3.3 Systems of ODEs

MATH 294 FALL 1981 PRELIM 1 # 1 294FA81P1Q1.tex

3.3.1 **a**) Convert the third order differential equation

$$y'" - 2y" - y' + 2y = 0$$

into a first oder system $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$, with

$$\underline{x} = \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}$$

b) The equation $y'" - 2y" - y' + 2y = 0$ has general solution

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}$$

Use this to find a basis $(\underline{x}^1(t), \underline{x}^2(t), \underline{x}^3(t))$ for the set of all solutions of $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$. The functions $\underline{x}^j(t)$ should be vector valued.

c) Show that the three solutions $\underline{x}^j(t)$ you found in (b) are linearly independent.

MATH 294 FALL 1982 FINAL # 3 294FA82FQ3.tex

3.3.2 Determine the solution to the initial value problem

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{f}, \qquad \underline{x}(0) = \underline{x}_0$$

where

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \underline{f} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \qquad \underline{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

MATH 294 FALL 1983 PRELIM 1 # 1 294SP83P1Q1.tex

3.3.3 Translate the following paragraph into a system of differential equations in matrix form. Clearly define all variables and constants. (Don't solve the equation).
"The rate that the temperature of a cup of tea decreases is proportional to the difference between its current temperature and the temperature of the room. The rate that the temperature of the room decreases is proportional to the difference between the current temperature of the room and the temperature outside. The temperature outside is 0^oC." (Note that the room is unheated and is unaffected by the tea.)

MATH 294 SPRING 1983 PRELIM 1 # 3 294SP83P1Q3abc.tex

3.3.4 Consider the system of ordinary differential equations given by

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\underline{A}}\ \underline{x}$$

a) Write this system in the form $ay" + by'cy = 0$ (Hint: $y = x_1,\ y' = \dot{x}_1 = x_2$)

b) Solve the second order equation above by any method you want.

c) Use your solution to part b above to find a basis for the vector space of solutions

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{of } \dot{\underline{x}} = \underline{\underline{A}}\ \underline{x}.$$

MATH 294 SPRING 1983 PRELIM 3 # 1 294SP83P3Q1.tex

3.3.5 Consider the system of equations given by

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{f}(t).$$

a) Find the general solution to the above system when $\underline{f}(t) = 0$.

b) Find the fundamental matrix solution $e^{\underline{\underline{A}}(t)}$ to the above system (still with $\underline{f}\ (t) = 0$)

c) Using any method, solve the above system when

$$\underline{f}(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \quad \textit{and} \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATH 294 SPRING 1983 MAKE UP PRELIM 3 # 3 294SP83MUP3Q3.tex

3.3.6 Consider the equation $\dot{\underline{x}} - \underline{\underline{A}}\ \underline{x} = \underline{f}(t)$ $(*)$,where

$$\underline{\underline{A}} = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$$

a) Find the general solution to (*) with $\underline{f}(t) = 0$

b) Find $e^{\underline{\underline{A}}(t)}$

c) for $\underline{f}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ and the initial condition $\underline{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

MATH 294 SPRING 1983 FINAL # 1 294SP83FQ1.tex

3.3.7 Solve for $\underline{x}(t)$ by any method:

$$\dot{\underline{x}} = \underline{\underline{A}}\ \underline{x} + \underline{f}(t), \quad \underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \underline{f}(t) = \begin{bmatrix} sin(t) \\ 0 \end{bmatrix}, \quad \underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

MATH 294 SPRING 1984 FINAL # 4 294SP84FQ4.tex

3.3.8 Find the general solution of $y" + ay' + by = 0$ as follows: Let $z_1 = y$ and $z_2 = y'$. Write the second order differential equation as a system. Solve the system and then convert the answer back in terms of y. Compare the result with the general solution of $y" + ay' + by = 0$ found using any other technique.

Hint: If λ is an eigenvalue of a 2x2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the eigenvector is $\begin{bmatrix} a_{21} \\ a_{11}^{-\lambda} \end{bmatrix}$

MATH 294 SPRING 1984 FINAL # 6 294SP84FQ6.tex

3.3.9 Find a solution of $\dot{x}(t) = Ax + f$ by "expanding" $x(t) = Pa(t)$ and solving for $a(t)$. P is the diagonalizing matrix for A, ie., $A = PDP^{-1}$. Here

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad f(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}$$

and

$$a = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

MATH 294 FALL 1984 FINAL # 2 294FA84FQ2.tex

3.3.10 **a**) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

b) Find A^{-1} and A^T.

c) Find the general solution of

$$\frac{d}{dx} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

MATH 294 SPRING 1985 FINAL # 3 294SP85FQ3.tex

3.3.11 For the case $\vec{b} \neq \vec{0}$, the vector $\vec{x} = \vec{0}$

a) Is always a solution.

b) May or may not be a solution depending on $\underline{\underline{A}}$.

c) Is always the only solution.

d) Is never a solution.

MATH 294 FALL 1985 FINAL # 6 294FA85FQ6.tex

3.3.12 Find the general solution of the system $Y' = AY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $Y' = \dfrac{dY}{dt}$, and $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$.

MATH 294 FALL 1986 FINAL # 5 294FA86FQ5.tex

3.3.13 Solve the system of differential equations $Y' = AY$,where Y is the vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and A is the matrix $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

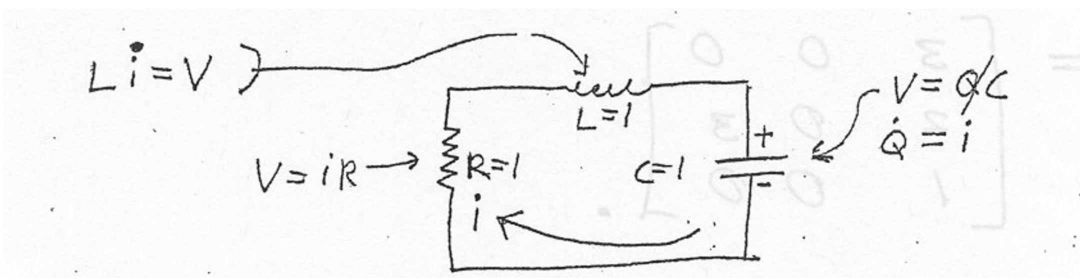
MATH 294 SPRING 1987 PRELIM 2 # 7 294SP87P2Q7.tex

