M294 SP87 P2#5

(15) $\dot{\underline{x}} = A\underline{x}$

$A = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ $\quad x(0) = \begin{pmatrix}1\\ 3\end{pmatrix}$

a. FIRST LOOK FOR EIGENVALUES, SINCE WE ASSUME A SOLN.

$\underline{x} = e^{\lambda t}\underline{v}$

$D\underline{x} = \lambda e^{\lambda t}\underline{v}$

$\lambda e^{\lambda t}\underline{v} = A e^{\lambda t}\underline{v}$

$A\underline{v} = \lambda \underline{v}$

GET EIGENVALUES FROM

$\det(A - \lambda I) = 0$

$\det\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix} = 0$

$(1-\lambda)^2 - 4 = 0$

$(1-\lambda) = \pm 2$

$\lambda = 3, -1$

So:

$\underline{x}_1 = e^{3t}\underline{v}_1 \quad ; \quad \underline{x}_2 = e^{-t}\underline{v}_2$

b. NOW WE FIND THE EIGEN VECTORS

SINCE $(A - \lambda I)\underline{v} = 0$

$\lambda = 3$

$\begin{pmatrix}1-3 & 2\\ 2 & 1-3\end{pmatrix}\underline{v} = 0$

$\begin{pmatrix}-2 & 2\\ 2 & -2\end{pmatrix}\underline{v} = 0$

TRIVIALLY

$\underline{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$

DOING THE REDUCTION ONCE FOR FUN

$\left(\begin{array}{cc|c}-2 & 2 & 0\\ 2 & -2 & 0\end{array}\right)$

$\Rightarrow 2 \to 1+2$

$\left(\begin{array}{cc|c}-2 & 2 & 0\\ 0 & 0 & 0\end{array}\right)$

$1 \to \frac{1}{2}$

$\left(\begin{array}{cc|c}-1 & 1 & 0\\ 0 & 0 & 0\end{array}\right)$ – REDUCED FORM

$\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\underline{v} = 0$

$v_1 + v_2 = 0$

$v_1 = -v_2 = a$

$\underline{v} = \begin{pmatrix}a\\ a\end{pmatrix}$

TAKE $a = 1$

$\underline{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda = -1$

$\begin{pmatrix}1-(-1) & 2\\ 2 & 1-(-1)\end{pmatrix}\underline{v} = 0$

$\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\underline{v}$

$\underline{v}_2 = \begin{pmatrix}1\\ -1\end{pmatrix}$ OR $\begin{pmatrix}-1\\ 1\end{pmatrix}$

TRY DOING THE REDUCTION AGAIN; GET THE SAME RESULT

c. SO NOW

$\underline{x} = c_1 e^{3t}\begin{pmatrix}1\\ 1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\ -1\end{pmatrix}$

$x(0) = \begin{pmatrix}1\\ 3\end{pmatrix}$ so

$\begin{pmatrix}1\\ 3\end{pmatrix} = \begin{pmatrix}c_1 e^0 + c_2 e^0\\ c_1 e^0 - c_2 e^0\end{pmatrix} = \begin{pmatrix}c_1 + c_2\\ c_1 - c_2\end{pmatrix}$

THIS IS IDENTICAL TO YOUR HW/

$c_1 = 2, \; c_2 = -1$ [CAN ALSO ROW REDUCE]

$\underline{x}(t) = \begin{pmatrix}2e^{3t} - e^{-t}\\ 2e^{3t} + e^{-t}\end{pmatrix}$

$\boxed{\underline{x}(2) = \begin{pmatrix}2e^{6} - e^{-2}\\ 2e^{6} + e^{-2}\end{pmatrix}}$

M294 SP87 P3 #2

(16)

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x \qquad \text{Writing out components, } \begin{cases} \dot{x}_1 = 2x_1 \\ \dot{x}_2 = 2x_2 \end{cases}$$

So $\begin{cases} x_1(t) = c_1 e^{2t} \\ x_2(t) = c_2 e^{2t} \end{cases}$ or $\boxed{x(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

Alternate solution:

Find eigenvalues and eigenvectors $\det\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2$ so $\lambda = 2, 2$ and $\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$ has two lin. indep. solns, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, whence

$$x(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

M294 SP87 F #1

(17)

d) $0 = \det[A - \lambda I] = \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$
$\Rightarrow \lambda = -1, 3$. Say $\lambda = -1 \Rightarrow \underline{[A - \lambda I]\underline{x} = \underline{0}} \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\Rightarrow \underline{x} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$ $\boxed{\lambda = -1 \text{ has e-vector } \underline{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

M294 SP88 P2 #7

(25)

$$\underline{x}(t) = \begin{bmatrix} e^{\lambda t} \\ e^{\lambda t} \\ e^{\lambda t} \end{bmatrix}$$

$$\dot{\underline{x}}(t) = \begin{bmatrix} \lambda e^{\lambda t} \\ \lambda e^{\lambda t} \\ \lambda e^{\lambda t} \end{bmatrix} = \lambda e^{\lambda t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \overset{\text{want}}{=} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} e^{\lambda t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= e^{\lambda t}\begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} \longleftarrow \text{compare} \longrightarrow \lambda\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

