2.8 Linear Transformation II

MATH 294 SPRING 1987 PRELIM 3 # 3

2.8.1 Consider the subspace of C^2_∞ given by all things of the form

$$\vec{x}(t) = \begin{bmatrix} a\sin t + b\cos t \\ c\sin t + d\cos t \end{bmatrix},$$

where a,b,c & d are arbitrary constants. Find a matrix representation of the linear transformation

$$T(\vec{x}) = D\vec{x}, \text{ where } D\vec{x} \equiv \dot{\vec{x}}.$$

carefully define any terms you need in order to make this representation. Hint: A good basis for this vector space starts something like this

$$\left\{ \begin{pmatrix} \sin t \\ 0 \end{pmatrix}, \dots \right\}.$$

MATH 294 SPRING 1987 PRELIM 3 # 5

2.8.2 The idea of eigenvalue λ and eigenvector $\mathbf{v}$ can be generalized from matrices and $\Re^n$ to linear transformations and their related vector spaces. If $T(\mathbf{v}) = \lambda\mathbf{v}$ (and $\mathbf{v} \neq 0$) then λ is an eigenvalue of T, and $\mathbf{v}$ is its associated eigenvector.
For the subspace of $\mathbf{x}(t)$ in C^1_∞ with $\mathbf{x}(0) = \mathbf{x}(1) = 0$ find *an* eigenvalue and eigenvector of $T(\mathbf{x}) = D^2\mathbf{x}$, where $D^2\mathbf{x} \equiv \ddot{\mathbf{x}} - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{x}$. What is the kernel of T?

MATH 294 spr97 FINAL # 2

2.8.3 T is linear transformation from C^2_∞ to C^2_∞ which is given by $T(\mathbf{x}) = \dot{\mathbf{x}}$

MATH 294 FALL 1987 PRELIM 3 # 14

2.8.4 Find the kernel of the linear transformation

$$T(\mathbf{x}(t)) \equiv \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where T transforms C^2_∞ into C^2_∞

MATH 294 FALL 1997 PRELIM 3 # 5

2.8.5 Define $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \equiv \begin{bmatrix} x+y \\ x-z \\ y+z \end{bmatrix}$, which is a linear transformation of $\Re^3$ into itself.

a) Is T 1-1?
b) Is T onto?
c) Is T an isomorphism?
Substantiate your answers.

MATH 294 FALL 1987 FINAL # 1

2.8.6 T is a linear transformation of $\Re^3$ into $\Re^2$ such that

$$T\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix},\ T\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\ T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}.$$

a) Is T 1-1?

b) Determine the matrix of T relative to the standard bases in $\Re^3$ and $\Re^2$.

MATH 294 FALL 1987 FINAL # 7a

2.8.7 Consider the boundary-value problem

$X'' + \lambda X = 0,\ \ 0 < x < \pi,\ X(0) = X(\pi) = 0$, where λ is a given real number.

a) Is the set of all solutions of this problem a subspace of $C_\infty[0,\pi]$? why?

b) Let W = set of all functions $X(x)$ in $C_infty[0,\pi]$ such that $X(0) = X(\pi) = 0$. Is $T \equiv D^2 - \lambda$ linear as a transformation of W into $C_\infty[0,\pi]$? Why?

c) For what values of λ is $Ker(T)$ nontrivial?

d) Choose one of those values of λ and determine $Ker(T)$

MATH 294 FALL 1989 PRELIM 3 # 3

2.8.8 Let W be the following subspace of $\Re^3$,

$$W = Comb\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}3\\3\\-3\end{bmatrix}\right).$$

a) Show that $\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}$ is a basis for W.

For b) and c) below, let T be the following linear transformation $T : W \to \Re^3$,,

$$T\left(\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)=\begin{bmatrix}1&0&-1&\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\\0&0&0&\\0&0&0&\end{bmatrix}$$

for those $w = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ in $\Re^3$ which belong to W.

[You are allowed to use a) even if you did not solve it.]

b) What is the dimension of $Range(T)$? (Complete reasoning, please.

c) What is the dimension of $Ker(T)$? (Complete reasoning, please.

MATH 294 FALL 1989 FINAL # 7

2.8.9 Let $T : \Re^2 \to \Re^2$ be the linear transformation given in the standard basis for $\Re^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 0 \end{bmatrix}.$$

a) Find the matrix of T in the standard basis for $\Re^2$

b) Show that $\beta = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ is also a basis for $\Re^2$.

In c) below, you may use the result of b) even if you did not show it.

c) Find the matrix of T in the basis β given in b). (I.e., in $T : \Re^2 \to \Re^2$ both copies of $\Re^2$ have the basis β.

