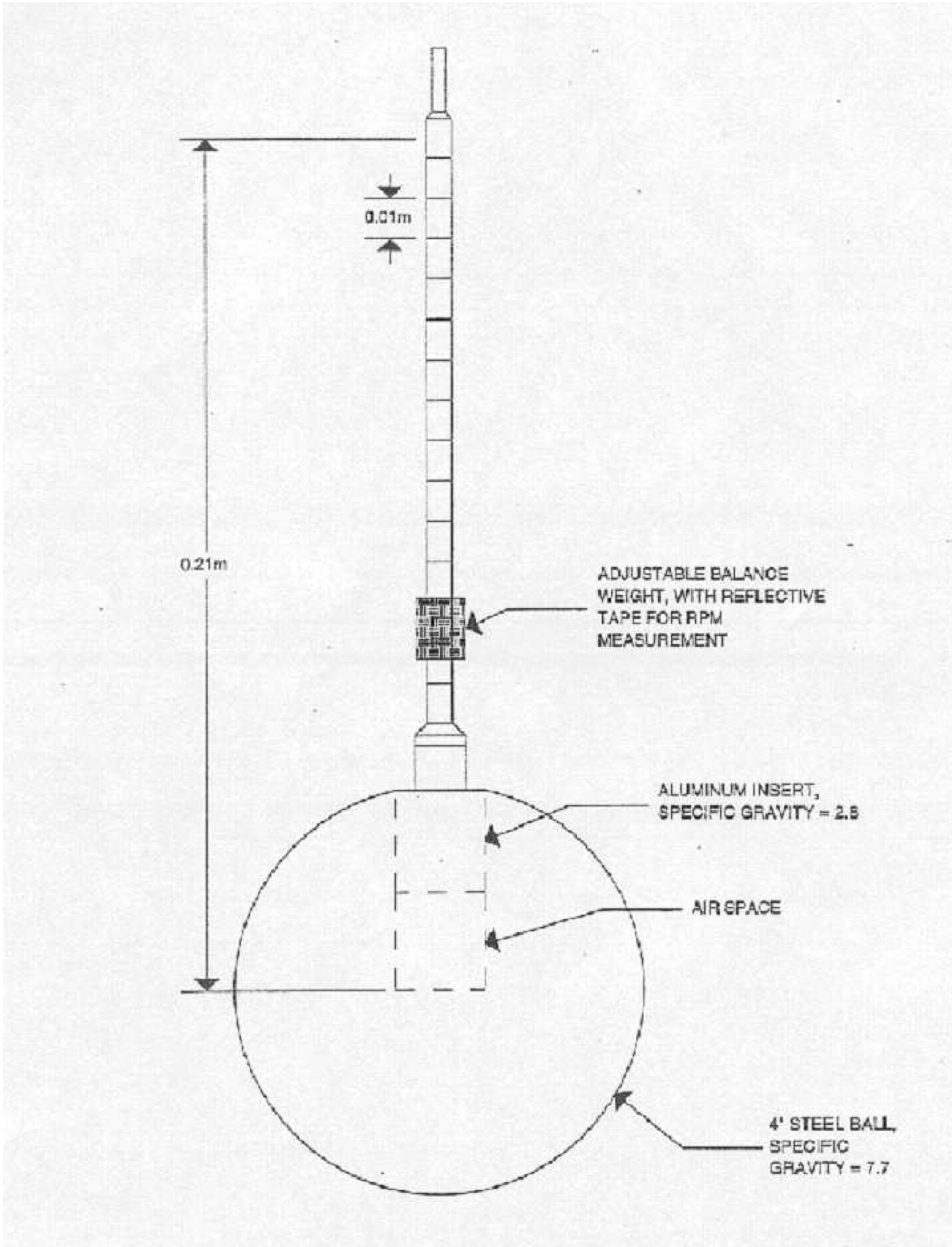

Figure 4.3: A diagram of the lab gyroscope.

CALCULATIONS & NOTES