$\therefore \boxed{\lambda = 6}$

M294 FA90 P2 #3

42

a)

n	t	x	y
0	1	1	2
1	1.25	3	8

$x_n = x_{n-1} + h\,(5x_{n-1}^2 + t_{n-1}y_{n-1} + t_{n-1}^3)$

$y_n = y_{n-1} + h\,(13x_{n-1}y_{n-1} - 2t_{n-1})$

$x_1 = 1 + 0.25\,(5+2+1) = \underline{3}$

$y_1 = 2 + 0.25\,(13(2) - 2) = \underline{8}$

b) Separate: $\int \frac{dy}{y^2} = 2\int \sin x\,dx$, $-\frac{1}{y} = -2\cos x + c$, $\frac{1}{y} = 2\cos x + C$

$\Rightarrow y = \frac{1}{2\cos x + c}$, $y(0) = 1 \Rightarrow 1 = \frac{1}{2(1)+c}$, $c = -1 \Rightarrow y = \frac{1}{2\cos x - 1}$

70 M294 FA92F #3

let $A = \begin{pmatrix}1&1\\4&1\end{pmatrix}$ then $\det(A-\lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$

and $\begin{pmatrix}1&1\\4&1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = -1\begin{pmatrix}1\\-2\end{pmatrix}$, $\begin{pmatrix}1&1\\4&1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = 3\begin{pmatrix}1\\2\end{pmatrix}$

so $x_h = c_1\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{3t}$ solves $Ax_h = x_h'$

Next Assume $x_p =$ constant $\begin{pmatrix}a\\b\end{pmatrix}$ then $0 = A\begin{pmatrix}a\\b\end{pmatrix} + \begin{pmatrix}3\\0\end{pmatrix}$ forces

$\begin{cases} a+b=-3 \\ 4a+b=0\end{cases}$ so $\begin{matrix}a=1\\b=-4\end{matrix}$ so $\boxed{x = \begin{pmatrix}1\\-4\end{pmatrix} + c_1\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e}$

71 M294 FA92 F #6

gen soln $y = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$ (if $\lambda > 0$)

$y'(0) = c_2\sqrt{\lambda} = 0$ so $c_2 = 0$

$y(1) = c_1\cos\sqrt{\lambda}$ will be 0 if $\sqrt{\lambda} =$

$$\boxed{\begin{matrix} y = c_1\cos\left(\frac{(2n+1)\pi}{2}x\right) \quad n = 0,1,2,3,\dots \\ \lambda = \left(\frac{(2n+1)\pi}{2}\right)^2 \end{matrix}}$$

one more case: $\lambda = 0$ gen soln $\left.\begin{matrix} y = mx+b \\ y'(0) = m = 0 \\ y(1) = b = 0\end{matrix}\right\}$ So

Section 3.3

72 M294 FA92 F #10

$x(0) = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$ is an eigenvector by problem 3,

$\lambda = -1$

$\sqrt{\lambda} = \pm i$

so

$$\boxed{\begin{array}{l} x(t) = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}(c_1 \cos t + c_2 \sin t) \\ \text{I.C. give} \quad c_1 = 1, \; c_2 = 0 \end{array}}$$

then

M294 FA93 P3 #2

$$\underline{x}' = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \underline{x}$$

102.) a.) $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \begin{pmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

e'values $\lambda^2 + \omega^2 = 0 \Rightarrow \underline{\underline{\lambda_{1,2} = \pm i\omega}}$

$\lambda_1 = i\omega$ $\quad \begin{pmatrix} -i\omega & \omega \\ -\omega & -i\omega \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow -i\cancel{\omega}x_1 + \cancel{\omega}x_2 = 0 \Rightarrow x_1 = -i x_2$

$\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_2 = -i\omega$ $\quad \begin{pmatrix} i\omega & \omega \\ -\omega & i\omega \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow i\cancel{\omega}x_1 - \cancel{\omega}x_2 = 0 \Rightarrow x_1 = i x_2$

$\begin{pmatrix} i \\ 1 \end{pmatrix}$

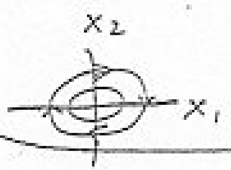
b.) $\underline{x}^{(1)} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{i\omega t}$ $\qquad \underline{x}^{(2)} = c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\omega t}$

$$\underline{x}^{(1)} = \begin{pmatrix} -i \\ 1 \end{pmatrix}(\cos\omega t + i \sin\omega t) = \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} + i\begin{pmatrix} -\cos\omega t \\ \sin\omega t \end{pmatrix}$$

So the real valued solution is $\boxed{\underline{x} = c_1 \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} + c_2 \begin{pmatrix} -\cos\omega t \\ \sin\omega t \end{pmatrix}}$

c.) Critical Points at $x_1' = \omega x_2 = 0 \Rightarrow x_2 = 0$

$x_2' = -\omega x_1 = 0 \Rightarrow x_1 = 0$



@ $x_2 = 1$ $\quad x_1' > 0$ $\quad \therefore \rightarrow$

$x_1 = 1$ $\quad x_2' < 0$ $\quad \downarrow$

Solutions are <u>stable</u> $\Rightarrow$ know direction of spin

d.) $\underline{x}' = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\underline{x} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \omega x_2 \\ -\omega x_1 \end{pmatrix}$

