4.7 Stokes Theorem

MATH 294 FALL 1984 FINAL # 3b 294FA84FQ3b.tex

4.7.1 Let $\vec{F}$ be defined as

$$\vec{F} = \text{ curl } \vec{G}$$

where $\vec{G} = x^2 z^2 \hat{\imath} + xy\hat{\jmath} + xz\hat{k}$.
Evaluate

$$\int\int_S \vec{F} \cdot \hat{n} d\sigma$$

if S is the surface

$$z = 4 - x^2 - y^2, \; z \geq 0.$$

and $\hat{n}$ is the unit normal on S.

MATH 294 SPRING 1985 FINAL # 19 294SP85FQ19.tex

4.7.2 Find the integral $\displaystyle\int\int_S (\nabla \text{ x } \vec{F}) \cdot \hat{n} d\sigma$ where $\vec{F}$ is the vector field $z(x^2 - y^2)\hat{\imath} + z^2(x^2 + y^2)\hat{\jmath} + (x^2 + y^2)\hat{k}$, S is the surface $z = \sqrt{1 - x^2 - y^2}$ (upper hemisphere of sphere with radius 1, centered at origin), and $\hat{n}$ is the unit normal that points away from the origin.
a) 2π
b) $-\dfrac{\pi}{2}$
c) 0
d) $\dfrac{1}{2\pi}$
e) none of these

MATH 294 SPRING 1985 FINAL # 20 294SP85FQ20.tex

4.7.3 Find the integral $\displaystyle\int\int_S (\nabla \text{ x } \vec{F}) \cdot \hat{n} d\sigma$ where $\vec{F}$ is the same vector field in the previous problem, but now S is the entire sphere $x^2 + y^2 + z^2 = 1$, and $\hat{n}$ is as above.
a) 2π
b) 4π
c) $-\pi$
d) $\dfrac{1}{\pi}$
e) none of these

MATH 294 FALL 1986 FINAL # 9 294FA86FQ9.tex

4.7.4 Let S be the portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies below the plane $z = 1$. Let $\hat{n}$ be the normal vector field on S which points away from the origin. Let $\vec{F}(x, y, z) = \dfrac{-yz}{x^2 + y^2 + 1}\hat{\imath} + \dfrac{xz}{x^2 + y^2 + 1}\hat{\jmath} - \dfrac{xyz}{x^2 + y^2 + 1}\hat{k}$. Compute $\displaystyle\int\int_S (\vec{\nabla} \text{ x } \vec{F}) \cdot \hat{n} d\sigma$.

MATH 294 SPRING 1987 PRELIM 1 # 6 294SP87P1Q6.tex

4.7.5 Consider the 3 dimensional vector field:

$$\mathbf{F} = (2x - y)\hat{i} + (x + z)\hat{j} + z^2\hat{k}$$

a) Calculate $curl(\mathbf{F})$ at (1,1,1).

b) Imagine this vector field represents the velocity field for fluid flow. A very small paddle wheel is inserted in the flow at the point (1,1,1) and held there with hands that don't upset the flow. Which direction should the axis of the wheel be oriented if it is to spin at a maximal rate? (indicate the direction with a unit vector).

MATH 294 SPRING 1987 FINAL # 5 294SP87FQ5.tex

4.7.6 Evaluate the integrals below by any means. In each case, $\mathbf{F} = (ye^z + x)\mathbf{i} + (2y - z)\mathbf{j} + 7k$, S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. D is the interior of the sphere, and $\mathbf{n}$ is the outward pointing unit normal of the sphere's surface. Hints: Each one of the integrals below is equal to at least one of the others. Volume of a sphere $= (4/3)\pi r^3$, surface area of a sphere $= 4\pi r^2$.

