2^{nd} and higher Order ODEs

M293 F FA92 #1

5) (b) $x\vec{i} + y\vec{j} = (1\vec{i} + 2\vec{j}) + \frac{\vec{i}+\vec{j}}{\sqrt{2}} s$, or $\left\{ \begin{array}{l} x = 1 + \frac{1}{\sqrt{2}}s \\ y = 2 + \frac{1}{\sqrt{2}}s \end{array} \right\}$ or $\boxed{y = x + 1}$

M294 P II SP96 #2

6) (a)

$$y^{iv} - 81y = 0$$

Try $y = Ce^{rx} \Rightarrow (r^4 - 81)Ce^{rx} = 0$

$$\therefore (r+3)(r-3)(r^2+9) = 0$$

$$\boxed{y_c = C_1 e^{3x} + C_2 e^{-3x} + C_3 \cos 3x + C_4 \sin 3x}$$

[illegible]

(b) Since neither a constant nor a pure trig function satisfies the homogeneous equation, for a particular solution we try

$$y_p = C_1 + C_2 \cos 2x + C_3 \sin 2x$$
$$y_p' = -2C_2 \sin 2x + 2C_3 \cos 2x$$
$$y_p'' = -4C_2 \cos 2x - 2C_3 \sin 2x$$

$\therefore y_p'' + 2y_p' + 2y_p = 2 + 10\cos 2x = 2C_1 + (2C_2 + 4C_3 - 4C_2)\cos 2x + (2C_3 - 4C_2 -$

$\therefore 2C_1 = 2 \Rightarrow \boxed{C_1 = 1}$

$\cos 2x$: $4C_3 - 2C_2 = 10$

$\sin 2x$: $-4C_2 - 2C_3 = 0 \Rightarrow C_3 = -2C_2 \Rightarrow -10C_2 = 10 \Rightarrow \boxed{C_2 = -1}$ $\boxed{C_3 = 2}$

$\therefore y = D_1 e^{-x} \sin x + D_2 e^{-x} \cos x + 1 - \cos 2x + 2\sin 2x$

$y(0) = 0 = D_2 + 1 - 1 \Rightarrow \boxed{D_2 = 0}$

$y' = -D_1 e^{-x} \sin x + D_1 e^{-x} \cos x + 2\sin 2x + 4\cos 2x$

$y'(0) = -2 = D_1 + 4$ $\boxed{D_1 = -6}$

$$\therefore \boxed{y = -6e^{-x} \sin x + 1 - \cos 2x + 2\sin 2x}$$

Check $y(0) = 0 + 1 - 1 + 0 = 0$

$y'(0) = 0 - 6 + 0 + 2(2) = -2$

M294 F1 FA93 #2

$$\underline{x}' = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \underline{x}$$

7) a.) $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \begin{pmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

e'values $\lambda^2 + \omega^2 = 0 \Rightarrow \underline{\underline{\lambda = \pm i\omega}}$

$\lambda_1 = i\omega$ $\quad \begin{pmatrix} -i\omega & \omega \\ -\omega & -i\omega \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow -i\cancel{\omega}x_1 + \cancel{\omega}x_2 = 0 \Rightarrow x_1 = -i\,x_2$ $\quad \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_2 = -i\omega$ $\quad \begin{pmatrix} i\omega & \omega \\ -\omega & i\omega \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow i\cancel{\omega}x_1 - \cancel{\omega}x_2 = 0 \Rightarrow x_1 = i\,x_2$ $\quad \begin{pmatrix} i \\ 1 \end{pmatrix}$

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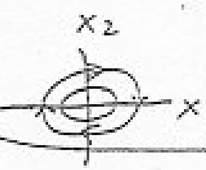
b.) $\underline{x}^{(1)} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{i\omega t}$ $\qquad \underline{x}^{(2)} = C_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\omega t}$

$$\underline{x}^{(1)} = \begin{pmatrix} -i \\ 1 \end{pmatrix}(\cos\omega t + i\sin\omega t) = \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} + i\begin{pmatrix} -\cos\omega t \\ \sin\omega t \end{pmatrix}$$