$x_1' = \omega x_2$

$x_1'' = \omega x_2' = \omega(-\omega x_1) = -\omega^2 x_1$

$\boxed{x_1'' + \omega^2 x_1 = 0}$

(72) M294 FA94 P2 #3

b) let $x_1 = y$, $x_2 = y'$, then $x_1' = x_2$

$x_2' = y'' = -2x_2 - 3x_1$

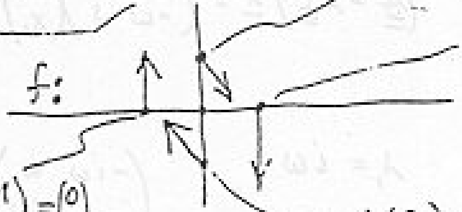
$\dot{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

So $\boxed{x' = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} x = f(x)}$

$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

f:

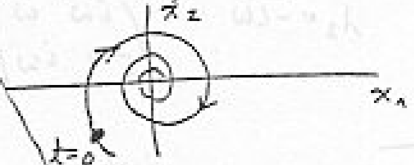


$f\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

c) for the initial conditions in (a),

$x(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ so $x(t)$ starts here, follows f, and $\to 0$ as $t \to \infty$



$y = e^{-t}\sin t$

(a) $y' = -y + e^{-t}\cos t \qquad y'' = -y' - e^{-t}\cos t - y = -y' - (y'+y$

$\boxed{y'' + 2y' + 2y = 0}$

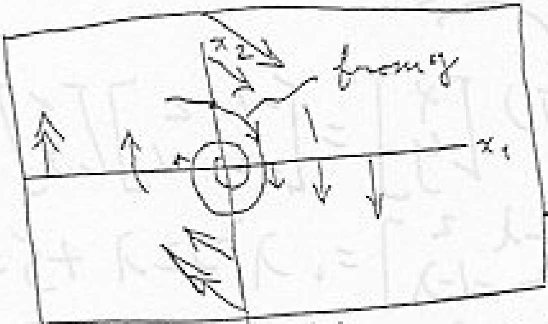
(b) let $x_1 = y$, $x_2 = y'$ then $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 2x_2 \end{cases}$

or $\boxed{x' = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}}_{A} x}$

$\det(A-\lambda) = \lambda^2 + 2\lambda + 2 \qquad \lambda = -1 \pm i$

$A - (-1+i) = \begin{pmatrix} 1-i & 1 \\ - & - \end{pmatrix} \qquad A\begin{pmatrix} -1 \\ 1-i \end{pmatrix} = (-1+i)\begin{pmatrix} -1 \\ 1-i \end{pmatrix}$

(c)



$y(0) = 0$
$y'(0) = 1$

← plotting several $A\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$

(d) let $x^{(1)} = \begin{pmatrix} -1 \\ 1-i \end{pmatrix} e^{(-1+i)t} = e^{-t}\begin{pmatrix} -(\cos t + i\sin t) \\ \cos t + i\sin t - i\cos t + \sin t \end{pmatrix}$

then $\operatorname{Re} x^{(1)} = e^{-t}\begin{pmatrix} -\cos t \\ \cos t + \sin t \end{pmatrix}$ and $\operatorname{Im} x^{(1)} = e^{-t}\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$

are solns

$x(t) = c_1 e^{-t}\begin{pmatrix} -\cos t \\ \cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$

$x(0) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \end{pmatrix} \overset{\text{want}}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $c_1 =$
$c_2 =$

$\boxed{x = -1e^{-t}\begin{pmatrix} -\cos t \\ \cos t + \sin t \end{pmatrix} - 2e^{-t}\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}}$

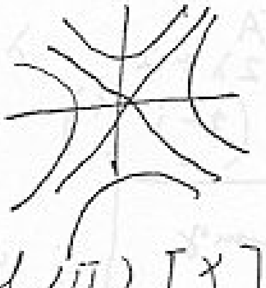
88 M294 SP95 P3 #3

(a) Phase portrait of (i) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ?$

Eigenvalues: $\det \begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda - 3 = 0$, $(\lambda+3)(\lambda-1) = 0$

$\lambda = 1, \lambda = -3$. 2 Real, different signs.

"Saddle Point"



must be $\underline{\underline{\gamma}}$.

(b) Phase portrait of (iii) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$?

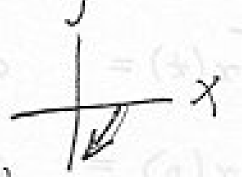
Eigenvalues: $\det \begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 5 = 0$

$\lambda = -1 \pm \frac{1}{2}\sqrt{4-20} = -1 \pm 2i$. Complex. Rotation.