3.3.14 The dynamics of the systems below can be put in the form:

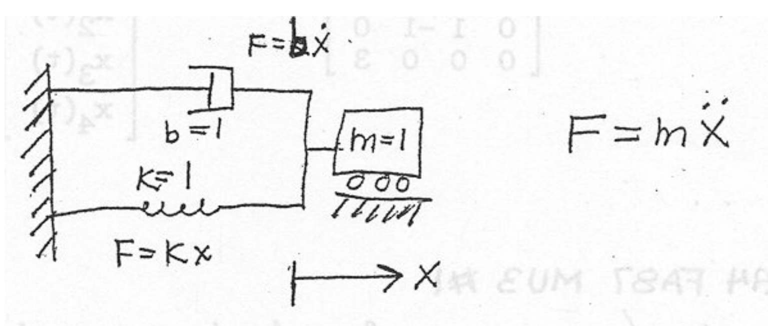
$$\dot{\mathrm{x}} = [A]\mathrm{x}$$

Using x as defined in one of the problems below, find the appropriate matrix $[A]$.

a) Electric circuit: $\mathrm{x} = \begin{bmatrix} Q \\ i \end{bmatrix}$ (Q is capacitor charge, i is current)



b) Mechanical system: $\mathrm{x} = \begin{bmatrix} x \\ v \end{bmatrix}$ (x = position, v = velocity)



MATH 294 SPRING 1987 PRELIM 2 # 5 294SP87P2Q5.tex

3.3.15 Find the value of x at $t = 2$ if

$$\dot{\mathrm{x}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x, \text{ with } x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

MATH 294 SPRING 1987 PRELIM 3 # 2 294SP87P3Q2.tex

3.3.16 Find the general solution of

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

MATH 294 SPRING 1987 FINAL # 1 294SP87FQ1.tex

3.3.17 Pick *any* value λ (a specific real or complex number) so that you can find a real non-zero solution to the equations below and then find *any* non-zero solution. If no such λ exists, say so.

a) $x' + \lambda x = 0$ with $x(1) = 1$

b) $x' + \lambda x = 0$ with $x(0) = 0$ and $x'(0) = 1$.

c) $x' + \lambda x = 0$ with $x(0) = 0$ and $x(7) = 0$

d) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathrm{x} = \lambda \mathrm{x}$

MATH 294 FALL 1987 PRELIM 2 # 6 294FA87P2Q6.tex

3.3.18 Find the general solution of x' = Ax, where $A = \begin{bmatrix} 5 & -1 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 3 \end{bmatrix}$.

MATH 294 FALL 1987 MAKE UP PRELIM 2 # 6 294FA87MUP2Q6.tex

3.3.19 Find the general solution of x' = Ax, where

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}.$$

MATH 294 FALL 1987 PRELIM 3 # 1 294FA87P3Q1.tex

3.3.20 Find the general solution of the system x' = Ax.
where

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \;\&\; x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

MATH 294 FALL 1987 MAKE UP PRELIM 3 # 1 294FA87MUP3Q1.tex

3.3.21 **a**) Find the general solution of the system $\dot{\underline{\mathrm{x}}} = A\underline{\mathrm{x}}$ if

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \underline{\mathrm{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

b) How many independent eigenvectors can we find for the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}?$$

MATH 294 FALL 1987 FINAL # 6 294FA87FQ6.tex

3.3.22 Consider $\underline{\mathrm{x}}' = A\underline{x}$, where A is a 3x3 matrix. If $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. find the general solution of the system of differential equations.

MATH 294 FALL 1987 MAKE UP FINAL # 7 294FA87MUFQ7.tex

3.3.23 Solve $\underline{\mathrm{x}}' = A\underline{\mathrm{x}}$ if $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\underline{\mathrm{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

MATH 294 SPRING 1988 PRELIM 2 # 6 294SP88P2Q6.tex

3.3.24 Translate the following situation to a system of first order Ordinary Differential equations in standard matrix form. Choose arbitrary values for any proportionality constants. x, y and z are functions of time t.
x increases at a rate proportional to y.
y increases at a rate proportional to z.
z increases at a rate proportional to $x + y + z$.

MATH 294 SPRING 1988 PRELIM 2 # 7 294SP88P2Q7.tex

3.3.25 Find λ so that

$$\mathbf{x}(t) = e^{\lambda t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ satisfies the equation } \frac{dx}{dt} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \mathrm{x}$$

MATH 294 SPRING 1989 PRELIM 2 # 4 294SP89P2Q4.tex

3.3.26 Consider the system of equations

$$\frac{dx}{dt} = x + 3y$$
$$\frac{dy}{dt} = x - y$$
$$\frac{dz}{dt} = -6y + 4z$$

a) Write the system in the form $D\underline{x} = A\underline{x}$

b) Find all the eigenvalues of A and their corresponding eigenvectors.

c) Let $\lambda_1, \lambda_2,$ and λ_3 be the eigenvalues, and $\underline{v}_1, \underline{v}_2,$ and $\underline{v}_3$ the corresponding eigenvectors found in Part b. Using the Wronskian, show that

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2 + c_3 e^{lambda_3 t}\underline{v}_3$$

is the general solution to the given homogeneous system

d) Consider the nonhomogeneous system $D\underline{x} = A\underline{x} + \underline{E}(t)$ with A as the matrix from Part a, and

$$\underline{E}(t) = \begin{bmatrix} -1-3t \\ t \\ 6t \end{bmatrix},$$

A particular solution of this system is given by

$$\underline{p}(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}.$$

Write down the general solution to the nonhomogeneous system.

MATH 294 SPRING 1989 PRELIM 2 # 5 294SP89P2Q5.tex

3.3.27 Consider a homogeneous system of differential equations, $D\underline{x} = A\underline{x}$ whose general solution is given by

$$\underline{x}(t) = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix}$$

Find the specific solution which satisfies the initial condition given by,

$$x_1(0) = 2, x_2(0) = 1, x_3(0) = -1.$$

That is, find the c's such that $\underline{x}(t) = c_1\underline{h}_1(t) + c_2\underline{h}_2(t) + c_3\underline{h}_3(t)$ at $t = 0$

MATH 294 SUMMER 1989 PRELIM 2 # 2 294SU89P2Q2.tex

3.3.28 **a**) Write $\dfrac{d^3x}{dt^3} - 2\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + 2x = 0$ as a linear system of ordinary differential equations.

$$D\underline{x} = A\underline{x}$$

b) Find the general solution $\underline{x}(t)$ to the above system.

c) Find a basis for the solution space of the system and show that the vectors constituting this basis are linearly independent.

