3-D Flux and Divergence

M294 PI SP87 #5

5)

$$div\,\vec F = \frac{\partial(2x-y)}{\partial x} + \frac{\partial(x+z)}{\partial y} + \frac{\partial(z^2)}{\partial z} = 2+2z$$

a) $\boxed{div\,\vec F(1,2,3) = 8}$

b) $\iint_S \vec F\cdot\hat n\,d\sigma = \iiint_{\text{solid ball}} div\,\vec F\,dV = \iiint(2+2z)\,dV$

Notice $2+2z > 0$ over the solid ball since there $6 \le z \le 8$, therefore the $\boxed{\text{integral is positive}}$.

M294 F SP87 #5

6) $F = (ye^z + x)\hat i + (2y - z)\hat j + 7\hat k;\quad \nabla\cdot F = 2+1 = 3$

a) $\iint_S F\cdot\hat n\,d\sigma = \iiint_V div\,F\,dV = 3\iiint_V dV = 3V_s = 4\pi R^3 = \boxed{4\pi}$

b) $\iint_S d\sigma = 4\pi R^2$ ← surface area of a sphere or $=\boxed{4\pi}$

odd fn. in even region

c) $\iiint_D z\,dV = \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}}\int_0^1\int_0^{2\pi} z\,r\,dr\,d\theta\,dz = \pi\int_{-\sqrt{\ }}^{\sqrt{\ }} z\,dz = \frac{z^2}{2}\Big|_{-\sqrt{\ }}^{\sqrt{\ }} = \boxed{0}$

d) $\iiint_D div(curl\,F)\,dV = \boxed{0}$ since $\nabla\cdot(\nabla\times F) = 0$ or

you can calculate $\nabla\times F = -\hat i + ye^z\hat j + -e^z\hat k$, $\nabla\cdot\nabla\times F = e^z - e^z = 0$

e) $\iint_S \nabla\times F\cdot\hat n\,d\sigma = \iiint_D \nabla\cdot(\nabla\times F)\,dV = \boxed{0}$ as in d above. Alt. methods: use Stokes Thm. subtly.

M294 PII SP88 #2

13) $$\iint_{x^2+y^2+z^2=25}\left((x - y\sin z)\hat i + (2y + \sin z)\hat j + (3z - \sin x)\hat k\right)\cdot\hat n\,d\sigma$$

$$\underset{\text{DIV. THM.}}{=} \iiint_{x^2+y^2+z^2\le 25}\left(\frac{\partial}{\partial x}(x - y\sin z) + \frac{\partial}{\partial y}(2y+\sin z) + \frac{\partial}{\partial z}(3z - \sin x)\right)dx\,dy\,dz$$

$$= \iiint_V (1+2+3)\,dx\,dy\,dz = \boxed{6\cdot\underbrace{\tfrac{4}{3}\pi 5^3}_{V}} = \boxed{1000\pi}$$

19) M294 P II FA90 #1

(a) (i). $F = y - z^2$, $\nabla F = j - 2zk$, $|\nabla F| = \sqrt{1+4z^2}$, $p = k$.

$|\nabla F \cdot p| = |\nabla F \cdot k| = |2z|$

$$\text{Area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA = \int_0^1 \int_0^{1-x} \frac{\sqrt{1+4y}}{2\sqrt{y}} \, dy \, dx$$

(a) (ii). $G = 2x + y + 2z$, $\nabla G = 2i + j + 2k$, $|\nabla G| = 3$, $p = k$.

$|\nabla G \cdot p| = 2$. $d\sigma = \frac{|\nabla G|}{|\nabla G \cdot p|} dA = \frac{3}{2} \, dx \, dy$.

$n = \frac{\nabla G}{|\nabla G|} = \frac{1}{3}(2i + j + 2k)$, $F \cdot n = \frac{xy}{3}$.

$$\text{Flux} = \iint_R F \cdot n \, d\sigma = \int_0^1 \int_0^{2-2x} \frac{xy}{2} \, dy \, dx$$

(b). $\nabla \cdot F = 2x$.

$$\text{Flux} = \iiint_B \nabla \cdot F \, dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} 2x \, dz \, dy \, dx$$

$$\overset{\text{change to polar}}{=\!=\!=} \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{1-r^2} 2r\cos\theta \, r \, dz \, d\theta \, dr$$

$$= \int_0^1 2r^2 dr \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta \int_0^{1-r^2} dz$$

$$= \int_0^1 2r^2(1-r^2) \, dr = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

M294 P I FA93 #4

29) $$\iint_{\text{cube}} \underline{F} \cdot \underline{n} \, d\sigma = \int_0^2\int_0^2\int_0^2 \text{div } \underline{F} \, dx dy dz = \int_0^2\int_0^2\int_0^2 7 \, dx dy dz = \boxed{7 \cdot 2^3}$$