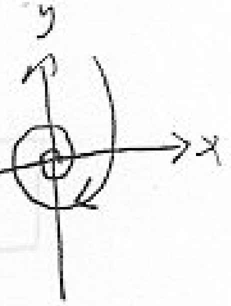
Non-zero real part. Spiral. $\operatorname{re}\lambda < 0$ inward asympt.

Which orientation of spiral? If $x = 1, y = 0$

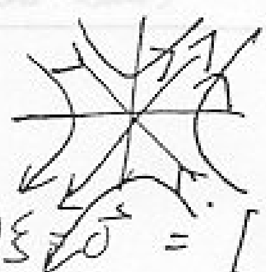
Then $x' = -1$, $y' = -2$



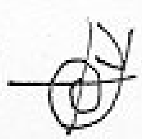
Check Chart. It's gotta be η



c) (i)



(ii)



Need Eigenvectors for

$(A - I)\xi_1 = \vec{0} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\xi_{-3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(i) Not all trajectories go towards one. (ii) all go to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

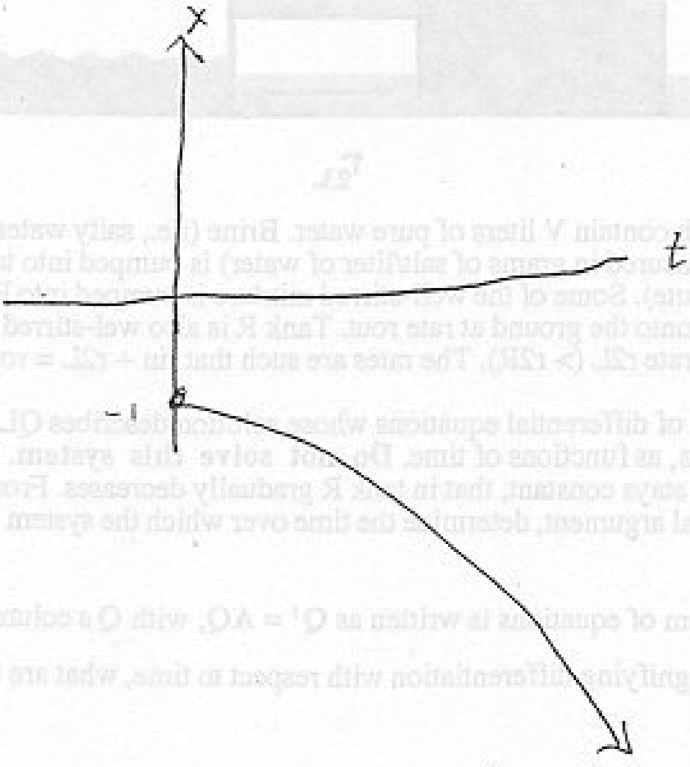
A74

(88) M294 SP95 P3 #3

d) general solution $= Ae^{t}\begin{bmatrix}1\\1\end{bmatrix} + Be^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}.$

$$\begin{bmatrix}x\\y\end{bmatrix} = -\tfrac{1}{2}e^{t}\begin{bmatrix}1\\1\end{bmatrix} + \tfrac{1}{2}e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}.$$

e) $x(t) = -\tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{-3t}.$

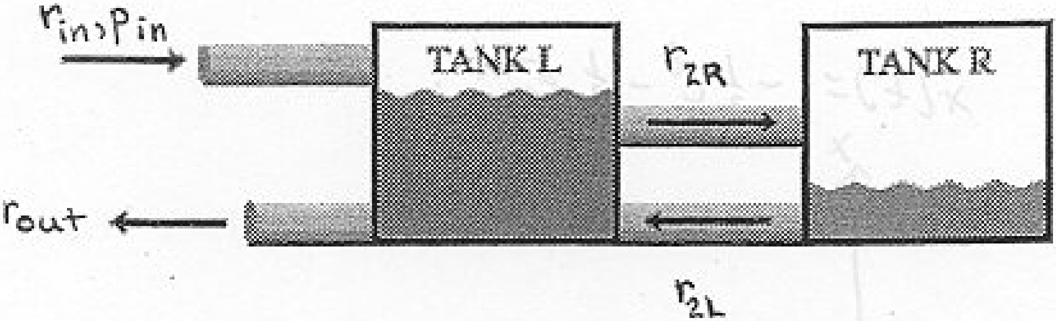


(86) M294 FA94 F #2

Solve $x' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$

Sketch the solution curve given parametically by $x_1(t), x_2(t), x_3(t)$.

(87) M294 SP95 P2 #3



Initially tanks L and R each contain V liters of pure water. Brine (i.e., salty water), with a concentration pin (e.g., measured in grams of salt/liter of water) is pumped into tank L at the rate of rin (measured in liters/minute). Some of the well-stirred mixture is pumped into R at the rate r, while some more is pumped out onto the ground at rate rout. Tank R is also wel-stirred and its contents are fed back into tank L at the rate r2L (> r2R). The rates are such that rin + r2L = rout + r2R.

(20 pts.) Set up the system of differential equations whose solution describes QL and QR, the amounts of salt in the tanks, as functions of time. **Do not solve this system.** Note that, while the amount of liquid in tank L stays constant, that in tank R gradually decreases. From both a mathematical and a physical argument, determine the time over which the system of differential equations is valid.