So the real valued solution is $\boxed{\underline{x} = C_1\begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} + C_2\begin{pmatrix} -\cos\omega t \\ \sin\omega t \end{pmatrix}}$

c.) Critical Points at $x_1' = \omega x_2 = 0 \Rightarrow x_2 = 0$

$x_2' = -\omega x_1 = 0 \Rightarrow x_1 = 0$



@ $x_2 = 1$ $\quad x_1' > 0 \quad \therefore \rightarrow$

$x_1 = 1$ $\quad x_2' < 0 \quad \downarrow$

Solutions are <u>stable</u> $\Rightarrow$ know direction of spin

d.) $\underline{x}' = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\underline{x} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \omega x_2 \\ -\omega x_1 \end{pmatrix}$

$x_1' = \omega x_2$

$x_1'' = \omega x_2' = \omega(-\omega x_1) = -\omega^2 x_1$

$\boxed{x_1'' + \omega^2 x_1 = 0}$

10) M293 PI FA95 #4b,c

b) $y'' + 2y' - 3y = 0$

$r^2 + 2r - 3 = 0$

$(r-1)(r+3) = 0 \qquad r = 1 \text{ or } -3 \qquad \boxed{y_c = c_1 e^{x} + c_2 e^{-3x}}$

c) $y' = c_1 e^{x} - 3c_2 e^{-3x}$

$y'' = c_1 e^{x} + 9c_2 e^{-3x}$

$y'' + 2y' - 3y = (c_1 + 2c_1 - 3c_1)e^{x} + (9c_2 - 2\cdot 3c_2 - 3c_2)e^{-3}$

$= 0 + 0$ $\boxed{\checkmark}$

M294 F SP95 #2c

"(e) $x' = 2x + 6y$ | $y = (x' - 2x)/6$, $y' = (x'' - 2x')/$
$y' = x + 3y$ | $y' = x + \frac{3}{6}(x' - 2x)$

$\dot{x}$ $\quad \frac{x'' - 2x'}{6} = x + \frac{3}{6}(x' - 2x)$

$x'' - 2x' = 6x + 3x' - 6x$

$x'' - 5x' = 0$

M294 PI FA92 #4

13) $y'' + 16y = 0$ gives $y = c_1 \cos 4x + c_2 \sin 4x$ and $y(0)$: gives $c_1 = 0$. So $y(L) = c_2 \sin 4L = 0$ only for $c_2 = 0$ or $\boxed{4L = \pi, 2\pi, 3\pi, 4\pi, \ldots}$ $\boxed{4b}$ $\boxed{y = C \sin 4x}$ C (const) $\boxed{4c}$ $y = 0$ is unique for all L $\underline{\text{except}}$ $\frac{\pi}{4}$.

M294 PII FA96 #1

15)

a) $x'' = -4x$ general soln is $c_1 \cos 2t + c_2 \sin 2t$
$x(0) = 0 = c_1$, $x'(0) = 4 = 2c_2$, $c_2 = 2$ $\boxed{x = 2 \sin 2t}$

If you did $r^2 + 4 = 0$ you get the same thing, or maybe

$\frac{e^{2it} - e^{-2it}}{i}$ which is the same.

b) For $x'' + 4x = \cos(ct)$, you nearly always find a particular solution of the form $A\cos(ct)$. The only exception is when $\boxed{c = 2}$ because then the right hand side is a solution to the homogeneous equation.

M294 PII FA94 #3

17a) $\left.\begin{array}{l} y'' + 3y' + 2y = 0 \\ y(0) = -1 \\ y'(0) = -1 \end{array}\right\}$ $r^2 + 3r + 2 = 0$ gives $r = -1, -2$ so $y = c_1 e^{-t} + c_2 e^{-2t}$

Then $y(0) = -1 = c_1 + c_2$
$y'(0) = -1 = -c_1 - 2c_2$
$c_2 = 2, c_1 = -3$

$\boxed{y = -3e^{-t} + 2e^{-2t}}$

M293 PI SP94 #1

19)

$y = e^{-t}\sin t$

(a) $y' = -y + e^{-t}\cos t \qquad y'' = -y' - e^{-t}\cos t - y = -y' - (y' + y)$

$$\boxed{y'' + 2y' + 2y = 0}$$

M294 F FA95 #2

21) (a) $y'' + a^2 y = \sin ax$

$y =$ homogeneous soln + particular solution

$= c_1 \cos ax + c_2 \sin ax + \underbrace{(bx^2 + cx + d)\sin ax + (ex^2 + fx + g)\cos ax}_{\text{trying } y_p}$,