M294 PI FA94 #2

33)

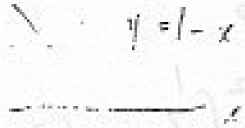
$$\underset{\sim}{n} = \frac{\hat{i}+\hat{j}+2\hat{k}}{[1+1+4]^{1/2}} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}} \longleftarrow \text{by inspection}$$

or, let $f = x+y-2z = 1$ $\quad \frac{\nabla f}{|\nabla f|} = \underset{\sim}{n} = \frac{\hat{i}+\hat{j}+2\hat{k}}{[1+1+4]^{1/2}}$ $\quad \underset{\sim}{p} = \hat{k}$

b) $d\sigma = \frac{dx\,dy}{\underset{\sim}{n}\cdot\underset{\sim}{p}} = \frac{dx\,dy}{2/\sqrt{6}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{2}\sqrt{2}}\,dx\,dy = \sqrt{3/2}\,dx\,dy$

c) $\underset{\sim}{F} = x\hat{i} - z\hat{j}$ $\quad \iint_S \underset{\sim}{F}\cdot\underset{\sim}{n}\,d\sigma = \iint_S (x\hat{i} - z\hat{j})\cdot\frac{(\hat{i}+\hat{j}+2\hat{k})}{\sqrt{6}}\,d\sigma$

$= \iint_S \frac{x-2z}{\sqrt{6}}\,d\sigma$; on C, $2z = 1-x-y$ $\therefore$ $x-2z = x-1+x+y = 2x+y-1$



$$= \iint_R \frac{2x+y-1}{\sqrt{6}}\sqrt{3/2}\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \frac{2x+y-1}{\sqrt{6}}\sqrt{3/2}\,dy\,dx = \frac{1}{2}\int_0^1 2xy + \frac{y^2}{2} - y\Big|_0^{1-x} dx$$

$\frac{\sqrt{3}}{\sqrt{12}} \cdot$ $\quad = \frac{1}{2}\int_0^1 \left[2x(1-x) + \frac{1-2x+x^2}{2} - 1 + x\right] dx$

$$= \frac{1}{2}\int_0^1 \left(2x - 2x^2 + \frac{1}{2} - x + \frac{x^2}{2} - 1 + x\right) dx$$

$$= \frac{1}{2}\int_0^1 \left(2x - \frac{1}{2} - \frac{3x^2}{2}\right) dx = \frac{1}{2}\left[\frac{2x^2}{2} - \frac{x}{2} - \frac{3x^3}{6}\right|_0^1$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{2}\right] = 0$$

M294 P I FA94 #3

34) $\underset{\sim}{F} = \hat{\imath} + y\hat{\jmath} - x\hat{k}$; $\operatorname{div}\underset{\sim}{F} = 1$

$\therefore \iint_{S} \underset{\sim}{F}\cdot\underset{\sim}{n}\, d\sigma = \iiint_{V} \operatorname{div} F\, dV = 1\cdot \text{Volume} = 1\cdot \frac{4}{3}\pi r^{2}$, where $r^2 = 5$

M294 PI SP95 #1

36)

a) on S_2: $\iint_{S_2} \vec{F}\cdot\vec{n}_2\, d\sigma = \int_0^2\int_0^{1-x/2} V_0 z\, dz\, dx = \int_0^2 \frac{V_0}{2}\left(1-\frac{x}{2}\right)^2 dx = \boxed{V_0/3}$

on S_1: surface S_1 given by $z + \frac{xy}{2} = f(x,y,z) = 1 \therefore \vec{n}_1 = \vec{\nabla} f = \frac{y}{2}\hat{\imath} + \frac{x}{2}\hat{\jmath} + \hat{k}$

$$\iint_{S_1} \vec{F}\cdot\vec{n}_1\, d\sigma = \int_0^2\int_0^1 \vec{F}\cdot\vec{\nabla} f\, \frac{dy\, dx}{\vec{\nabla} f\cdot\hat{k}} = \int_0^2\int_0^1 \frac{V_0 z x}{2}\, dy\, dx = \int_0^2\int_0^1 \frac{V_0\left(1-\frac{xy}{2}\right)x}{2}\, dy\, dx$$

$$= \frac{V_0}{2}\int_0^2\int_0^1 \left(x - \frac{x^2 y}{2}\right) dy\, dx = \frac{V_0}{2}\int_0^2 \left(x - \frac{x^2}{4}\right) dx = \boxed{\frac{2V_0}{3}}$$

b) $\operatorname{div}\vec{F} = 0$, thus flux across a closed surface is zero.

c) $-\iint_{S_3} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma$ since flux across the other 3 sides of the shape drawn is zero since their normal vectors are perpendicular to $\vec{F}$.