MATH 294 FALL 1989 PRELIM 3 # 2 294FA89P3Q2.tex

3.3.29 Consider the 2 x 2 system of differential equations,

$$\text{(I)} \qquad \mathbf{Dx(t)} + \mathbf{Ax(t)}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

a) Verify that

$$\mathbf{h_1(t)} = \mathbf{e^t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{h_2(t)} = \mathbf{e^{-t}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

are two linearly independent solutions of (I).
[For part b and c below, you may use the result of part a even if you did not show it]

b) Find a particular solution for

$$\mathbf{Dx(t)} = \mathbf{Ax(t)} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

c) Taking on faith that $\begin{bmatrix} -t+1 \\ t-1 \end{bmatrix}$ is a particular solution of

$$\text{(II)} \qquad \mathbf{Dx(t)} = \mathbf{Ax(t)} + \begin{bmatrix} -1 \\ t \end{bmatrix},$$

find the solution of (II) that satisfies the initial condition

$$\mathbf{x(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

MATH 294 FALL 1989 PRELIM 2 # 3 294FA89P2Q3.tex
3.3.30 Consider the system of ordinary differential equations

$$\mathbf{Dx} = \mathbf{Ax},$$

$$\text{where } \mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

a) Find the general solution in terms of real-valued functions
b) Find the solution satisfying the initial conditions

$$\mathbf{x(0)} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

MATH 294 FALL 1989 FINAL # 2 294FA89FQ2.tex
3.3.31 Consider the homogeneous system of differential equations

$$(*) \qquad \mathbf{Dx} = \mathbf{Ax},$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

a) Show that $\lambda = 2$ is a double eigenvalue of $\mathbf{A}$ and that $\lambda = 1$ is a simple eigenvalue.
In part b below, you may use the result of part a even if you did not show it.
b) Find the general solution of (*).

MATH 294 FALL 1989 FINAL # 3 294FA89FQ3.tex
3.3.32 Consider the inhomogeneous system of differential equations of $x_1(t)$ and $x_2(t)$

$$\frac{dx_1}{dt} = x_1 - x_2 + e^t.$$

$$\frac{dx_2}{dt} = x_1 + x_2$$

.

a) Find the real-valued general solution of the corresponding homogeneous problem.
b) Find a particular solution for the given inhomogeneous problem.
c) Find the solution of the inhomogeneous problem which satisfies the initial condition $x_1(0) = 1,\ x_2(0) = 1$

MATH 294 SPRING 1990 PRELIM 2 # 2 294SP90P2Q2.tex

3.3.33 Find the solution of the initial-value problem

$$\frac{dx_1}{dt} = x_1 + x_2$$
$$\frac{dx_2}{dt} = x_1 + x_2$$

subject to

$$x_1(0) = 1, x_2(0) = 5$$

MATH 294 SPRING 1990 PRELIM 3 # 1 294SP90P3Q1.tex

3.3.34 Find the general solution of

$$x_1' = x_1 - 2x_2 + 13e^{2t}$$
$$x_2' = 5x_1 - x_2 + 5\sin 2t$$

MATH 294 SPRING 1990 PRELIM 3 # 2 294SP90P3Q2.tex

3.3.35 Solve the initial-value problem

$$x_1' = 4x_1 - x_2$$
$$x_2' = 4x_1$$
$$x_1(0) = 1, x_2(0) = 0.$$

MATH 294 SPRING 1990 PRELIM 3 # 3 294SP90P3Q3.tex

3.3.36 Find the general solution $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Hint: the eigenvalues of $\mathbf{A}$ are -4, 0, 2

MATH 294 SPRING 1990 FINAL # 3 294SP90FQ3.tex

3.3.37 Find the general solution of

$$x_1' = x_2 + t,$$
$$x_2' = -2x_1 + 3x_2 - 1$$

MATH 294 SPRING 1990 FINAL # 8 294SP90FQ8.tex

3.3.38 Find the general solution of $\mathbf{x}" = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} \equiv \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Hint: $\phi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$

MATH 294 SUMMER 1990 PRELIM 2 # 2 294SU90P2Q2.tex

3.3.39 Consider the following system of differential equations"

$$\frac{dx}{dt} = 4x + 6y + g(t)$$

and

$$\frac{dy}{dt} = -x - 3y.$$

a) Find the complementary solution, that is, the solution of the homogeneous system with $g(t) = 0$.

b) Find the particular solution for $g(t) = 2 + e^t$

MATH 294 SUMMER 1990 PRELIM 2 # 3 294SU90P2Q3.tex

3.3.40 Find the general solution of the following second order system:

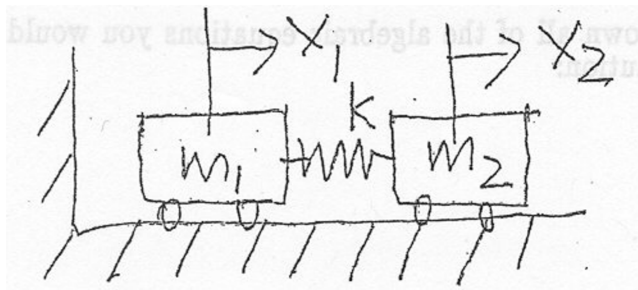
$$\frac{d^2x_1}{dt^2} = 4x_1,$$

$$\frac{d^2x_2}{dt^2} = 3x_1 + 4x_2 + 2x_3,$$

and

$$\frac{d^2x_3}{dt^2} = 2x_1$$

MATH 294 SUMMER 1990 PRELIM 2 # 4 294SU90P2Q4.tex



3.3.41 In the system shown, x_1, is the displacement of mass $m1, x2$ is the displacement of mass $m2$. When $x_2 - x_1 = 0$ the spring is relaxed. The spring constant is k. Under the usual assumptions (no friction; mass of spring relatively small, etc...),

a) derive the equations of motions for the system, and put your solution in the form of a 2 x 2 second order homogeneous system;

b) show that the eigenvalues for this system are $\lambda = 0$ and $\lambda = -k(\frac{m_1+m_2}{m_1 m_2})$;

c) explain the physical meaning of the zero eigenvalue. What could be added to the system to get rid of the zero eigenvalue?