$y_p'' + a^2 y_p - \sin ax = 2b\sin ax + 2(2bx + c)a\cos ax$
$+ (bx^2 + cx + d)a^2(-\sin ax) + 2e\cos ax$
$+ 2(2ex + f)(-a\sin ax) + (ex^2 + fx + g)(-a^2\cos ax)$
$+ a^2(bx^2 + cx + d)\sin ax + a^2(ex^2 + fx + g)\cos ax - \sin ax$

$= \cos(ax)\big((4b - a^2 f + a^2 f)x + (2ac + 2e - a^2 g + a^2 g) + x^2(-ea^2 + ea^2)$
$+ \sin(ax)\big((2b - a^2 d - 2af + da^2 - 1) + x(-ca^2 - 4ae + ca^2) + x^2(-ba^2 + a^2 b)\big)$

$= \cos ax\,(4abx + (2ac + 2e)) + \sin ax\,((2b - 2af - 1) - 4aex)$

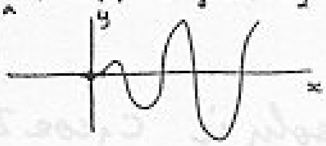
so we need $b = 0$, $e = -ac$, $af = -1/2$ or $f = -1/2a$, $e = 0$

$$\boxed{y = c_1\cos ax + c_2 \sin ax + \frac{-1}{2a}x\cos ax}$$ (may as well absorb $d + g$ into $c_2 + c_1$)

with the I.C.,

$y(0) = c_1 = 0$, $y'(0) = ac_2 + \frac{-1}{2a}\cdot 1\cdot\cos(0) = 0$ gives $c_2 = \frac{1}{2a^2}$

$$\boxed{y = \frac{1}{2a^2}\sin ax + \frac{-1}{2a}x\cos ax}$$



(b) $y'' - a^2 y = \sin ax$

$y = c_1 e^{ax} + c_2 e^{-ax} + A\sin ax$

$y'' - a^2 y - \sin ax = (-a^2 A - a^2 A - 1)\sin ax$, so $A = \frac{-1}{2a^2}$

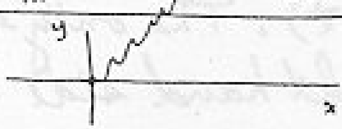
$$\boxed{y = c_1 e^{ax} + c_2 e^{-ax} + \frac{-1}{2a^2}\sin ax}$$

with the I.C., $y(0) = c_1 + c_2 = 0$, $c_2 = -c_1$

$y'(0) = ac_1 - ac_2 - \frac{a}{2a^2} = 0$, $2ac_1 = \frac{a}{2a^2}$

$c_1 = \frac{1}{4a}$

$$\boxed{y = \frac{1}{4a}(e^{ax} - e^{-ax}) + \frac{-1}{2a^2}\sin ax}$$



M294 SP97 PI #1

25) $$\ddot{x} + 4x = 2\cos t + 3\sin 3t$$

(a) The general solution is composed of 2 parts i.e. homogeneous & particular

$$x(t) = x_h(t) + x_p(t)$$

$x_h(t)$ is obtained as:

$$\ddot{x}_h + 4x_h = 0 \quad \text{i.e.} \quad x_h(t) = A\cos 2t + B\sin 2t$$

$x_p(t)$ is obtained as:

$$x_p(t) = c_1\cos t + c_2\sin 3t$$

Substituting into o.d.e.:

$$-c_1\cos t - 9c_2\sin 3t + 4c_1\cos t + 4c_2\sin 3t = 2\cos t + 3\sin$$

$$\Rightarrow \quad 3c_1\cos t - 5c_2\sin 3t = 2\cos t + 3\sin 3t$$

$$\Rightarrow \quad c_1 = \frac{2}{3} \quad \text{and} \quad c_2 = -\frac{3}{5}$$

General Solution is:

$$x(t) = A\cos 2t + B\sin 2t + \frac{2}{3}\cos t - \frac{3}{5}\sin 3t$$

M294PI FA92 #3

34) a) $y'' + y' + 3y = 0$

$\lambda^2 + \lambda + 3 = 0$ has roots $\lambda_{1,2} = \frac{-1 \pm \sqrt{1-12}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$

so $\boxed{y = c_1 e^{-\frac{1}{2}x} \cos\frac{\sqrt{11}}{2}x + c_2 e^{-\frac{1}{2}x} \sin\frac{\sqrt{11}}{2}x}$

b) Solve first $y'' + y' + 3y = e^{i2x}$ by assuming $y = Ae^{i2x}$ th

$$(-4A + 2iA + 3A)e^{i2x} = e^{i2x}$$

$$-A + 2iA = 1, \quad A = \frac{1}{-1+2i} = \frac{-1-2i}{5}$$

$$y = \frac{-1-2i}{5}(\cos 2x + i \sin 2x) = -\frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x + i(\cdots)$$

