5.4 Heat Equation

MATH 294 SPRING 1985 FINAL # 1

5.4.1 **a**) Find the solution to the partial differential equation given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \text{where} u = u(x,t)$$

with boundary conditions
$u(0,t) = 0$
$u(L,t) = 0$
and initial condition
$u(x,0) = 1;\ 0 < x < L$

b) What does the solution $u(x,t)$ approach as $t \to \infty$. Briefly explain this answer. (No credit will be given for guessing.)

MATH 294 FALL 1986 FINAL # 10

5.4.2 Solve for $u = u(x,t)$ where

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, (0 < x < 2, t > 0),$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(2,t) = 0 \end{array}\right\} t > 0$$

$$u(x,0) = \cos(2\pi x), 0 \leq x \leq 2.$$

MATH 294 FALL 1986 FINAL # 13

5.4.3 Find the solution to the initial/boundary value problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < L, t > 0$$

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, t > 0$$

$$u(x,0) \equiv 1, 0 < x < L.$$

You may use symmetry to solve a more familiar problem on $0 < x < 2L$.

MATH 294 FALL 1987 PRELIM 1 # 8

5.4.4 Find **any** non-zero solution to the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

that satisfies the boundary conditions $u(0,t) = u(5,t) = 0$. (You need not go into great detail explaining how you find this solution.)

MATH 294 SPRING 1983 PRELIM 3 # 3

5.4.5 Let $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < 10$ have boundary conditions

$$u_x(0,t) = u_x(10,t) = 0$$

a) What is the most general solution to this boundary value problem you can find.
b) Given $u(x,0) = x^2$, what is $u(5, t \to \infty)$ in the above problem?

MATH 294 SPRING 1983 PRELIM 3 # 4

5.4.6 Let $u_t = u_{xx}, 0 < x < \pi$ have boundary conditions $u_x(0,t) = 0, u(\pi,t) = 0$.
a) What is the most general solution you can find to this equation and the given boundary conditions.
b) For the initial condition $u(x,0) = x^2 - e^x \sin x$ what is $u(1, t \to \infty)$?

MATH 294 SPRING 1983 FINAL # 4

5.4.7 A bat of length 1 is assumed to satisfy the heat equation with $\alpha^2 = 1$. The ends of the bar are in ice water at temperature $u = 0$. At time $t = 0$ the temperature of the bar is $u(x,0) = 100 \sin \pi x$. What is the temperature in the middle of the bar at $t = 2$?

MATH 294 SPRING 1984 FINAL # 16

5.4.8 Heat conduction in a closed-loop wire of radius 1 can be described by $u(t,x)$ where

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\ 0 \le x < 2\pi,\ t > 0$$

and $u(t, x = 0) = u(t, x = 2\pi)\ \ t \ge 0$. The initial distribution of temperature is

$$u(t=0,x) = \begin{cases} 0 & 0 < x \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ 0 & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

find $u(t,x)$ by separation of variables, what is the temperature distribution as $t \to \infty$?

MATH 294 FALL 1984 FINAL # 14

b) Solve the following initial-boundary value problem for the heat equation

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \; 0 \leq x \leq L, \; t \geq 0$

$\frac{\partial u}{\partial x}(0,t) = 0, \; \frac{\partial (}{\partial L}, t) = 0$

$u(x,0) = f(x)$ where $f(x)$ is the function given in IV A.

c) Determine the value of $H(t)$ given by $H(t) = \int_0^L u(x,t)dx$. If you have been unable to find the solution in IV B, then form $\frac{\partial H}{\partial t}$ and use the differential equation for u to find an ordinary differential equation for H.

