2.9 Orthogonality

MATH 294 SPRING 1987 PRELIM 3 # 8

2.9.1 Find c_3 so that:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ -4 \\ -4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

Note that the four vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and $\vec{v}_4$ are mutually orthogonal.

MATH 294 SPRING 1992 FINAL # 6

2.9.2 Given $A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

a) Find an orthogonal matrix C such that $C^{-1}AC$ is diagonal. (The columns of an orthogonal matrix are orthonormal vectors.)

b) If $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 + d\vec{v}_3$

where $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are the columns of C, find the scalars a, b and d.

MATH 293 FINAL SPRING 1993 # 3

2.9.3 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

a) Find the vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that a solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ exists.

b) Find a basis for the column space $\mathcal{R}(A)$ of A

c) It is claim that $\mathcal{R}(A)$ is a plane in $\Re^3$. If you agree, find a vector n in $\Re^3$ that is normal to this plane. Check your answer.

d) Show that n is perpendicular to each of the columns of A. Explain carefully why this is true.

MATH 293 FALL 1994 PRELIM 3 # 5

2.9.4 True/False

Answer each of the following as True or False. If False, explain, by an example.

a) Every spanning set of $\Re^3$ contain at least three vectors.
b) Every orthonormal set of vectors in $\Re^5$ is a basis for $\Re^5$.
c) Let A be a 3 by 5 matrix. Nullity A is at most 3.
d) Let W be a subspace of $\Re^4$. Every basis of W contain at least 4 vectors.
e) In $\Re^n, ||cX|| = |c|||X||$
f) If A is an $n \times n$ symmetric matrix, then rank $A = n$.

MATH 294 FALL 1997 PRELIM 3 # 4

2.9.5 Consider $\mathcal{W}$, a subspace of $\Re^4$, defined as $\sqsupseteq\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$\mathcal{W}$ is a "plane" in $\Re^4$.

a) Find a basis for a subspace $\mathcal{U}$ of $\Re^4$ which is orthogonal to $\mathcal{W}$.

Hint: Find *all* vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that are perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

b) What is the geometrical nature of $\mathcal{U}$?

c) Find the vector in $\mathcal{W}$ that is closest to the vector $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

MATH 294 unknown unknown # ?

2.9.6 Let W be the subspace of $\Re^3$ spanned by the orthonormal set $\left\{\frac{(1,2,-1)}{\sqrt{6}}, \frac{(1,0,1)}{\sqrt{2}}\right\}$. Let $X = (1, 1, 1)$. Find a vector Z, in W, and a vector Y, perpendicular to every vector in W, such that $X = Z + Y$. What is the distance from X to W?

MATH 294 SPRING 1999 PRELIM 3 # 1

2.9.7 Let the functions $f_1 = 1, f_2 = tf_3 = t^2$ be three "vectors" which span a subspace, S, in the vector space of continuous functions on the interval $-1 \leq t \leq 1 (C[-1, 1])$, with inner product

$$< f, g >\equiv \int_{-1}^{1} f(t)g(t)dt.$$

Find three orthogonal vectors, $u_1 = 1, u_2 =?, u_3 =?$ that span S.

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2.9.8 Consider the vector space $C_0(-\pi,\pi)$ of continuous functions in the interval $-\pi \leq x \leq \pi$, with inner product conjugation. Consider the following set of functions $b = \left\{\dots e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \dots\right\}$.

a) Are they linearly independent? (Hint: Show that they are orthogonal, that is $\left(e^{inx}, e^{imx}\right) = 0$ for $n \neq m$.
$\left(e^{inx}, e^{imx}\right) \neq 0$ for $n = m$.

b) Ignoring the issue of convergence for the moment, let $f(x)$ be in $C_0(-\pi,\pi)$. Express $f(x)$ as a linear combination of the basis B. That is,

$$f = \dots a_{-2}e^{-2ix} + a_{-1}e^{-ix} + a_0 + a_1e^{ix} + a_2e^{2ix} + \dots$$

find the coefficients $\{a_n\}$ of each of the basis vectors. Use the results from (a).

c) How does this relate to the Fourier series? Are the coefficients $\{a_n\}$ real or complex? What if B is a set of arbitrary orthogonal functions?

MATH 294 SPRING 1999 PRELIM 2 # 2a

2.9.9 **a**) Three matrices A, B, and P have:

i) $A = P^{-1}BP$,

ii) B is symmetric ($B^T = B$), and

iii) P is orthogonal ($P^T = P^{-1}$).

Is it necessary true that A is symmetric? If so, prove it. If not, find a counter example (say three 2×2 matrices A, B and P where (i) - (iii) above are true and A is not symmetric).

MATH 294 SPRING 1999 PRELIM 3 # 4

2.9.10 The temperature, $u(x,y)$, in a rectangular plate was measured at six locations. The (x,y) coordinates and measured temperatures, u, are given in the table below.

x	y	u
0	0	11
$\frac{\pi}{2}$	0	19
0	1	1
$\frac{\pi}{2}$	1	14

Assume that $u(x,y)$ is supposed to obey the equation (this is *not* a PDE question)

$$u(x,y) = \beta_0 + \beta_1 e^{-y}\sin x.$$

Set up, but do not solve, a system of equations for the parameters, β_0, β_1, that provide the least-squares best fit of the measured data to the equation above.
Extra credit Neatly write out a sequence of Matlab commands that will give you the parameters β_0, β_1.