MATH 294 SPRING 1990? PRELIM 2 # 4

2.8.10 Let A be a linear transformation from a vector space V to another vector space U. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a basis for V and let $(\vec{u}_1, \ldots, \vec{u}_n)$ be a basis for U.
Suppose it is known that

$$\begin{aligned} &A(\vec{v}_1) = 2u_2 \\ &A(\vec{v}_2) = 3u_3 \\ &\vdots \\ &A(\vec{v}_i) = (i+1)\vec{u}_{i+1} \\ &\vdots \\ &A(\vec{v}_{n-1}) = n\vec{u}_n \end{aligned}$$

and $A(\vec{v}_n) = 0 \leftarrow$ zero vector in U.
Can you find $A(\vec{v})$ in terms of the $\vec{u}_i$'s where

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_n = \sum_{i=1}^{n} \vec{v}_i$$

MATH 294 FALL 1991 FINAL # 8

2.8.11 T/F

c) If $T : V \to W$ is a linear transformation, then the range of T is a subspace of V.

d) If the range of $T : V \to W$ is W, then T is 1-1.

e) If the null space of $T : V \to W$ is $\{0\}$, then T is 1-1.

f) Every change of basis matrix is a product of elementary matrices.

g) If $T : U \to V$ and $S : V \to W$ are linear transformations, and S is not 1-1, then $ST : U \to W$ is not 1-1.

h) If V is a vector space with an inner product, (,), if $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is an orthonormal basis for V, and if $\vec{v}$ is a vector in V, then $\vec{v} = \sum_{i=1}^{n} (\vec{v}, \vec{w}_i)\vec{w}_i$.

i) $T : V_n \to V_n$ is an isomorphism if and only if the matrix which represents T in any basis is non-singular.

j) If S and T are linear transformations of V_n into V_n, and in a given basis, S is represented by a matrix A, and T is represented by a matrix B, then ST is represented by the matrix AB

-note- Matrices are not necessarily square.

MATH 294 SPRING 1992 PRELIM 3 # 5

2.8.12 The vector space V_3 has the standard basis $S = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and the basis $B = \left(2\vec{e}_2, -\frac{1}{2}\vec{e}_1, \vec{e}_3\right)$.

a) Find the change of basis matrices $(B : S)$ and $(S : B)$. If a vector $\vec{v}$ has the representation $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the standard basis, find its representation $\beta(\vec{v})$ in the B basis.

b) A transformation T is defined as follows: $T\vec{v} =$ the reflection of $\vec{v}$ across the $x-z$ plane in the standard basis. (For reflection, in V_2 the reflection of $a\vec{i} + b\vec{j}$ across the x axis would be $a\vec{i} - b\vec{j}$. Find a formula for T in the standard basis. Why is T a linear transformation?

c) Find T_B, the matrix of T in the B basis.

d) Interpret T geometrically in the B basis, i.e., describe T_B in terms of rotations, reflections, etc.

MATH 294 FALL 1992 FINAL # 6

2.8.13 Let $C^2(-\infty, \infty)$ be the vector space of twice continuously differentiable functions on $-\infty < x < \infty$ and $C^0(infty, \infty)$ be the vector space of continuous functions on $-\infty < x < \infty$.

a) Show that the transformation $L : C^2(\infty, \infty) \to C^0(-\infty, \infty)$ defined by $Ly = \frac{\partial^2 y}{\partial x^2} - 4y$ is linear.

b) Find a basis for the null space of L. Note: You must show that the vectors you choose are linearly independent.

MATH 293 SPRING 1995 FINAL # 2

2.8.14 Let P^3 be the vector space of polynomials of degree ≤ 3, and let $L : P^3 \to P^3$ be given by

$$L(p)(t) = t\frac{\partial^2 p}{\partial t^2}(t) + 2p(t).$$

a) Show that L is a linear transformation.
b) Find the matrix of L in the basis $\left(1, t, t^2, t^3\right)$.
c) Find a solution of the differential equation

$$t\frac{\partial^2 p}{\partial t^2} + 2p(t) = t^3.$$

Do you think that you have found the general solution?

MATH 293 SPRING 1995 FINAL # 3

2.8.15 Let V be the vector space of real 3×3 matrices.
a) Find a basis of V. What is the dimension of V?
Now consider the transformation $L : V \to V$ given by $L(A) = A + A^T$.
b) Show that L is a linear transformation.
c) Find a basis for the null space (kernel) of L.