MATH 294 FALL 1987 PRELIM 3 # 3

5.4.9 Find the solution of the initial-boundary-value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; 0 < x < \frac{\pi}{2}, \; t > 0,$

$u(0,t) = \frac{\partial u}{\partial x}\left(\frac{\pi}{2}, t\right) = 0 \; t > 0,$

$u(x,0) = 7\sin x - 14 \sin 9x$

(Hint: symmetry and superposition)

MATH 294 FALL 1987 PRELIM 2 # 5

5.4.10 Find the solution of the initial-boundary-value problem

5.4.11 Find the solution of the initial-boundary-value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; 0 < x < \pi, \; t > 0,$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \; t > 0,$

$u(x,0) = 10 \sin^2 4x$

MATH 294 FALL 1987 PRELIM 3 # 4

5.4.12 a) Find the solution of the initial-boundary-value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; 0 < x < \pi, \; t > 0,$

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \; t > 0,$

$u(x,0) = 1 + x \;\; 0 < x < \pi$

b) What is $\lim_{t\to\infty} u(x,t)$, where $u(x,t)$ is the solution of part (a).

c) Show that your result from part (b) is a time-independent (equilibrium) solution of the first two equations in part (a).

MATH 294 FALL 1988 PRELIM 3 # 5

5.4.13 Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $\frac{\partial 0,t}{\partial =}0$ and $\frac{\partial u}{\partial x}(\pi, t) = 0$ and initial condition $u(x,0) = 4\cos(x) - 5\cos(4x)$.

MATH 294 FALL 1989 FINAL # 5

5.4.14 Let $\alpha > 0$. Find the solution $u(x,t)$ of

$u_{xx} = \frac{1}{\alpha} u_t, \; t > 0 \;, 0 < x < 1,$

$u_x(0,t) = u_x(1,t) = 0, \; t > 0,$

$u(x,0) = 1 + 3\cos(2\pi x) - 2\cos(5\pi x), \; 0 < x < 1.$

MATH 294 SPRING 1991 FINAL # 2

5.4.15 Consider the conduction of heat through a wire of unit length that is insulated on its lateral surface <u>and at its ends</u>.

a) Use the method of separation of variables to show that the solution of the initial value problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $0 \leq 1 \leq 0, 0 \leq t \leq \infty$;

with $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$; and $u(x,0) = f(x)$

is given in the form

$u(x,t) = \frac{a_0}{2} + \sum_{i=1}^{n} a_n e^{-n^2\pi^2 t} \cos n\pi x.$

Hint: The equation $X'' + \lambda X = 0, 0 \leq x \leq 1$, with $X'(0) = X'(1) = 0$ has nonzero solutions only for an infinite number of constants $\lambda = n^2\pi^2$, for $n = 0, 1, 2, 3, \ldots$. The corresponding solutions are $X_n(x) = A_n \cos n\pi x$.

MATH 294 FALL 1994 FINAL # 5

5.4.16 Solve

$$\begin{cases} u_t = u_{xx} \text{on} 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \begin{cases} 1 & \frac{\ell}{4} \leq x \leq \frac{3\ell}{4} \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

You may leave Fourier coefficients unsimplified after doing the integrals.

MATH 294 SPRING 1993 FINAL # 5

5.4.17 **a**) The solution to

$$\begin{aligned} U_{tt} &= u_{xx}; \quad -\infty < x < \infty \\ u(x,0) &= e^{-x^2} \\ u_t(x,0) &= 0 \end{aligned}$$

is the form $u(x,t) = \varphi(x+t) + \varphi(x-t)$. Find the solution without using Fourier series.

b) Find the solution of

$$\begin{aligned} u_{xx} &= u_t \quad 0 \leq x \leq 1 \\ u(0,t) &= 1 \\ u(1,t) &= 2 \\ u(x,0) &= 1 + x \end{aligned}$$

Hint: The solution may be time-independent.

MATH 294 SPRING 1994 FINAL # 7

5.4.18 Consider

$$\begin{aligned} u_{xx} &= u_t, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) &= 0, \\ u(1,t) &= -1, \\ u(x,0) &= 0 \end{aligned}$$

a) Find $v(x)$ if $u(x,t) \to v(x)$ as $t \to \infty$

b) Find a Fourier series solution for $u(x,t)$.

MATH 294 SPRING 1995 PRELIM 3 # 2

5.4.19 Consider the condition of heat through a wire of unit length that is insulated on its lateral surface and at its ends. This implies boundary conditions $u_x(0,t) = 0 = u_x(1,t), t \geq 0$.

a) Verify that solutions $u(x,t)$ to the heat equation with the initial condition $u(x,0) = f(x)$ piecewise continuous first derivatives may be given in the form

$$u(x,t) = \frac{a_0}{2} + \sum_{i=1}^{n} a_n e^{-n^2\pi^2 t} \cos(n\pi x)$$

b) Find $u(x,t)$ when $f(x) = 2 + 5\cos(3\pi x)$.?