a) $\displaystyle\int\int_S \mathbf{F}\cdot\mathbf{n}d\sigma.$

b) $\displaystyle\int\int_S d\sigma.$

c) $\displaystyle\int\int_D\int zdV.$

d) $\displaystyle\int\int_D\int div(curl(\mathbf{F}))dV.$

e) $\displaystyle\int\int_S curl(\mathbf{F})\cdot\mathbf{n}d\sigma.$

MATH 294 FALL 1987 PRELIM 1 # 5 294FA87P1Q5.tex

4.7.7 Evaluate $\displaystyle\int\int_S (\nabla \text{ x } \mathbf{F})\cdot\mathbf{n}d\sigma$, where S is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$, with <u>outward</u> unit normal $\mathbf{n}$ (from the z-axis), and $\mathbf{F}(x, y, z) = x\cos(xz^2)\mathbf{i} + 3x\mathbf{j} + e^{xy}\sin x\mathbf{k}$.

MATH 294 FALL 1987 MAKE UP FINAL # 5 294FA87MUFQ5.tex

4.7.8 Evaluate $\displaystyle\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x, y, z) = -y\hat{i} + x\hat{j} + e^{\cos z^2}\hat{k}$, and C is the closed curve (ellipse) of intersection of the <u>cylinder</u> $x^2 + y^2 = 4$, $-\infty < z < \infty$, with the plane $x + y + z = 5$. The curve is oriented counterclockwise when viewed from above. (Hint: draw a picture.)

MATH 294 SPRING 1988 PRELIM 2 # 8 294SP88P2Q8.tex

4.7.9 Evaluate the path integral $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{R}$ with

$$\mathbf{F} = (x + e^{y^2})\mathbf{j}$$

for the curve parameterized by $\mathbf{R} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + (\cos\theta)\mathbf{k}$ with $0 \leq \theta \leq 2\pi$.

MATH 294 FALL 1988 PRELIM 3 # 2 294FA88P3Q2.tex

4.7.10 Evaluate $\int\int_{S_1} \nabla \text{ x } \mathbf{F}\cdot\mathbf{n}d\sigma$ where $\mathbf{F} = y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$, S_1 is the path of the paraboloid $z = x^2 + y^2$ below the plane $z = x + \frac{3}{4}$, and $\mathbf{n}$ is a unit vector that is normal to the surface and has a positive x-component, i.e., $\mathbf{n}\cdot\mathbf{k} > 0$.

MATH 294 FALL 1988 PRELIM 3 # 3 294FA88P3Q3.tex

4.7.11 Using the same $\mathbf{F}$ as in 2 above, evaluate $\int\int_{S_2} \nabla \text{ x } \mathbf{F}\cdot\mathbf{n}d\sigma$, where S_2 is the part of the plane $z = x + \frac{3}{4}$ inside the paraboloid $z = x^2 + y^2$.

MATH 294 FALL 1989 PRELIM 2 # 4 294FA89P2Q4.tex

4.7.12 Calculate the circulation of the field

$$x\sin(x^2)\mathbf{i} + x^2e^y\mathbf{j} + (z^5 + x - y)\mathbf{k}$$

around the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$ when it is traveled counterclockwise as seen from the point (1,0,1).
(Hint: Stokes' Theorem may be helpful.)

MATH 294 SPRING 1990 PRELIM 2 # 1 294SP90P2Q1.tex

4.7.13 Consider the vector field $\mathbf{F}(x, y) = (2xy^3 - \sin^3 x)\mathbf{i} + (3x^2y^2 + 3x)\mathbf{j}$.

a) Find the curl of $\mathbf{F}$ (∇ x $\mathbf{F}$).

b) Compute the circulation of $\mathbf{F}$ for the counterclockwise path around a square with vertices (1,0), (2,0), (2,1) and (1,1).

MATH 294 FALL 1990 PRELIM 2 # 2 294FA90P2Q2.tex

4.7.14 **a)** Show that the curl of the vector field $\mathbf{F} = y\sin z\mathbf{i} + x\sin z\mathbf{j} + xy\cos z\mathbf{k}$ vanishes.

b) Determine a potential f for this vector field.

c) Use the potential to evaluate the integral.