MATH 294 SPRING 1997 FINAL # 10

2.8.16 Let P_2 be the vector space of polynomials of degree ≤ 2, equipped with the inner product

$$< p(t), q(t) >= \int_{-1}^{1} p(t)q(t)dt$$

Let $T : P_2 \to P_2$ be the transformation which sends the polynomial $p(t)$ to the polynomial

$$(1 - t^2)p''(t) - 2tp'(t) + 6p(t)$$

a) Show that T is linear.
b) Verify that $T(1) = 6$ and $T(t) = 4t$. Find $T(t^2)$.
c) Find the matrix A of T with respect to the standard basis $\epsilon = \left(1, t, t^2\right)$ for P_2.
d) Find the basis for $Nul(A)$ and $Col(A)$.
e) Use the Gram-Schmidt process to find an orthogonal basis B for P_2 starting form ϵ.

MATH 294 FALL 1997 PRELIM 3 # 5

2.8.17 Let $T : \Re^2 \rightarrow \Re^2$ be the linear transformation that rotates every vector (starting at the origin) by θ degrees in the counterclockwise direction. Consider the following two bases for $\Re^2$:

$$B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

and

$$C = \left(\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right).$$

a) Find the matrix $[T]_B$ of T in the standard basis B.
b) Find the matrix $[T]_C$ of T in the basis C. Does $[T]_C$ depend on the angle α?

MATH 294 FALL 1997 FINAL # 9

2.8.18 Consider the vector space V of 2 matrices. Define a transformation $T : V \rightarrow V$ by $T(A) = A^T$, where A is an element of V (that is, it is a 2×2 matrix), and A^T is the transpose of A.
a) Show that T is linear transformation.
The value λ is an *eigenvalue* for T, and $\vec{v} \neq 0$ is the corresponding eigenvector, if $T(\vec{v}) = \lambda\vec{v}$. (*Note*: here $\vec{v}$ is a 2×2 matrix).
b) Find an eigenvalue of T (You need only find one, not all of them). (*Hint*: Search for matrices A such that $T(A)$ is a scalar multiple of A.)
c) Find an eigenvector for the particular eigenvalue that yl=ou found in part (b).
d) Let W be the complete eigenspace of T with the eigenvalue from part (b) above. Find a basis for W. What is the dimension of W?

MATH 294 SPRING 1998 FINAL # 6

2.8.19 Let $T : P^2 \rightarrow P^3$ be the transformation that maps the second order polynomial $p(t)$ into $(1 + 2t)p(t)$,
a) Calculate $T(1), T(t)$, and $T(t^2)$.
b) Show that T is a linear transformation.
c) Write the components of $T(1), T(t), T(t^2)$ with respect to the basis $C = \{1, t, t^2, 1 + t^3\}$.
d) Find the matrix of T relative to the bases $B = \{1, t, t^2\}$ and $C = \{1, t, t^2, 1 + t^3\}$.

MATH 294 FALL 1998 PRELIM 3 # 1

2.8.20 Consider the following three vectors in $\Re^3$

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{and} \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

[Note: $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.].

a) Find the orthogonal projection of $\vec{y}$ onto the subspace of $\Re^3$ spanned by $\vec{u}_1$ and $\vec{u}_2$.

b) What is the distance between $\vec{y}$ and $span\{\vec{u}_1, \vec{u}_2\}$?

c) In terms of the standard basis for $\Re^3$, find the matrix of the linear transformation that orthogonally projects vectors onto $span\{\vec{u}_1, \vec{u}_2\}$.

MATH 294 FALL 1998 FINAL # 4

2.8.21 Here we consider the vector spaces P_1, P_2, and P_3 (the spaces of polynomials of degree 1,2 and 3).

a) Which of the following transformations are linear? (Justify your answer.)

i) $T : P_1 \to P_3,\ T(p) \equiv t^2 p(t) + p(0)$

ii) $T : P_1 \to P_1,\ T(p) \equiv p(t) + t$

b) Consider the linear transformation $T : P_2 \to P_2$ defined by $T(a_0 + a_1 t + a_2 t^2) \equiv (-a_1 + a_2) + (-a_0 + a_1)t + (a_2)t^2$. with respect to the standard basis of $P_2, \beta = \{1, t, t^2\}$, is $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that an eigenvalue/eigenvector pair of A is $\lambda = 1, v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Find an eigenvaue/eigenvector (or eigenfunction) pair of T. That is, find λ and $g(t)$ in P_2 such that $T(g(t)) = \lambda g(t)$.

c) Is the set of vectors in $P_2\{3+t, -2+t, 1+t^2\}$ a basis of P_2? (Justify your answer.)

MATH 293 SPRING 19? PRELIM 2 # 4

2.8.22 Let M be the transformation from P^n to P^n such that

$$Mp(t) = \frac{1}{2}[p(t) + p(-t)](\text{t real})$$

a) If $n = 3$ find the matrix of this transformation with respect to the basis $\{1, t, t^2, t^3\}$.

b) Let $N = I - M$. What is $Np(t)$ in terms of $p(t)$? Show that $M^2 = MM = M$, $MN = MN = 0$

MATH 294 FALL 1987 PRELIM 2 # 3 MAKE-UP

2.8.23 a) If A is an $n \times n$ matrix with $rank(A) = r$, then what is the dimension of the vector space of all solutions of the system of linear equations $A\vec{x} = \vec{0}$

b) What is the dimension of the kernel of the linear transformation from $\Re^n$ to $\Re^n$ which has A for its matrix in the standard basis.