(10 pts.) If the above system of equations is written as Q' = AQ, with Q a column vector (containing elements QL and QR) and ' signifying differentiation with respect to time, what are the elements of the 2x2 matrix A?

38 M294 SP95 P3 #3

Consider the following systems of first order differential equations

(i)
$$x' = -x + 2y$$
$$y' = 2x - y$$

(ii)
$$x' = -x + 2y$$
$$y' = -2x - y$$

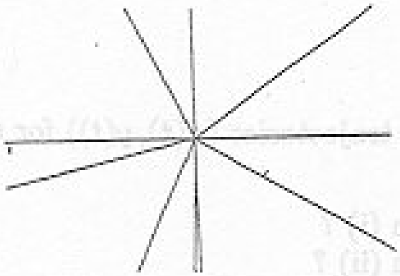
Consider also the following phase plane pictures of typical trajectories $(x(t), y(t))$ for solutions of some first order linear systems.

(a) Which phase plane picture corresponds to system (i) ?
(b) Which phase plane picture corresponds to system (ii) ?
(c) Sketch your chosen pictures in your exam book, placing "arrowheads" on the trajectories to indicate the direction in which the solutions move (along the trajectories) as $t \to \infty$. For each system, state whether all solutions approach one particular solution as $t \to \infty$.
(d) Give the solution to system (i) satisfying the initial condition

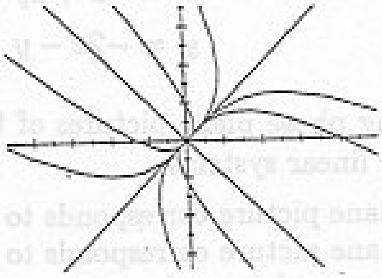
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

(e) Sketch the graph of $x(t)$ from your solution to (d) (in the (t, x)-plane.

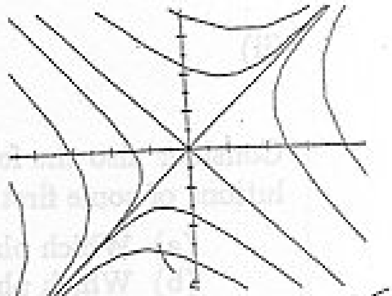
First-Order Menagerie



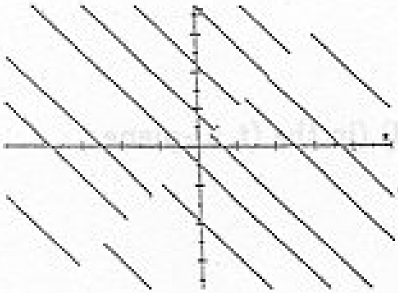
α



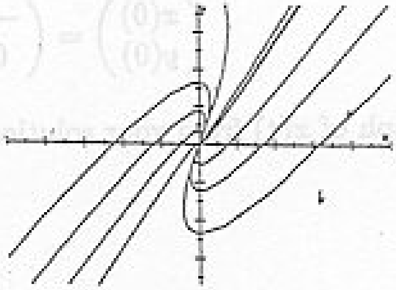
β



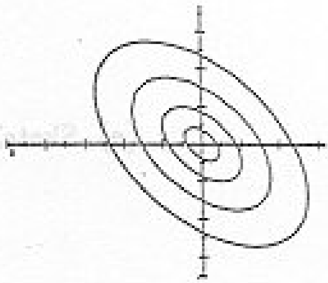
γ ✓ i



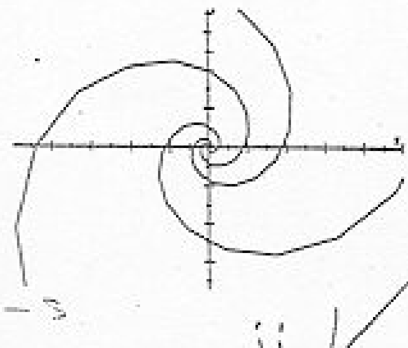
δ



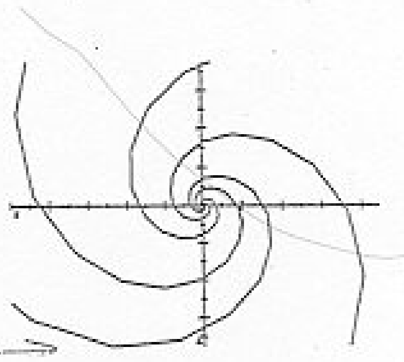
ε



ζ



η ii ✓



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(89) M294 SP95 F #2

(a) $\det\begin{bmatrix} 2-\lambda & 6 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 5\lambda = 0, \quad \lambda = 5, \lambda = 0.$

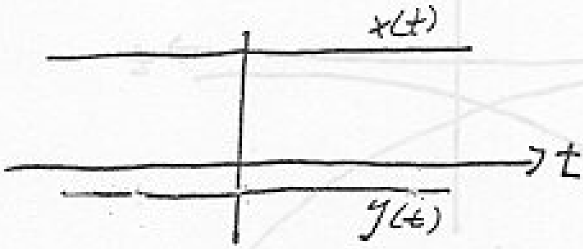
$\vec{\xi}_5$: $\begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \xi_1 - 2\xi_2 = 0. \quad \vec{\xi}_5 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