MATH 294 FALL 1990 FINAL # 2 294FA90FQ2.tex

4.7.15 Consider the portion of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant and the vector field $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to calculate the circulation of the vector field around the edge of this surface in a counter-clockwise direction when viewed from the first octant.

MATH 294 SPRING 1991 PRELIM 3 # 3 294SP91P3Q3.tex

4.7.16 Calculate the circulation of the vector field $\mathbf{F} = xz\mathbf{i}+yz\mathbf{j}+z^2\mathbf{k}$ around the boundary of the triangle cut from the plane $x+y+z = 1$ by the first octant, counterclockwise when viewed from above, in two different ways:

a) by direct calculation of the circulation around the edges;

b) using Stokes' Theorem.

MATH 294 SPRING 1991 FINAL # 2 294SP91FQ2.tex

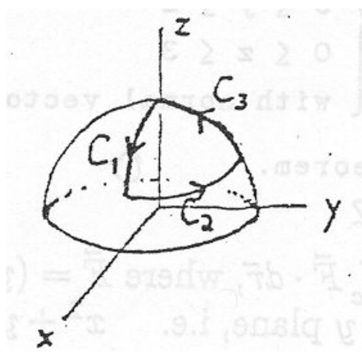
4.7.17 Consider the portion of the plane $x + y + 2z = 2$ in the first octant and the vector field $\mathbf{F} = (x - y)\mathbf{k}$. Use Stokes' Theorem to calculate the circulation of the vector field around the edges of this surface in a counter-clockwise direction when viewed from above the plane in the first octant.

MATH 294 FALL 1991 FINAL # 2 294FA91FQ2.tex

4.7.18 Let S be the portion of the spherical surface $x^2 + y^2 + z^2 = 1$ in the first octant and let C be the boundary of S. Determine, by any means, the circulation of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ about the circuit C in a counterclockwise direction when viewed from the first octant.

MATH 294 SPRING 1992 PRELIM 3 # 5 294SP92P3Q5.tex

4.7.19 Let C be the curve on the sphere $x^2 + y^2 + z^2 = 9$ made up of the three curves C_1, C_2, and C_3 as shown.

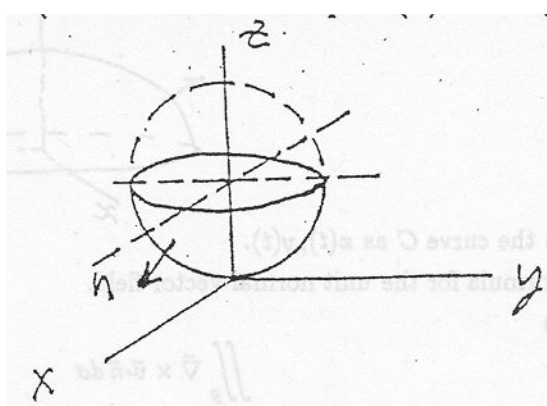


The curve C_1 lies in the xz-plane, $z = \sqrt{5}$, and C_3 in the yz-plane. Calculate the circulation of the vector field $\vec{F} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$ around the curve C in the direction indicated in the picture.

MATH 294 FALL 1992 FINAL # 5 294FA92FQ5.tex

4.7.20 **a**) Evaluate $\displaystyle\int\int_S (\nabla \text{ x } \mathbf{F}) \cdot \mathbf{n} d\sigma$, where S is the bottom half of the sphere $x^2 + y^2 + (z-1)^2 = 1$, where $\mathbf{n}$ denotes the downward unit normal, $\nabla \text{ x } (\cdot) \equiv \text{ curl}(\cdot)$, and $\mathbf{F} = x\cos(xz^2)\mathbf{i} + 3x\mathbf{j} + e^{xy}\sin x\mathbf{k}$.

b) Repeat part (a) when S is now the complete sphere $x^2 + y^2 + (z-1)^2 = 1$ and $\mathbf{n}$ is the outward unit normal. (Hint: the answer to part (b) is independent of $\mathbf{F}$.)