MATH 294 FALL 1990 PRELIM 2 # 3 294FA90P2Q3.tex

3.3.42 Determine the analytic solution of $\dfrac{dy}{dx} = 2y^2 sinx, \;\; y(0) = 1$

MATH 294 FALL 1990 PRELIM 3 # 1 294FA90P3Q1.tex

3.3.43 Determine the solution to the initial value problem

$$\frac{dx_1}{dt} = x_1 + x_2 + 12e^{-t}, \; x_1(0) = 2$$

$$\frac{dx_2}{dt} = -2x_1 + 4x_2 + 6, \; x_2(0) = 3$$

Hint: the eigenvectors are (1,2) and (1,1)

MATH 294 FALL 1990 PRELIM 3 # 2 294FA90P3Q2.tex

3.3.44 Determine the most general solution of the system

$$\frac{dx_2}{dt} = 4x_1 - 5x_2$$

$$\frac{dx_2}{dt} = x_1 + 2x_2$$

Hint: the eigenvalues are 3 ± i2. You may express your solution as the sum of two complex conjugate functions.

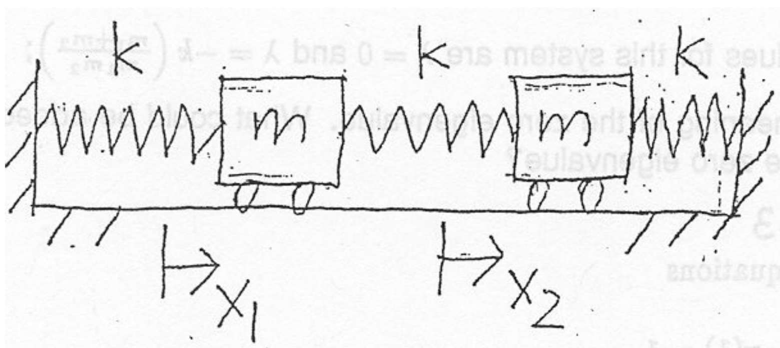
MATH 294 FALL 1990 PRELIM 3 # 3 294FA90P3Q3.tex

3.3.45 For the system below, write down all of the algebraic equations you would have to solve to determine a particular solution:

$$\frac{dx_1}{dt} = 4x_1 - 5x_2 + 4\sin 3t$$
$$\frac{dx_2}{dt} = x_1 + 2x_2 + 2 + 3t.$$

MATH 294 FALL 1990 PRELIM 3 # 4 294FA90P3Q4.tex

3.3.46 Consider a physical system consisting of two identical masses and three identical springs



The equations of motions are

$$m\text{x"}_1 = -k\text{x}_1 + k(\text{x}_2 - \text{x}_1)$$
$$m\text{x"}_2 = -k(\text{x}_2 - \text{x}_1) - k\text{x}_2.$$

a) Express the most general solution of this system in terms of sines an cosines. Hint: α^2 =-(k/m) and -(3k/m)

b) Identify the displacements associated with each normal mode.

MATH 294 FALL 1990 FINAL # 4 294FA90FQ4.tex

3.3.47 Determine the analytic solution of the initial value problem

$$\text{x"}_1 = -2\omega^2\text{x}_1 + \omega^2\text{x}_2 + \omega^2\text{x}_3 \quad \text{x'}_1(0) = 0 \quad \text{x}_1(0) = 0$$
$$\text{x"}_2 = \omega^2\text{x}_1 - \omega^2\text{x}_2 \quad \text{x'}_2(0) = -2 \quad \text{x}_2(0) = 0$$
$$\text{x"}_3 = \omega^2\text{x}_1 - +\omega^2\text{x}_3 \quad \text{x'}_3(0) = 2 \quad \text{x}_3(0) = 0$$

Hint: The eigenvectors are (0,1,-1), (1,1,1), and (-2,1,1).

MATH 294 FALL 1990 MAKE UP FINAL # 4 294FA90MUFQ4.tex

3.3.48 Determine the analytic solution of the initial value problem

$$\mathrm{x''}_1 = -2\omega^2 \mathrm{x}_1 + \omega^2 \mathrm{x}_2 + \omega^2 \mathrm{x}_3 \quad \mathrm{x'}_1(0) = 2 \quad \mathrm{x}_1(0) = 0$$
$$\mathrm{x''}_2 = \omega^2 \mathrm{x}_1 - \omega^2 \mathrm{x}_2 \quad \mathrm{x'}_2(0) = -1 \quad \mathrm{x}_2(0) = 0$$
$$\mathrm{x''}_3 = \omega^2 \mathrm{x}_1 - +\omega^2 \mathrm{x}_3 \quad \mathrm{x'}_3(0) = -1 \quad \mathrm{x}_3(0) = 0$$

Hint: The eigenvectors are (0,1,-1), (1,1,1), and (-2,1,1).

MATH 294 SPRING 1991 PRELIM 1 # 4 294SP91P1Q4.tex

3.3.49 **a)** Consider the initial value problem

$$\frac{d^2y}{dx^2} + xy^2 - 3x^3\frac{dy}{dx} - x = 0 \qquad with \qquad y(1) = 0 \; and \; \frac{dy}{dx}(1) = 1$$

b) Write the equation as a first order system.

c) Use Euler's method with a step size of $\frac{1}{4}$ on the system to obtain the approximate solution for y at $x = \frac{3}{2}$

MATH 294 SPRING 1991 PRELIM 2 # 1 294SP91P2Q1.tex

3.3.50 Consider the inhomogeneous system

$$\frac{dx_1}{dt} = 2x_1 + x_2 + 12e^{-t}$$
$$\frac{dx_2}{dt} = 5x_1 - 2x_2 + 12$$

a) Determine the general solution of the homogeneous system.

b) Determine a particular solution

c) Determine the solution of the inhomogeneous system that satisfies the initial conditions $x_1(0) = x_2(0) = \frac{-5}{6}$

MATH 294 SPRING 1991 PRELIM 2 # 2 294SP91P2Q2.tex

3.3.51 Find a particular solution of the system

$$\frac{dx}{dt} = -2x + 2y + \cos t$$
$$\frac{dy}{dt} = -3x + 2y.$$

MATH 294 SPRING 1991 PRELIM 2 # 3 294SP91P2Q3.tex

3.3.52 Consider the second order system

$$x" = -2x + y + z$$
$$y" = x - y$$
$$z" = x - z$$

a) Write down the corresponding characteristic equation and show that its roots are 0, -1, and -3.
b) Determine the general solution of the homogeneous system and express it in terms of real-valued functions.
c) Determine the circular frequency of the normal mode that the initial conditions $x(0) = -2,\ y(0) = z(0) = 1,\ x'(0) = y'(0) = 0$. will excite.

