5.3 Laplace Equation

MATH 294 FALL 1982 FINAL # 6

5.3.1 **a)** Determine the solution to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $0 < x < 1,\ 0 < y < 1$, subject to the boundary conditions $u(0, y) = u(x, 0) = u(1, y) = 0$, $u(x, 1) = 2x$

b) Use this solution and the linearity of Laplace's equation to obtain the solution to the boundary value problem $u(0, y) = u(x, 0) = 0$, $u(1, y) = 2y$, $u(x, 1) = 2x$.

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5.3.2 Solve Laplace's equation on the rectangle $0 \le x \le 4, 0 \le y \le 3$ with the given boundary conditions.

$$u_{xx} + u_{yy} = 0,$$
$$u(x, 0) = u(0, y) = 0,$$
$$u(4, y) = 2\sin\left(\frac{\pi y}{3}\right),$$
$$u(x, 3) = 5\sin\left(\frac{\pi x}{4}\right)$$

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5.3.3 **a)** Solve Laplace's equation $\delta^2 u = 0$ on the square $0 < x < \pi, 0 < y < \pi$, subject to $u = 0$ on the three sides $x = 0, y = 0$ and $x = \pi$, and $u(x, \pi) = g(x)$, where g is defined in 4(b). (Hint: $u_n(x, y) = \sin(nx)\sinh(ny), n = 1, 2, \ldots$)

b) Repeat (a) if the b.c. $u(\pi, y) = 0$ is replaced by $u(\pi, y) = 2\sin 3y - 14\sin 9y$

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5.3.4 The solution to any partial differential equation depends on the domain in which the solution is to be valid as well as the boundary conditions and/or initial conditions that the solution must satisfy. We wish to consider physically plausible (i.e. u does not approach ∞) solutions to Laplace's equation in circular regions, $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$, for various boundary conditions. Below you are given some functions that satisfy the Laplace equation and certain conditions. You are asked to choose (and to give arguments that led to that choice) what problem is being discussed. <u>You do not necessarily have to solve any equation</u>.

a) $u(r,\theta) = fracC_n2 + \sum_{i=1}^{n} r^{-n}(C_n \cos n\theta + K_n \sin n\theta)$

$C_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$; $K_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$

i) a disk $(r \leq a)$ with $u(a,\theta) = f(\theta)$

ii) an annulus of inner radius a $(a \leq r < infty)$ with $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$

iii) an annulus of inner radius a $(a \leq r < infty)$ with $u(a,\theta) = f(\theta)$

iv) none of above

b) $u(r,\theta) = \sum_{i=1}^{n} C - nr^n \sin n\theta$; $C_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta d\theta$

i) An annulus $(a \leq r < \infty)$, satisfying $u(a,\theta) = f(\theta),\ 0 \leq \theta < \pi$

ii) A half-disk $(r \leq a;\ 0 \leq \theta \leq \pi)$, with $u(r,\pi) = 0, u(a,\theta) = f(\theta)$

iii) A disk $(r \leq a)$ with $u(a,\theta) = f(\theta)$ for $0 \leq \theta < \pi$

iv) none of the above

c) $u(r,\theta) = \frac{u_b \ln\left(\frac{r}{a}\right) + u_b \ln\left(\frac{b}{r}\right)}{\ln\left(\frac{b}{a}\right)}$

i) a pie-shaped wedge $(0 \leq r \leq a)$ of angle $\tan\theta_0 = \frac{b}{a}$ with $u(a,\theta) = u_a\theta + u_b\left(\frac{b}{r}\right)$

ii) An annulus $(a \leq r \leq b)$ with constant specified values on the inner and outer bounding circles

iii) a disk $(r \leq a)$ satisfying $u(a,\theta) = u_a + \theta u_b$

iv) none of the above

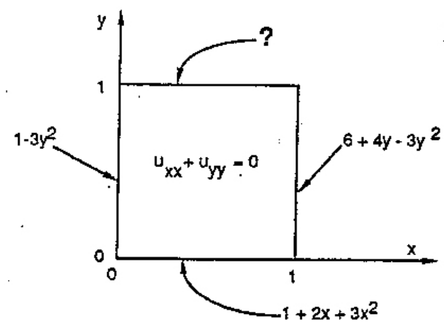
d) Describe a situation where Laplace's equation arises in cartesian coordinates. Discuss the meaning of the appropriate boundary and initial conditions for the problem that you have chosen.

MATH 294 SPRING 1988 PRELIM 2 # 4

5.3.5 **a**) Find constants $A, B, C, D, E,$ and F (real numbers) so that the function

$$u = A + Bx + Cx^2 + Dx^3 + Ey^2 + Fxy$$

is
i) a solution to Laplace's equation in the rectangle shown, and
ii) satisfies the three boundary conditions shown.
b) What boundary condition is satisfied on the fourth boundary?



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5.3.6 Solve: $U_{xx} + u_{yy} = 0, 0 < x < \pi, 0 < y < 1$ subject to the boundary conditions $u_x(0, y) = u_x(\pi, y) = 0, 0 \leq y \leq 1, u(x, 0) = 4\cos(6x) + \cos(7x), 0 \leq x \leq \pi, u(x, 1) = 0, 0 \leq x \leq \pi.$

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5.3.7 Consider the Laplace equation in the semi-infinite strip $0 < x < L, y > 0$, with boundary conditions $u(0, y) = 0, u(L, y) = 0, u(x, 0) = f(x)$, and u must not approach ∞ as y approaches ∞.
a) Find a general solution for these conditions.
b) Write out the solution for this problem in the case that $f(x) = x$. You are given the Fourier sine and cosine series for $x, (0 < x < L)$

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$x = \frac{2L}{\pi} \sum_{i=1}^{n} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

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5.3.8 Find the solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\left\{\begin{array}{l}0<x<\pi\\0<y<\pi\end{array}\right.,$$

$$\frac{\partial u}{\partial x}(0,y)=\frac{\partial u}{\partial x}(\pi,y)=0,\ 0<y<\pi,$$

$$u(x,0)=0,u(x,\pi)=9,\ 0<x<\pi.$$