MATH 294 SPRING 1993 FINAL # 6 294SP93FQ6.tex

4.7.21 Use Stokes' Theorem to evaluate

$$\oint_C -zdy + ydz$$

where C is the circle of radius 3 on the plane $x + y + z = 0$ and centered at the origin.

MATH 294 FALL 1993 PRELIM 1 # 5 294FA93P1Q5.tex

4.7.22 Evaluate $\int_C (\mathbf{a} \text{ x } \mathbf{r}) \cdot d\mathbf{r}$ if $\mathbf{a}$ is a constant vector and C is the boundary of the rectangle

$$\begin{cases} x = 0 \\ 0 \leq y \leq 2 \\ 0 \leq z \leq 3 \\ \text{with normal vector } \mathbf{i} \end{cases}$$

You may use Stokes' Theorem.

MATH 294 SPRING 1995 PRELIM 1 # 2 294SP95P1Q2.tex

4.7.23 Evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (y + x)\hat{\imath} - x\hat{\jmath} + zx^3y^2\hat{k}$, and C is the unit circle in the x, y plane, i.e., $x^2 + y^2 = 1,\ z = 0$.

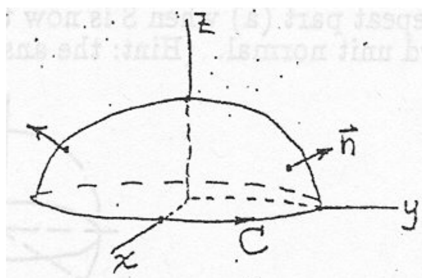
MATH 294 FALL 1995 PRELIM 2 # 1 294FA95P2Q1.tex

4.7.24 **a**) Evaluate $\int\int_{S_1} \text{curl } \vec{F} \cdot \hat{n} d\sigma$ where S_1 is the hemisphere $x^2 + y^2 + z^2 = 9,\ z \geq 0,\ \hat{n}$ points toward positive z, and $\vec{F} = y\hat{\imath} + 8x\hat{\jmath}$.

b) Evaluate $\int_C y^2z^2dx + 2xyz^2dy + 2xy^2zdz$ where C is a path from the origin to the point (5,2,-1).

MATH 294 SPRING 1996 FINAL # 2 294SP96FQ2.tex

4.7.25 S is the surface $z = 4 - 4x^2 - y^2$ (between $z = 0$ and $z = 4$) with unit normal vector field as shown.



a) Describe the curve C as $x(t),\ y(t)$.
b) Find a formula for the unit normal vector field.
c) Evaluate

$$\int\int_S \vec{\nabla} \text{ x } \vec{v} \cdot \hat{n} d\sigma$$

where $\vec{v} = x^3\hat{j} - (z+1)\hat{k}$. You may need: $\int \sin^3 x dx = \frac{1}{3}\cos^3 x - \cos x + c$, $\int \cos^3 x dx = \frac{1}{3}\sin^3 x + \sin x + c$, $\int \cos^4 x dx = \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$, $\int \sin^4 x dx = \frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$.

MATH 294 FALL 1996 PRELIM 1 # 5 294FA96P1Q5.tex

4.7.26 Evaluate $\int\int_S (\nabla \text{ x } \mathbf{F} \cdot \hat{n} d\sigma$, where S is the open bottom half of the sphere $x^2 + y^2 + z^2 = a^2$, and $\hat{n}$ is the (outward) downward unit normal, and $\mathbf{F} = x\cos z\hat{i} + y\hat{j} + e^{xy}\hat{k}$.

MATH 293 FALL 1998 FINAL # 4 293FA98FQ4.tex

4.7.27 Use Stokes' Theorem to calculate the outward flux of ∇ x $\mathbf{F}$ over the cylinder $x^2 + y^2 = 4$ that has an open bottom at $z = 0$ and a closed top at $z = 3$, where

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}.$$