\[ u_t = u_{xx} - u \]

Some solutions are (depending on guess):

- Solns indep. of \( x \):
  \[ u_t = -u \]
  \[ u = e^{-t} \]
  \[ u = 3e^{-t} \]
  \[ u = e^x \]
  \[ u = e^{-x} \]
  \[ u = 10e^{-x} - 3e^x \]
  \[ u = 7 \sin x + 2 \cos x \]

- Solns indep. of \( t \):
  \[ 0 = u_{xx} - u \]

Sols found by rep. of variables:

\[ u = X(x) T(t) \]
\[ XT' = X''T - XT' \]
\[ \frac{T'}{T} = \frac{X''}{X} \]
\[ T = e^{\lambda t} \]
\[ X'' = (1 + \lambda)X \]

With \( \lambda = 1 \) you get \[ X'' = 2X \] so \[ X = e^{\sqrt{2}x} \]

With \( \lambda = -1 \) you get \[ X'' = 0 \] so \[ X = ax + b \]

With \( \lambda = -2 \) you get \[ X'' = -X \] so \[ X = \sin x, \cos x \]

etc. the results are:

\[ u(x, t) = \begin{cases} e^{\sqrt{2}x} & e^{-\sqrt{2}x} \\ e^{t} & e^{-t} \\ e^{-x} & (3x-15) \\ e^{-2t} & \sin x \\ e^{-2t} & (3\sin x + 5\cos x) \end{cases} \]

Solutions found by educated guessing:

Try \( u(x, t) = e^{\alpha t} \sin bx \), then \( u_t = u_{xx} - u \) becomes \( a u = (-b^2 - 1) u \).

So \( a = -b^2 - 1 \).

So \[ u(x, t) = \begin{cases} e^{-(b^2+1)t} & \sin bx \\ e^{-5t} & \sin 2x \\ e^{-10t} & \sin 3x \end{cases} \]

And there are others.
12) \( \alpha^2 u_{xx} = u_{tt} \quad 0 < x < l, \quad t > 0 \)

a) Try \( u = X(x) T(t) \)

\[ \alpha^2 X'' T = X T'' \]

or

\[ \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\alpha^2 \]

Since LHS is \( f(x) \) alone and RHS is \( g(t) \) alone, they must each equal a constant which we are told is \( -\alpha^2 \)

\[ \frac{X''}{X} = -\alpha^2 \quad \text{or} \quad \left( X'' + \alpha^2 X = 0 \right) \]

just the same sign that we found for the heat problem.

b) Either recognize soln

or try:

Apply the B.C. \( X(0) = A(0) + B(0) = A = 0 \)

\[ X(l) = B \sin \alpha l \]

For non-trivial solns \( \alpha = \frac{n \pi}{l} \)

\[ \therefore X(x) = B \sin \frac{n \pi x}{l} \]

c) Using this \( \alpha \) in the t equation:

\[ \frac{T''}{\alpha^2} = \left( \frac{n \pi}{l} \right)^2 \quad \Rightarrow \quad \left( \frac{n \pi}{l} \right)^2 T = 0 \]

\[ \therefore T = C \sin \frac{n \pi x}{l} + D \cos \frac{n \pi x}{l} \]

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15. (a) \( u = X(x) Y(y) \) Then

\[ \frac{X'}{X} + \frac{Y''}{Y} + 1 = 0 \]

\[ \frac{X'}{X} = \frac{Y''}{Y} = -c \]

\[ X' = c Y, \quad X = e^{c x} \]

\[ Y'' = (-1-c) Y \]

\[ \therefore Y = \cos bx, \quad c = b, \quad X = e^{bx}, \quad u = e^{bx} \cos bx \]

(b) one soln is with

\[ Y = \cos 3y, \quad c = 8, \quad X = e^{8x}, \quad u = e^{8x} \cos 3y \]
15) \( X = \sin(\sqrt{\lambda} x) \) since \( X'' = -\lambda X \) and \( X(0) = 0 \) (Note the \( X(0) = 0 \) rules out cosine).
Then \( X'(1) = \sqrt{\lambda} \cos(\sqrt{\lambda}) \) is 0 if \( \sqrt{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \)
\[
X(x) = \sin\left(\frac{n\pi x}{2}\right) \\
\lambda = \left(\frac{n\pi}{2}\right)^2 \\
n = 1, 3, 5, \ldots
\]