MATH 294 SPRING 1991 FINAL # 4 294SP91FQ4.tex

3.3.53 Express the general solution of the system

$$\frac{d^2x}{dt^2} = -x + 3y \text{ and } \frac{d^2y}{dt^2} = 2x - 2y$$

In terms of real-valued functions. Then determine the solution that satisfies the initial conditions

$$x(0) = 1, y(0) = -1, \frac{dx}{dt}(0) = 0, \frac{dy}{dt}(0) = 0.$$

Hint: The eigenvectors of the matrix of coefficients are (1,1) and ($\frac{3}{2}$, 1)

MATH 294 SPRING 1991 FINAL # 5 294SP91FQ5.tex

3.3.54 Find a particular solution of

$$\frac{d^3x}{dt^3} - x + e^t.$$

Hint: The characteristic equation factors to $(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$.
a) Determine the unique solution of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \text{ , where } \mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}, \text{ with } \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

MATH 294 FALL 1991 PRELIM 2 # 2 294FA91P2Q2.tex
3.3.55 Consider the system of first order equations

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = x - 2t + z + e^t$$
$$\frac{dz}{dt} = y - z - 2e^t$$

Given that the eigenvectors of the coefficient matrix are (1,1,1), (1, 0, -1) and (1,-2,1).,

a) Determine the general solution of the homogeneous system.
b) Determine a particular solution.
c) Determine the solution of the inhomogeneous system that satisfies the inital conditions $x(0) = 1, y(0) = 0, z(0) = 0$.

MATH 294 FALL 1991 PRELIM 2 # 3 294FA91P2Q3.tex
3.3.56 Determine the general solution of the second order system

$$x" = 3x + 7y$$
$$y" = x - 3y$$

and express it in terms of real-valued functions.

MATH 294 FALL 1991 FINAL # 4 294FA91FQ4.tex
3.3.57 Consider the inhomogeneous system

$$\frac{dx_1}{dt} = 4x_1 - x_2 + 3e^{2t}$$
$$\frac{dx_2}{dt} = 5x_1 - 2x_2 + 10\sin t$$

a) Determine the general solution of the homogeneous system.
b) Determine a particular solution.
c) Determine the solution of the inhomogeneous system that satisfies the initial conditions $x_1(0) = x_2(0) = 0$

MATH 294 SPRING 1992 PRELIM 1 # 4 294SP92P1Q4.tex
3.3.58 Describe the following situation by a set of differential equations:
"A barrel of water has temperature $T_b(t)$.
It sits in a room with temperature $T_r(t)$.
Outside the room the temperature is a constant 20^oF.
The barrel cools at a rate proportional to the difference between its temperature and the room temperature.
The room cools due to heat flow to the outside at a rate proportional to the difference between the inside and outside temperature.
The room is simultaneously heated by the barrel at a rate proportional to the difference between the barrel temperature and the room temperature."
Clearly define the constants and state which are positive in your description.

MATH 294 SPRING 1992 PRELIM 2 # 1 294SP92P2Q1.tex

3.3.59 Consider the system of differential equations

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = 6x + y$$

a) Find the general solution.

b) Find the solution which satisfies the initial conditions $x(1) = 0$ and $y(1) = 0$.

MATH 294 SPRING 1992 PRELIM 2 # 2 294SP92P2Q2.tex

3.3.60 Consider the initial value problem $\frac{d^2y}{dt^2} - ty\frac{dy}{dt} + y = t,\ y(0) = 0\ \ and\ \ \frac{dy}{dt}(0) = 1$

a) Rewrite this problem as an initial value problem for a first order system of two differential equations.

b) Write down Euler's method for the system in part a) using step size h.

c) Use part b) with h = .1 to calculate the approximate solution at t = .1 and t = .2

MATH 294 SPRING 1992 PRELIM 2 # 3 294SP92P2Q3.tex

3.3.61 Find a particular solution of the system

$$\frac{dx}{dt} = -3x - e^t$$

$$\frac{dy}{dt} = z + 3\sin 2t$$

$$\frac{dz}{dt} = -y + 5$$

Note: <u>You are not asked to find the general solution of the system.</u>

MATH 294 SPRING 1992 PRELIM 2 # 4 294SP92P2Q4.tex

3.3.62 Given the vector field $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ and the closed curve C shown below, compute:

a) the (counter-clockwise) circulation of $\mathbf{F}$ around C;

b) the flux of $\mathbf{F}$ across C.

MATH 294 SPRING 1992 PRACTICE PRELIM 2 # 3 294SP92PP2Q3.tex

3.3.63 Find a particular solution of the system

$$\frac{dx}{dt} = -2x + 2y + \cos t + 3$$

$$\frac{dy}{dt} = -3x + 2y + 3e^{-t}$$

MATH 294 SPRING 1992 FINAL # 4 294SP92FQ4.tex

3.3.64 Find the general solution of

$$\frac{dx}{dt} = 2x - y + 2\sin t + 2\cos t$$
$$\frac{dy}{dt} = 3x - 2y + 1.$$

MATH 294 SPRING 1992 FINAL # 5 294SP92FQ5.tex

3.3.65 **a**) Find the general solution of

$$\frac{d^2x}{dt^2} = x + y$$
$$\frac{d^2y}{dt^2} = 4x - 2y.$$

Write your solution in terms of real valued functions.

b) Find the solution satisfying initial conditions $x(0) = 2,\ y(0) = 2,\ x'(0) = y'(0) = 0$.

MATH 294 FALL 1992 PRELIM 2 # 1 294FA92P2Q1.tex

3.3.66 **a**) Find the general solution of

$$\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} e^{-t} \\ 4 \end{bmatrix},\ \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

b) Find the solution satisfying the initial condition $\mathbf{x}(0) = 0$.

MATH 294 FALL 1992 PRELIM 2 # 2 294FA92P2Q2.tex

3.3.67 Consider a second-order system, **x**" = **Ax**, where **A** is a 3x3 symmetric matrix, and **x** is 3-vector. Although **A** is not specified, it is known that

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}.$$

Find the solution satisfying the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, **x**'(0) = 0.