37)

M294 PI F95 #4

Field $\underset{\sim}{F}$ is a radial field, and it is parallel to the three faces $x=0$, $z=0$, $y=0$,

i.e. on $x=0$ face, $\underset{\sim}{n} = -\hat{\imath}$ $\quad \underset{\sim}{F}\cdot\underset{\sim}{n} = (0\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot(-\hat{\imath}) = 0$

thus flux $= \iint\limits_{S(x=0,\,y=0,\,z=0)} \underset{\sim}{F}\cdot\underset{\sim}{n}\, d\sigma = 0$ on these 3 faces.

$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{(1+1+1)^{\frac{1}{2}}}$

We need to calculate only on surface $x+y+z=a$

On this surface, $\underset{\sim}{n} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$, $\underset{\sim}{F}\cdot\underset{\sim}{n} = (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$

$= \frac{x+y+z}{\sqrt{3}}$

$\iint \underset{\sim}{F}\cdot\underset{\sim}{n}\, d\sigma = \iint\limits_S \frac{x+y+z}{\sqrt{3}}\, d\sigma \xrightarrow{\text{thus}} \iint\limits_S \frac{a}{\sqrt{3}}\, d\sigma$

(+z=a)

but on this surface, $x+y+z=a$,

$= \frac{a}{\sqrt{3}}\cdot(\text{Area of face}) = \frac{a}{\sqrt{3}}\underbrace{\frac{\sqrt{3}a^2}{2}}_{\text{given}} = a^3/2$

$\iint \underset{\sim}{F}\cdot\underset{\sim}{n}\, d\sigma = \iiint\limits_V \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\, dV = \iiint\limits_V 3\cdot dV = 3\cdot\text{Volume} = \frac{3a^3}{6} = a^3/2$

t. surface

es, consistent: Total flux calculated directly $= 0+0+0+a^3/2$.

Total flux using div. thm $= a^3/2$

M294 F FA95 #6

38) $S: x^2-y^2+z=0 \quad , x^2+y^2 \le 25$

(a) let $f(x,y,z) = x^2-y^2+z$ then S is part of $f=0$ so a normal is $\vec{\nabla} f = 2x\vec{i} - 2y\vec{j} + \vec{k}$, and $\vec{n} = \frac{2x\vec{i}-2y\vec{j}+\vec{k}}{\sqrt{4x^2+4y^2+1}}$

There are 2 correct answers; this $\vec{n}$, and its negative.

(b) projecting to the (x,y) plane, $d\sigma = \frac{dxdy}{|\cos\measuredangle(\vec{k},\vec{n})|}$

$= \frac{dxdy}{|\vec{k}\cdot\vec{n}|} = \sqrt{4x^2+4y^2+1}\, dxdy$

(c) area of $S = \iint_S d\sigma = \iint_{x^2+y^2\le 25} \sqrt{4x^2+4y^2+1}\, dxdy$

$= \int_0^{2\pi}\int_0^5 \sqrt{4r^2+1}\, r\,dr\,d\theta = \left[\frac{2}{3\cdot 8}(4r^2+1)^{3/2}\right]_{r=0}^{5} \cdot 2\pi$

$= \frac{(101)^{3/2}-1}{6}\pi$

(d) flux $= \iint_S (300\vec{k})\cdot\vec{n}\, d\sigma = \iint_S \frac{300}{\sqrt{4x^2+4y^2+1}}\, d\sigma$.

$= \iint_{x^2+y^2\le 25} 300\, dxdy = 300\,(\text{area of } \{x^2+y^2\le 25\})$

$= 300\pi\cdot 25$ -OR- the negative of this if you used the other normal field.

M294 PI SP96 #3

39)

a) $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_{x^2+y^2+z^2\le 16} \text{div}\vec{F}\, dxdydz$ by Divergence Thm but $\text{div}\vec{F} = 5$

$= 5\cdot\frac{4}{3}\pi 4^3$

b) $\vec{\nabla}\times(\vec{\nabla}f) = \text{curl}(\vec{\nabla}f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}\left(f_{zy}-f_{yz}\right) + \dots =$

$\vec{\nabla}\cdot f$ makes no sense at all since div must act on a vector.

$\text{div}(\vec{\nabla}\times\vec{F}) = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = (P_y-N_z)_x + (M_z-P_x)_y + (N_x-M_y)_z$
$= 0$

$(\vec{\nabla}f)\cdot(\vec{\nabla}g)$ makes sense but is nothing special