So a particular soln to $y'' + y' + 3y = \cos 2x$ is the real part $\boxed{-\frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x}$

c) $\boxed{y(x) = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{11}}{2}x + c_2 \sin\frac{\sqrt{11}}{2}x\right) - \frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x}$

d) $y(0) = c_1 - \frac{1}{5} = 1$ gives $c_1 = \frac{6}{5}$

$y'(0) = -\frac{1}{2}c_1 + \frac{\sqrt{11}}{2}c_2 + \frac{2 \cdot 2}{5} = -2$ gives $c_2 = \dfrac{-2 + \frac{4}{5} + \frac{3}{5}}{\frac{\sqrt{11}}{2}} = \dfrac{-6}{5\sqrt{11}}$

$\boxed{y(x) = e^{-\frac{1}{2}x}\left(\frac{6}{5}\cos\frac{\sqrt{11}}{2}x - \frac{6}{5\sqrt{11}}\sin\frac{\sqrt{11}}{2}x\right) - \frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x}$

e) $\boxed{y_p = x^2(A_0 + A_1 x + A_2 x^2)e^x + B_1 \sin 5x + B_2 \cos 5x + C_1 + C_2 x}$

M 244 FA96 P2 #3

38) Consider the ODE to be

$$y'' + by' + cy = 0$$

We know the solution is

$$y = 4e^{-t}\sin 2t$$

$$y' = -4e^{-t}\sin 2t + 8e^{-t}\cos 2t$$

$$y'' = 4e^{-t}\sin 2t - 8e^{-t}\cos 2t - 8e^{-t}\cos 2t - 16e^{-t}\sin 2t$$

$$= -12e^{-t}\sin 2t - 16e^{-t}\cos 2t$$

Substituting these back into the general equation:

$$y'' + by' + cy = -12e^{-t}\sin 2t - 16e^{-t}\cos 2t + b(-4e^{-t}\sin 2t + 8e^{-t}\cos 2t) + 4ce^{-t}\sin 2t$$

$$= (-12 - 4b + 4c)e^{-t}\sin 2t + (-16 + 8b)e^{-t}\cos 2t$$

If this is to satisfy the eqn, each coefficient must vanish

$$0 = -16 + 8b \Rightarrow b = 2$$

$$0 = -12 - 4b + 4c \Rightarrow c = 3 + b = 5$$

$\therefore$ The equation is $\boxed{y'' + 2y' + 5y = 0}$

Test by characteristic eqn $r_{1,2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} = -1 \pm \sqrt{1-5}$

$= -1 \pm 2i$ OK

Alternatively, $-\frac{b}{2} = -1 \Rightarrow b = 2$

$\left(\frac{b}{2}\right)^2 - c = -4 \Rightarrow c = 4 + 1 = 5$

ⓑ general soln is

$$y = Ae^{-t}\sin 2t + Be^{-t}\cos 2t$$

$$y(0) = A(1)(0) + B(1)(1) = B$$

but soln has no term in $\cos 2t$ $\therefore$ $y(0) = 0$

$$y' = \frac{d}{dt}(Ae^{-t}\sin 2t) = -Ae^{-t}\sin 2t + 2Ae^{-t}\cos 2t$$

$$y'(0) = -A(1)(0) + 2A(1)(1)$$

$$\therefore A = \frac{y'(0)}{2}$$

For our problem $A = 4 \Rightarrow y'(0) = 8$

Alternatively, we have the soln so it must match the I.C.
$y(0) = 4e^{0}\sin 2(0) = 0$
$y'(t) = -4e^{-t}\sin 2t + 8e^{-t}\cos 2t$
$y'(0) = 0 + 8 = 8$

ⓒ Terms in particular soln must be those on RHS, plus those that would arise from differentiation plus extras for repeated roots.

$$\therefore y_p = A + B\sin t + C\cos t + De^{-t}\cos 2t + Ee^{-t}\sin 2t + Fte^{-t}\cos 2t + Gte^{-t}\sin 2t$$

M294 P II FA94 # 2

39) a) $y'' + 2y' + 3y = 0$, $y(0) = -1$, $y'(0) = -1$

$r^2 + 2r + 3 = 0$ gives $r = -1 \pm i\sqrt{2}$

so $y = (c_1 \cos\sqrt{2}\,t + c_2 \sin\sqrt{2}\,t)e^{-t}$

$y(0) = -1 = c_1$

$y'(0) = -1 = -c_1 + \sqrt{2}\,c_2$

$c_2 = -\sqrt{2}$

$\boxed{y = e^{-t}(-\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t)}$

M293 F SP96 #29

45) False.