15) (a) \( yu_x x + y_3 = y X'' Y + X Y' \) is 0 if \( \frac{X''}{X} = -\frac{Y'}{y Y} \)
\( X'' = -c X \)
\( Y' = c y Y \)
(b) \( X = \cos(\sqrt{c} x) \) \( Y = e^{\sqrt{c} y^{3/2}} \) any \( c > 0 \) will work
(c) \( u(x,y) = \cos(3x) e^{\sqrt{c} y^{3/2}} \) will work, and many others.
17: \[ L = \pi \]
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi} \int_{0}^{\pi} 3 \, dx = 3 \]
\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} 3 \cos(nx) \, dx = \frac{3}{\pi} \left[ \frac{\sin(nx)}{n} \right]_{0}^{\pi} = 0 \]
\[ b_n = \frac{1}{\pi} \int_{0}^{\pi} 3 \sin(nx) \, dx = \frac{3}{\pi} \left[ -\frac{\cos(nx)}{n} \right]_{0}^{\pi} = \frac{3}{\pi} \frac{\cos(n\pi) - 1}{n} \]
\[ = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{6}{n\pi} & \text{if } n \text{ is odd} \end{cases} \]

\[ f(x) \approx \frac{3}{2} + \frac{6}{\pi} \sin(x) + \frac{6}{3\pi} \sin(3x) + \ldots \]

Series converges to \( \frac{3}{2} \) at \( x = 0 \).

b \[ u_t + u = 3u_x \quad \text{Define} \ u(x,t) = X(x)T(t) \text{ then} \]
\[ XT' + XT = 3X'T \]

\[ \frac{T'}{T} + 1 = \frac{3X'}{X} = \text{constant} \lambda \]

\[ 3X' = \lambda X \quad \text{and} \quad T' + T = \lambda T \]
(a) Equation: $$T'' = -\lambda T$$ and $$a^2 X'' = (b^2 - \lambda)X$$; solution is $$u(x,t) = \sum_{n=1}^{\infty} c_n \cos \left( \sqrt{b^2 + \frac{(a \pi)^2}{L^2}} x \right) \sin \left( \frac{n \pi x}{L} \right)$$.

(ii) $$u(x,0) = 0$$ rules out cosines; $$u \to 0$$ at $$\pm \infty$$ rules out $$r^n$$.

(b) $$u(x,t) = 3 \cos \left( \sqrt{b^2 + \frac{(a \pi)^2}{L^2}} x \right) \sin \left( \frac{n \pi x}{L} \right)$$.

(i) (a) Because it is like $$\frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct)$$ where $$f(x) = y(x)$$.

(ii) (a) does not have $$0^0$$ on boundary; (b) is not steady state.

(c) does not involve $$x_1$$.

(i) (d) None of the above.

$$M294 = 92$$

(a) $$-1, -\frac{\pi}{3}, \frac{\pi}{3}, 0, 0, 0, 1, 0.2, 0.2$$.

(b) $$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \left( (2n-1)x \right)$$, $$-\pi \leq x \leq \pi$$.

$$M294 = 92$$

Using $$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \left( (2n-1)x \right)$$, $$0 \leq x \leq \pi$$.

You get $$\frac{3 \pi}{2} - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \left( (2n-1)x \right)$$.

So $$T(x,t) = \frac{3 \pi}{2} - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \left( (2n-1)x \right) e^{-\frac{a^2 \pi^2}{L^2} t}$$.

It works because (i) $$\frac{\pi}{2}$$ and each product $$\cos(x)e^{-\frac{a^2 \pi^2}{L^2} t}$$ are solutions to heat eqn, which is linear.

(ii) $$T_x = \sum$$ dines which are at ends.

(iii) $$T(x,0) = 3x$$ because we're given the

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