MATH 294 FALL 1992 FINAL # 3 294FA92FQ3.tex

3.3.68 Find the general solution of

$$x_1' = x_1 + x_2 + 3.$$
$$x_2' = 4x_1 + x_2$$

MATH 294 FALL 1992 FINAL # 6 294FA92FQ6.tex

3.3.69 Find the eigenvalues and eigenfunctions (nontrivial solutions) of the two-point boundary-value problem

$$y" + \lambda y = 0,\ 0 < x < 1, (\text{ assume } \lambda \geq 0)$$

$$y'(0) = y(1) = 0.$$

MATH 294 FALL 1992 FINAL # 10 294FA92FQ10.tex

3.3.70 Solve the initial-value problem $\mathbf{x}" = \mathbf{A}\mathbf{x}$ (where $\mathbf{x}$ is a 3-vector), $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}, \mathbf{x}'(0) = 0,$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Hint: If you think they are required, all eigenvalues/eigenvectors can be found as follows: one eigenpair follows by inspection; the other two eigenpairs were essentially computed in problem 3 (see. Q68)

MATH 294 SPRING 1993 PRELIM 2 # 1 294SP93P2Q1.tex

3.3.71 **a**) Find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

b) Find the particular solution with

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

MATH 294 SPRING 1993 PRELIM 2 # 4 294SP93P2Q4.tex

3.3.72 A damped spring-mass system satisfies $u" + 2u' + 10u = 0$. Define $x_1(t) = u(t),\ x_2(t) = u'(t),\ x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

a) Show that x satisfies a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and find A.

b) Find the general solution of the system in (a).

MATH 294 SPRING 1993 MAKE UP PRELIM 2 # 2 294SP93MUP2Q2.tex

3.3.73 Solve

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

MATH 294 SPRING 1993 MAKE UP PRELIM 2 # 4 294SP93MUP2Q4.tex

3.3.74 **a)** If you were to try to solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{iwt}$$

by substituting $\mathbf{x}(t) = \mathbf{v}e^{iwt}$, what algebraic equation would $\mathbf{v}$ have to satisfy? (You are not asked to solve it.)

b) Give an example of a 4x4 matrix all of whose eigenvalues are real.

c) If $\mathbf{x}' = 3\mathbf{x} + ie^{it}$ and $\mathbf{r}(t)$ is the real part of $\mathbf{x}(\mathbf{t})$, what first order ODE does $\mathbf{r}(t)$ solve? (You are not asked to solve it.)

MATH 294 SPRING 1993 FINAL # 3 294SP93FQ3.tex

3.3.75 Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \mathbf{x}.$$

MATH 294 FALL 1993 PRELIM 2 # 5 294FA93P2Q5.tex

3.3.76 Find a second order differential equation which $x_2(t)$ solves whenever $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x}$. Here x_2 is the second component of the vector $\mathbf{x}$.

MATH 294 FALL 1993 MAKE UP PRELIM 2 # 5 294FA93MUP2Q5.tex

3.3.77 Reduce

$$u" + 0.5u' + 2u = 3\sin t$$

to a system of first order equations and express in matrix form.

MATH 294 FALL 1993 PRELIM 3 # 2 294FA93P3Q2.tex

3.3.78 **a)** Find the eigenvalues and eigenvectors of the system

$$\mathbf{x}' = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \mathbf{x},$$

b) Find the general real-valued solution for the system in part a.

c) Sketch the phase portrait. Are these solutions stable?

d) Write the second-order differential equation that corresponds to the system in part a.

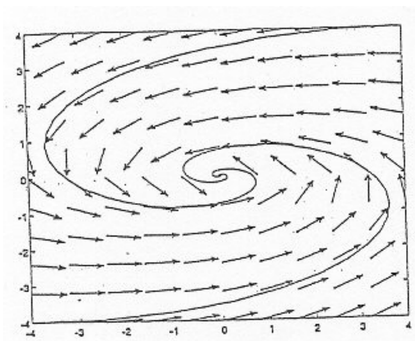
MATH 294 FALL 1993 PRELIM 3 # 4 294FA93P3Q4.tex

3.3.79 **a**) Find the solution of

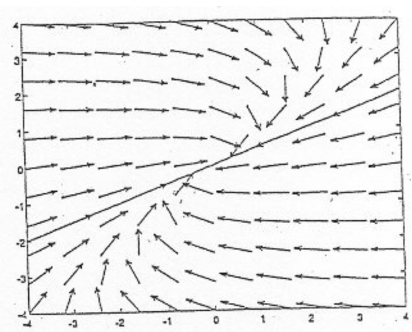
$$\mathbf{X}' = \begin{bmatrix} 1 & -4 \\ 4 & 7 \end{bmatrix} \mathbf{X}$$

$$\text{for } \mathbf{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

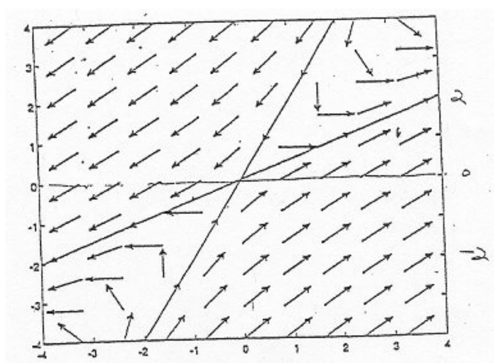
b) The phase portraits near equilibrium for three systems of two 1st order equations are sketched below. Match the portraits with the eigenvalue pairs listed.



A



B



C

(i) $\lambda_1 = -1 + 2i,\ \lambda_2 = -1 - 2i$
(ii) $\lambda_1 = 1 + 2i,\ \lambda_2 = 1 - 2i$
(iii) $\lambda_1 = -1,\ \text{ no } \lambda_2$
(iv) $\lambda_1 = -2,\ \lambda_2 = -1$
(v) $\lambda_1 = 1,\ \lambda_2 = -1$
(vi) $\lambda_1 = -1,\ \lambda_2 = 0$

MATH 294 FALL 1993 FINAL # 2 294FA93FQ2.tex

3.3.80 **a)** Solve $y"(t) + 2y'(t) + 2y(t) = 0, \quad y(0) = 1, \quad y'(0) = 1$

b) Put the equation in (a) into the form $x' = Ax$, where $x_1 = y$, $x_2 = y'$. Put the initial conditions in vector form too. Sketch several trajectories in the phase plane.

c) For the forced equation $y" + 3y' + 2y = coswt$, find a value for w and a particular solution such that the friction term $3y'$ matches the force $coswt$, while the acceleration $y"$ and spring force term $2y$ cancel each other.