M293 P II SP98 #2

49) $\lambda^4 + 6\lambda^2 + 5 = 0 \Rightarrow (\lambda^2+1)(\lambda^2+5) = 0 \Rightarrow \lambda = \pm i$ or $\pm\sqrt{5}i \Rightarrow$

$X(t) = c_1 e^{it} + c_2 e^{-it} + c_3 e^{\sqrt{5}it} + c_4 e^{-\sqrt{5}it}$

$X(t) = C_1 \sin t + C_2 \cos t + C_3 \sin\sqrt{5}t + C_4 \cos\sqrt{5}t.$

60 $y'' + \lambda y = 0 \qquad y(0) + y'(0) = 0 \quad , \quad y(1) = 0$

$\because$ eigenvalues are nonnegative $\Rightarrow$ consider 2 cases

1.) $\lambda = 0$

2.) $\lambda > 0 \quad (\lambda = +\alpha^2 \ (\alpha > 0))$

<u>Case 1.)</u> $\lambda = 0$

$$\Rightarrow y'' = 0 \Rightarrow y = Ax + B \Rightarrow y'(x) = A$$

$$y(0) + y'(0) = B + A = 0 \Rightarrow A = -B$$

$$y(1) = A + B = 0 \Rightarrow A = -B$$

i.e. $y(x) = Ax - A = A(x-1)$

i.e. $\lambda = 0$ is an eigenvalue with $y \equiv (x-1)$ as eigenfunct

<u>Case 2.)</u> $\lambda = +\alpha^2 \ (\alpha > 0)$

$$y'' + \alpha^2 y = 0 \Rightarrow y(x) = A\cos\alpha x + B\sin\alpha x$$

$$y' = -A\alpha\sin\alpha x + B\alpha\cos\alpha x$$

$$y(0) + y'(0) = A + B\alpha = 0 \Rightarrow B\alpha = -A$$

$$y(1) = A\cos\alpha + B\sin\alpha = 0$$

$$-B\alpha\cos\alpha + B\sin\alpha = 0$$

$$B\left[\sin\alpha - \alpha\cos\alpha\right] = 0$$

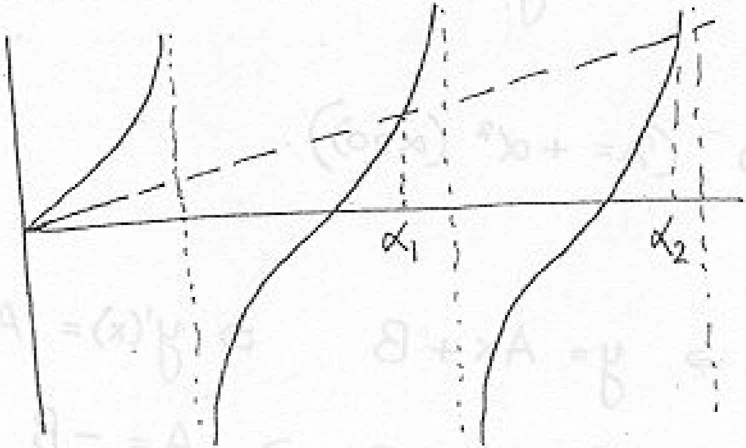
For $B \neq 0$ (Non-Trivial solutions)

$$\sin\alpha - \alpha\cos\alpha = 0$$

M294 PI SP97 #2

Section 3.2

6.0 cont'd) i.e. $\tan\alpha = \alpha$ $(\cos\alpha \neq 0)$



Eigenvalues: $\lambda_n = \alpha_n^2$

Eigenfunction: $y_n = -\alpha_n \cos\alpha_n x + \sin\alpha_n x$

M294 F SP98 #1

6.2) $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1 \Rightarrow \left.\begin{array}{l} x(t) = c_1 e^t + c_2 t e^t \\ x(0) = 2\ ;\ x'(0) = 3 \end{array}\right\} \Rightarrow x(t) = 2e^t + t e^t$

(note: $x'(t) = (c_1 + c_2)e^t + c_2 t e^t$)

#