$\vec{\xi}_0$: $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \xi_1 + 3\xi_2 = 0 \quad \vec{\xi}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$

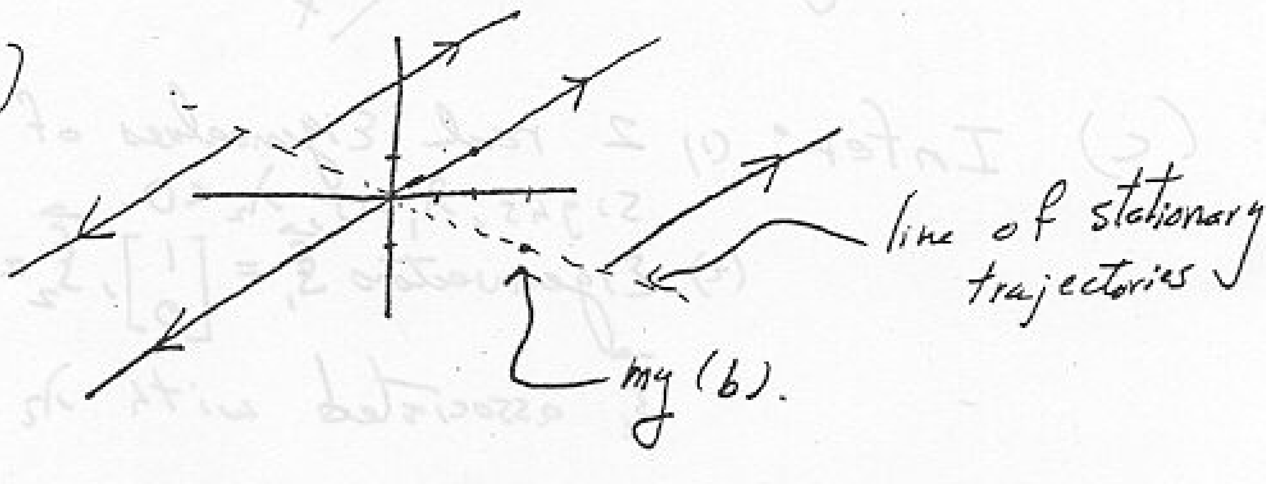
Solutions : $e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)



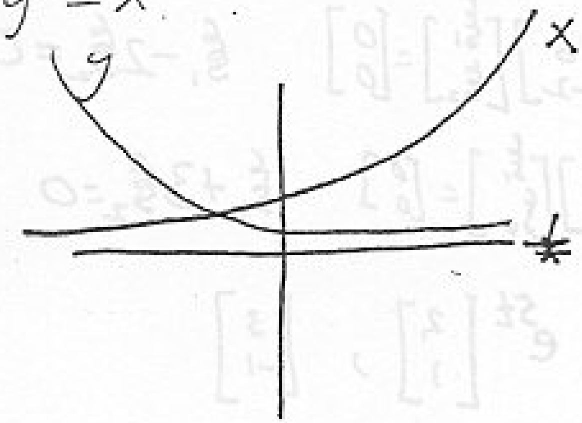
(d)



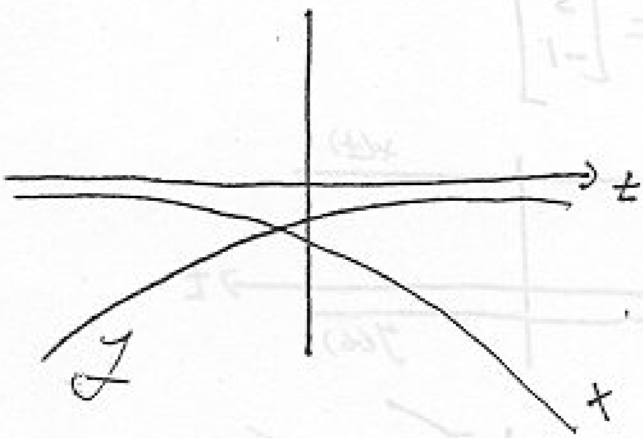
(90) M294 F SP95 #3

$y' - 3y = x$

(a)



(b)



(c) Infer: (1) 2 real Eigenvalues of opposite signs, $\lambda_1 > 0$, $\lambda_2 < 0$.

(2) Eigenvectors $\vec{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{\xi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{\xi}_i$ associated with λ_2.

(d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4. There are no oscillatory solutions.