MATH 294 SPRING 1996 PRELIM 3 # 3

5.4.20 Consider the heat equation

$$\begin{aligned} u_t &= 0.04 u_{xx} \\ u(x,0) &= \sin\left(\frac{\pi x}{2}\right) - \frac{1}{2}\sin(\pi x) \\ u(0,t) &= 0 \\ u(2,t) &= 0 \end{aligned}$$

a) Find the solution to this problem.

b) Verify by substitution that your answer to part (a) does in fact satisfy all four of these equations. (You can get full credit for this part by checking everything, even if your answer to (a) is *wrong*.

MATH 294 FALL 1996 PRELIM 3 # 3

5.4.21 **a**) Find the full solution to

$$\begin{aligned} u_{xx} &= ut \;\; 0 < x < \pi, \;\; t > 0 \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= x \end{aligned}$$

You may find problem 2 helpful in solving this.

b) Using only the first two terms in your solution, write out $u(x,0)$ and $u(x,1)$. Sketch these terms and their sum. Comment on your plot.

$$\begin{aligned} e^0 &= 1 \\ e^{-1} &= .368 \\ e^{-2} &= .135 \\ e^{-3} &= .050 \\ e^{-4} &= 0.018 \\ e^{-5} &= .007 \end{aligned}$$

MATH 294 SPRING 1998 PRELIM 1 # 3

5.4.22 Consider the one dimensional heat transfer problem

$$u_{xx} = u_t, \; 0 \le x \le 1$$

with boundary conditions $u(0,t) = 0, u(1,t) = 1, t > 0$,
and initial conditions $u(x,0) = 0, \; 0 \le x \le 1$.

a) Find the long time, i.e. time independent, solution reached as $t \to \infty$.

b) Find the time-dependent solution $u(x,t)$ that satisfies the given boundary and initial conditions.

MATH 294 SPRING 1998 FINAL # 2

5.4.23 Find the solution of the initial value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \text{on} 0 < x < 1 \text{for} t > 0,$$

with

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

and

$$u(x,0) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

MATH 294 FALL 1998 FINAL # 2

5.4.24 Solve, for $t > 0$ and $0 < x < \pi$, the partial differential equation $2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the boundary conditions that $u(0,t) = 0,\ u(\pi,t) = \pi$,
and the initial condition that $u(x,0) = x + \sin(x)$.
(This problem can be solved completely without any integrations.)

MATH 294 UNKNOWN 1990 UNKNOWN # 5

5.4.25 Solve the initial boundary value problem

$$u_t = u_{xx},\ 0 < x < \pi, T > 0,$$

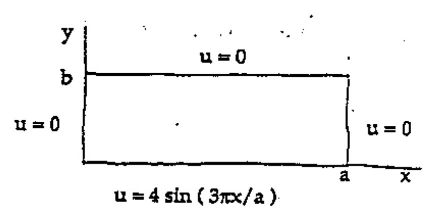
$$u(0,t) = u_x(\pi,t) = 0,$$

$$u(x,0) = 18\sin\left(\frac{9x}{2}\right).$$

(Hint: $u(x,y) = X(x)T(t)$; you may use the fact that the only nontrivial solution here occurs when X is a linear combination of a cosine and a sine.)

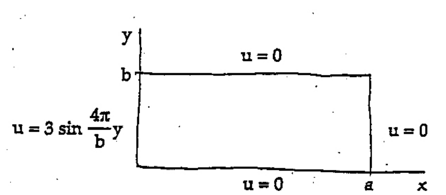
MATH 294 FALL 1990 FINAL # 5

5.4.26 Find the steady state temperature distribution in the plate shown.



MATH 294 FALL 1990 FINAL # 4 MAKE-UP

5.4.27 Find the steady state temperature distribution in the plate shown.



MATH 294 FALL 1994 PRELIM 1 # 5

5.4.28 Find the steady state temperature distribution in the plate shown.