MATH 294 SPRING 1994 FINAL # 6 294SP94FQ6.tex

3.3.81 Suppose that $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then find $\mathbf{x}(t)$ for the following systems of ODE's:

a) $\mathbf{x}' = 2\mathbf{x}$

b) $\mathbf{x}' = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

c) $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}$

Hint: It may help to write the ODE's for x_1 and x_2 separately in some cases.

MATH 294 FALL 1994 PRELIM 2 # 2 294FA94P2Q2.tex

3.3.82 Solve the following

a) $y" + 3y' + 2y = 0, \qquad y(0) = -1,\ y'(0) = -1.$

b) $y' = \dfrac{y^2 cosx}{3}$, find general solution.

MATH 294 FALL 1994 PRELIM 3 # 1 294FA93P3Q1.tex

3.3.83 Given $y(t) = e^{-t}\sin(t)$,

a) Find an ordinary differential equation $(y" + by' + cy = 0)$, solved by y.

b) Find an equivalent system of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

c) Sketch the phase portrait of the system of equations. Show the curve corresponding to y as a curve on the phase plane.

d) Solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

MATH 294 SPRING 1995 FINAL # 2 294SP95FQ2.tex

3.3.84 Let

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

a) Write a fundamental set of solutions to the system.

b) Give the solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

to the system such that $x(0) = 3,$ and $y(0) = -1.$

c) Sketch your solutions $x(t)$ and $y(t)$ to part (b) in the $x - t,$ and $y - t$ planes.

d) Sketch the phase plane for this system showing characteristic trajectories, including that of your solution to (b) (indicate it), with arrowheads indicating direction of movement as $t \to \infty$.

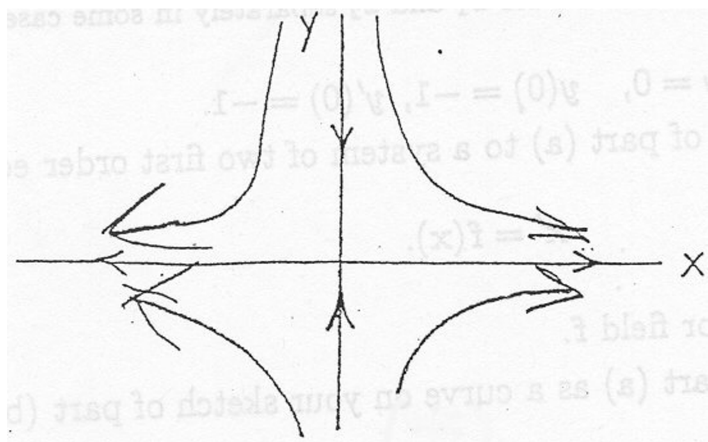
e) Transform this system of equations to a second order equation in x plus a first order equation in x and y that may be solved successively to obtain the same solutions as the original system.

MATH 294 SPRING 1995 FINAL # 3 294SP95FQ3.tex

3.3.85 Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Suppose the phase plane picture of trajectories looks like this:



a) If

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is a solution with $x(0) > 0$ and $y(0) > 0$, sketch the graphs of $x(t)$ and $y(t)$ for $-\infty < tM\infty$.

b) Same as (a) of $x(0) < 0$, $y(0) < 0$.

c) What information can you infer about the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from the phase plane picture?

d) Give an example of a matrix such that the system will have this phase plane picture.

MATH 294 SUMMER 1995 QUIZ 3 # 2 294SU95Quiz3Q2.tex

3.3.86 **a**) Convert $\frac{1}{2}y" + y' + 5y = 0$ to a system of first order differential equations.

b) Write the above in a matrix form, i.e.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

MATH 294 SUMMER 1995 QUIZ 3 # 3 294SU95Quiz3Q3.tex

3.3.87 **a**) Solve the system of linear, first order differential equations,

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}, \text{ with } \mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

b) Sketch the phase plane portrait corresponding to the above system of differential equations.

c) Show the solution corresponding to the given initial condition on the phase plane portrait.

d) Sketch the solutions x_1 and x_2 vs t, i.e. $x_1(t)$ and $x_2(t)$ for the above I.C.

MATH 294 FALL 1995 PRELIM 2 # 3 294FA95P2Q3.tex

3.3.88 Write a system of 1st order equations which is equivalent to the equation $y" + 3y' + y + y^3 = 0$. Don't try to solve.

MATH 294 FALL 1995 PRELIM 3 # 1 294FA95P3Q1.tex

3.3.89 For each of the systems find the general solution and sketch the phase plane. Also show the solution passing through $x(0) = 1,\ y(0) = 1$ both on your phase plane sketch and on plots of $xvst$ and $yvst$ with as much detail as you can give. Note that you are given most of the eigenvalue and eigenvector information you need.

a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

given $\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1+i \\ i \end{pmatrix} = (-2+i) \begin{pmatrix} 1+i \\ i \end{pmatrix}$ and $\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1-i \\ -i \end{pmatrix} = (-2-i) \begin{pmatrix} 1-i \\ -i \end{pmatrix}$

b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

given $\begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = -3 \begin{pmatrix} 1 \\ i \end{pmatrix}$, repeated eigenvalue, one linearly independent eigenvector

c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

given $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = - \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

MATH 294 FALL 1995 FINAL # 3 294FA95FQ3.tex

3.3.90 This concerns the system of ODE's $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

a) Find the general solution.

b) Solve for $x(t)$ if $x(0) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and describe the behavior of this solution as $t \to \infty$.

MATH 294 SPRING 1996 PRELIM 3 # 1 294SP96P3Q1.tex
3.3.91 Solve the system

$$\begin{cases} x' = -x + 2y \\ y' = -2x - y \end{cases}.$$

with the initial conditions $x(0) = 4,\ y(0) = 5$. You may use the fact that the associated matrix has eigenvalues $-1+2i$ and $-1-2i$. Sketch the solution curve in the (x, y) plane. Also find the critical point (i.e. equilibrium or constant solution) of this system.

MATH 294 SPRING 1996 MAKE UP FINAL # 4 294SP96MUFQ4.tex
3.3.92 Consider the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} \qquad (1)$$

a) Write the corresponding second-order ordinary differential equation.
b) Find the eigenvalues of (1) and the character of the critical point(s).
c) Find the eigenvalues for

$$\mathbf{x}' = \begin{pmatrix} \epsilon & 1 \\ 1 & \epsilon \end{pmatrix} \mathbf{x} \qquad (2)$$

What is the character of the critical point(s) of (2), and how does the stability depend on the sign of ϵ?