93 M294 FA95 P2 #3

let $x_1 = y$, $x_2 = y'$. Then $x_2' = y'' = -3y' - y - y^3 = -3x_2$

so $\boxed{\begin{cases} x_1' = x_2 \\ x_2' = -3x_2 - x_1 - x_1^3 \end{cases}}$ other correct answers are possible

94 M294 FA95 P3 #1

$$e^{(-2+i)t}\begin{pmatrix}1+i\\ i\end{pmatrix} = e^{-2t}\begin{pmatrix}\cos t + i\cos t + i\sin t - \sin t\\ i\cos t - \sin t\end{pmatrix}$$

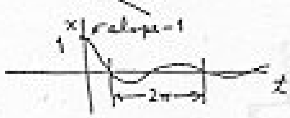
$$= e^{-2t}\begin{pmatrix}\cos t - \sin t\\ -\sin t\end{pmatrix} + i e^{-2t}\begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix}$$

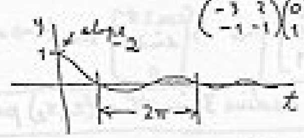
$\therefore$ gen. soln $\begin{pmatrix}x\\ y\end{pmatrix} = c_1 e^{-2t}\begin{pmatrix}\cos t - \sin t\\ -\sin t\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix}$ inward spiral

$$\begin{pmatrix}x'(0)\\ y'(0)\end{pmatrix} = \begin{pmatrix}-3 & 2\\ -1 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}-1\\ -2\end{pmatrix}$$

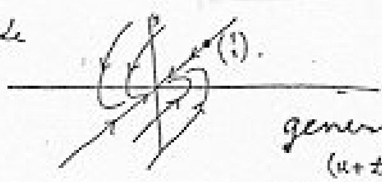
plot other field vectors: $\begin{pmatrix}-3 & 2\\ -1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}-3\\ -1\end{pmatrix}$

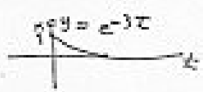
$\begin{pmatrix}-3 & 2\\ -1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}2\\ -1\end{pmatrix}$





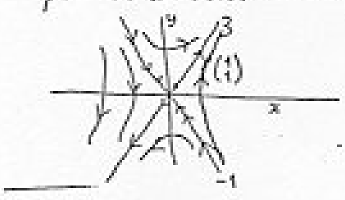
b) This is a node

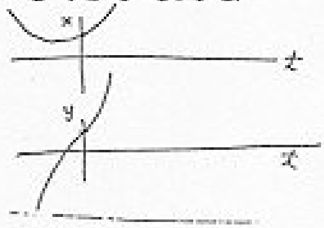




general soln is of the form $(u + tv)e^{-3t}$ where $v = c_1\begin{pmatrix}1\\ 1\end{pmatrix}$ and $\left(\begin{pmatrix}1 & -4\\ 4 & -7\end{pmatrix} - (-3)I\right)u = v$

(c) gen soln is $c_1 e^{-t}\begin{pmatrix}-1\\ 2\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}1\\ 2\end{pmatrix}$. Since one eigenvalue is positive and one is negative, this is a saddle





95 M294 FA95 F #3

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \det(A-\lambda I) = (-5-\lambda)\det\begin{pmatrix} -\lambda & -2 \\ 2 & -\lambda \end{pmatrix}$$
$$= (-5-\lambda)(\lambda^2+2)$$
$$= -(\lambda+5)(\lambda+2i)(\lambda-2i)$$

eigenvectors: $(A-2iI)u = \begin{bmatrix} -2i & -2 & 0 \\ 2 & -2i & 0 \\ 0 & 0 & -5-2i \end{bmatrix} u = 0$ if $\boxed{u_3 = 0 \text{ and } u_2 = -iu_1 \text{ take } u = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}}$

$A\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = 2i\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$ then since A is real,

$$A\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = -2i\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$ ← call it v

$(A-5I)w = \begin{bmatrix} -5 & -2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} w$ so take $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $Aw = -5w$

(a) gen soln to $x' = Ax$ is $x(t) = \underbrace{c_1 e^{2it}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}}_{\begin{bmatrix} \cos 2t + i\sin 2t \\ \sin 2t - i\cos 2t \\ 0 \end{bmatrix}} + c_2 e^{-2it}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} + c_3 e^{-5t}$

or, using real solutions,

$$\boxed{x(t) = a_1\begin{bmatrix} \cos 2t \\ \sin 2t \\ 0 \end{bmatrix} + a_2\begin{bmatrix} \sin 2t \\ -\cos 2t \\ 0 \end{bmatrix} + a_3 e^{-5t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

(b) $x(0) = a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ if $a_1 = 3$, $a_2 = 0$, $a_3 = 1$

$\boxed{x(t) = 3\begin{bmatrix} \cos 2t \\ \sin 2t \\ 0 \end{bmatrix} + e^{-5t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \sim 3\begin{bmatrix} \cos 2t \\ \sin 2t \\ 0 \end{bmatrix}$ as $t \to \infty$ so this curve $\boxed{\text{approaches a circle of radius 3 in the } (x_1, x_2) \text{ plane}}$

Section 3.3

(96) M294 SP96 P3 #1

Let $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$.

Using $\lambda = -1+2i$, $(A-\lambda I)v = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is 0 if $2v_2 = 2iv_1$. So take $v = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Then $A\begin{bmatrix} 1 \\ i \end{bmatrix} = (-1+2i)\begin{bmatrix} 1 \\ i \end{bmatrix}$

and

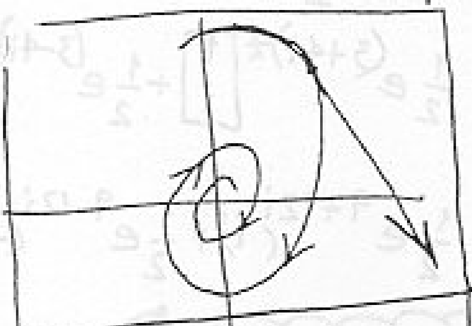
$$ve^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+2i)t} = e^{-t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + ie^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

So gen. soln is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

This an inward spiral; to see which way it goes try the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ say, where $A\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -13 \end{bmatrix}$

so it is clockwise.



critical point is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ so } c_1 = 4,\ c_2 = 5$$

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 4e^{-t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + 5e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}$$

M294 FA96 P2 #3

98) **Solution By Hand** Write $z = \begin{bmatrix} x \\ y \end{bmatrix}$ then $z' = Az$, $A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

$\det(A - \lambda I) = (3-\lambda)^2 + 4^2$ is 0 if $3 - \lambda = \pm 4i$ ~~so~~ $\lambda = 3 \pm 4i$

If $\underline{\lambda = 3+4i}$: $(A - \lambda I)v = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is 0 if $-4iv_1 + 4v_2 = 0$

$v_2 = iv_1$

set $v = \begin{bmatrix} 1 \\ i \end{bmatrix}$ check: $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} \overset{?}{=} (3+4i)\begin{bmatrix} 1 \\ i \end{bmatrix}$ ✓

If $\underline{\lambda = 3-4i}$: replace i by $-i$ and find

$$\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = (3-4i)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Then — OR

$$e^{(3+4i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{3t}\begin{bmatrix} \cos 4t + i\sin 4t \\ i\cos 4t - \sin 4t \end{bmatrix}$$

So

$$z(t) = e^{3t}\left(c_1\begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + c_2\begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}\right)$$

$$z(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$z(t) = e^{3t}\begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix}$$

$$\boxed{y(3) = -e^9 \sin(12)}$$

$$z(t) = a_1 e^{(3+4i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + a_2 e^{(3-4i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$z(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ ia_1 - ia_2 \end{bmatrix} \rightarrow a_1 = a_2$$

$\therefore a_1 = a_2 = \frac{1}{2}$

$$z(t) = \frac{1}{2}e^{(3+4i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{(3-4i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$y(3) = \frac{1}{2}e^{9+12i}(i) + \frac{1}{2}e^{9-12i}(-i)$$

check this if you have time

these are the same

$\boxed{3}$ **Solution by MATLAB**

rhs.m

```
function zdot = rhs(z)
zdot(1) = 3*z(1) + 4*z(2);
zdot(2) = -4*z(1) + 3*z(2);
% z(1) means x, z(2) means y
```

```
[t, z] = ode23('rhs', 0, 3, [1 0]);   % z is now a list of vectors
z(length(z))(2)                        % and you want the y coordinate of the
                                       % last one.
```

99 M294 FA96 P3 #1

$$\underline{x}' = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\underline{x}$$

(a) or

$$\begin{pmatrix}x_1'\\ x_2'\end{pmatrix} = \begin{pmatrix}-x_2\\ x_1\end{pmatrix} \quad \therefore\ x_1' = -x_2 \text{ and } x_1'' = -x_2' = -x_1$$

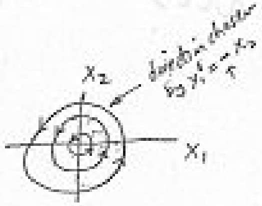
$$\boxed{\therefore\ x_1'' + x_1 = 0}$$

(b) E'values

$$\det\begin{pmatrix}0-\lambda & -1\\ 1 & 0-\lambda\end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \boxed{\lambda = \pm i}$$

Critical point at $x_2 = 0$, $x_1 = 0$, i.e., origin

Point is a center & is stable.



(c)

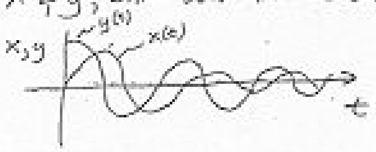
$$\det\begin{pmatrix}(\epsilon-\lambda) & -1\\ 1 & (\epsilon-\lambda)\end{pmatrix} = (\epsilon-\lambda)^2 + 1 = 0 \Rightarrow \epsilon - \lambda = \pm i$$

$$\therefore\ \lambda = +\epsilon \mp i$$

Origin is still critical point, but now a spiral point. It will be stable if $\epsilon < 0$, unstable if $\epsilon > 0$.

These represent decaying oscillations in both x & y, which both have the same amplitude and period.

y leads x by 90° (¼ period) in phase



100 M293 FA96 P3 #4

(a) $\left.\begin{array}{l} x(5-y) = 0\\ y(5-x) = 0\end{array}\right\}$ give $\boxed{(x,y) = (0,0) \text{ or } (x,y) = (5,5)}$

only THE SECOND graph has these critical points, and they are BOTH UNSTABLE because in both cases there are solutions trying to get away