MATH 293 FALL 1996 PRELIM 2 # 3 293FA96P2Q3.tex
3.3.93 Find $y(3)$ given that $x(0) = 1,\ y(0) = 0$, and:

$$\dot{x} = 3x + 4y$$

$$\dot{y} = -4x + 3y$$

MATLAB option. You can get full credit for this problem by writing a set of MATLAB commands and/or functions and/or script files which would give $y(3)$. You can get full credit by solving by hand also.

MATH 294 FALL 1996 PRELIM 3 # 1 294FA96P3Q1.tex

3.3.94 **a**) Write the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{x} \text{ where } \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

as a second-order ordinary differential equation.

b) Find the eigenvalues for the problem given in part a. Locate the critical point(s) of this system and describe its (their) character. Sketch a few trajectories in the phase plane.

c) Find the eigenvalues of

$$\frac{dx}{dt} = \begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} \underline{x}; \epsilon << 1$$

What is the character of the critical point(s) of this equation and (how) does the stability depend on ϵ's sign? Sketch a few trajectories in the phase plane for $\epsilon < 0$, and plot $x_1(t)$ and $x_2(t)$.

MATH 293 FALL 1996 PRELIM 3 # 4 293FA96P3Q4.tex

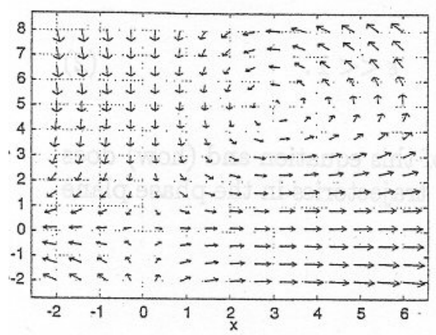
3.3.95 No partial credit. You need correct reasoning and the correct answer to get credit for each part. No credit for part(b) unless part(a) is correct. Consider the pair of coupled ordinary differential equations:

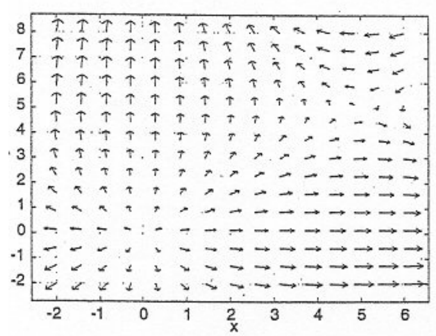
$$\frac{dx}{dt} = x(5-y)$$

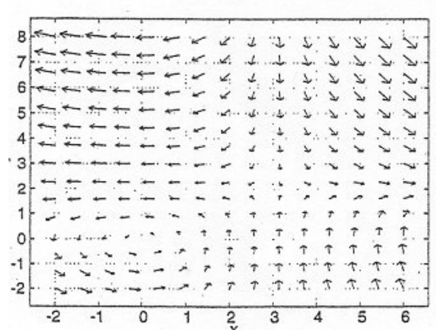
$$\frac{dy}{dt} = y(5-x)$$

a) What are the critical points for these equations?

b) One and only one of the three pictures below (made by PPLANE) is the vector field for the equations above. Pick and use the correct picture to determine for each critical point whether it is stable or unstable.







MATH 294 FALL 1996 MAKE UP PRELIM 3 # 1 294FA96MUP3Q1.tex

3.3.96 **a**) Show that

$$\underline{x} = \underline{\xi} t^r,$$

where ξ a constant vector and r a constant, will give non-trivial solutions to

$$t\underline{x}' = \underline{Ax}$$

if ξ and r satisfy $(\underline{A} - r\underline{I})\underline{\xi} = 0$.

b) Find a general solution for the system of equations, assuming $t > 0$

$$t\underline{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \underline{x}.$$

MATH 293 FALL 1996 FINAL # 2 293FA96FQ2b.tex

3.3.97 $\dfrac{dx}{dt} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

% This is a diary of a MATLAB session"
% It does all the calculations needed for this problem
¿¿ [V,D] = eig([1 1 ; 1 1]);
¿¿ V = sqrt(2) * V ; % this tidies up V
¿¿ [V,D]
ans =
1.0000 1.0000 0 0
-1.0000 1.0000 0 2.0000
¿¿ V/[3 7]'
ans =
-2
5

MATH 294 SPRING 1997 PRELIM 2 # 3 294SP97P2Q3.tex

3.3.98 You are told that for a certain matrix B it is true that $B\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $B\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Solve the initial value problem

$$\mathbf{x}' = \mathbf{Bx}$$
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You are not asked to find $\mathbf{B}$. (Notation: $x =$ x $= \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $' = \frac{d}{dt}$)

MATH 294 FALL 1997 PRACTICE PRELIM 3 # 2 294FA97PP3Q2.tex

3.3.99 Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \qquad (2)$$

It is given that the characteristic equation of **A** is: $-\lambda(\lambda-1)^2 = 0$

a) Find all the eigenvalues of **A**.

b) Find all possible linearly independent eigenvectors of **A**.

c) Is **A** diagonalizable? (i.e. Is **A** similar to a diagonal matrix **D**?) If so, write down a diagonal **D** that is similar to **A**.

Now consider the system of ordinary differential equations:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad (3)$$

with

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad (4)$$

where $\mathbf{x}(t)\epsilon R^3$ and $\mathbf{x} = \dfrac{dx}{dt}$

d) Find a general solution to (3)

e) Find the unique solution to $\mathbf{x}(t)$ of (3,4).

f) Verify that $\mathbf{x}(t)$ satisfies (3,4).

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3.3.100 A certain real 3x3 matrix **A** has the following properties:

$$\mathbf{A}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{A}\begin{bmatrix} 3i \\ 2+i \\ i \end{bmatrix} = \begin{bmatrix} 3i \\ 2i-1 \\ i \end{bmatrix}$$

Without finding the matrix **A**, find the real form of the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

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3.3.101 Construct the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ involving complex eigenfunctions and then obtain the general real solution when

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}.$$

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3.3.102 Determine the general solution of the sytem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, when

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}.$$

Hint: The eigenvalues are -3, -1, and 5.

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3.3.103 Determine the solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix},$$

that satisfies the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

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3.3.104 Determine the solution of the initial-value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0,$$

where

$$\mathbf{A} \equiv \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix} \text{ and } \mathbf{x}_0 \equiv \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

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3.3.105 Give the general solution, in terms of real valued functions, of the system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{x}.$$