MATH 294 FALL 1987 PRELIM 2 # 14 MAKE-UP

2.8.24 Show that if $T : V \to W$ is a linear transformation from V to W, and kernel$(T) = \vec{0}$, then T is 1-1. (Recall: kernel$(T) = \left\{ \vec{v} \in V |\ T(\vec{v}) = \vec{0} \right\}$.)

MATH 294 FALL 1987 FINAL # 6 MAKE-UP

2.8.25 Let $T : \Re^2 \to \Re^4$ be a linear transformation.

a) If $T \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ and $T \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, what is $T \begin{bmatrix} -9 \\ 26 \end{bmatrix}$?

b) What are $T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

c) What is the matrix of T in the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\Re^2$, and the standard basis for $\Re^4$?

MATH 294 SUMMER 1989 PRELIM 2 # 1

2.8.26 **a)**

b) Find a basis for $ker(L)$, where L is linear transformation from $\Re^4$ to $\Re^3$ defined by

$$L(\vec{x}) \equiv \begin{bmatrix} 1 & 2 & -4 & 3 \\ 1 & 2 & -2 & 2 \\ 2 & 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

c) What is the dimension of $ker(L)$?

d) Is the vector $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $range(L)$? (Justify your answer.) If so, find all vectors $\vec{x}$ in $\Re^4$ which satisfy $L(\vec{x}) = \vec{y}$

MATH 294 SUMMER 1989 PRELIM 2 # 4

2.8.27 Let P be the linear transformation from $\Re^3$ to $\Re^3$ defined by

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

a) Find a basis for $ker(P)$.

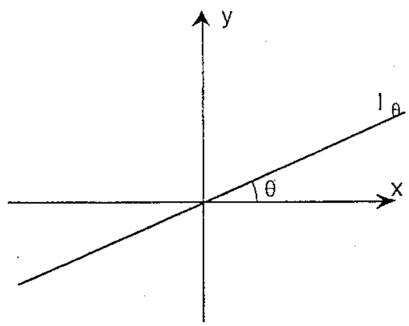
b) Find a basis for $range(P)$.

c) Find all vectors $\vec{x}$ in $\Re^3$ such that $P\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

d) Find all vectors $\vec{x}$ in $\Re^3$ such that $P\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

MATH 293 SPRING 1995 PRELIM 3 # 4

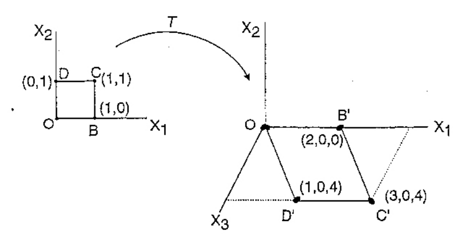
2.8.28 Let $L_\theta : \Re^2 \to \Re^2$ be the linear transformation which represent orthogonal projection onto the line ℓ_θ forming angle θ with the x-axis.



a) Find the matrix T of L_θ (with respect to the standard basis of $\Re^2$).
b) Is L_θ invertible. Explain your answer geometrically.
c) Find all the eigenvalues of T.

MATH 294 FALL 1998 PRELIM 2 # 1

2.8.29 The unit square $OBCD$ below gets mapped to the parallelogram $OB'C'D'$ (on the $x_1 - x_3$ plane) by the linear transformation $T : \Re^2 \to \Re^3$ shown.



Problems (b) - (e) below can be answered with or without use of the matrix A from part (a).
a) Is this transformation one-to-one? For this and all other short answer questions on this test, some explanation is needed.)
b) What is the null space of A?
c) What is the column space of A?
d) Is A invertible? (No need to find the inverse if it exists.)

MATH 294 FALL ? FINAL # 1 MAKE-UP

2.8.30 Consider the homogeneous system of equations $B\vec{x} = \vec{0}$, where

$$B = \begin{bmatrix} 0 & 1 & 0 & -3 & 1 \\ 2 & -1 & 0 & 3 & 0 \\ 2 & -3 & 0 & 0 & 4 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

a) Find a basis for the subspace $W \subset \Re^5$, where $W =$ set of all solutions of $B\vec{x} = \vec{0}$.

b) Is B 1-1 (as a transformation of $\Re^5 \to \Re^3$)? Why?

c) Is $B : \Re^5 \to \Re^3$ onto? Why?

d) Is the set of all solutions of $B\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ a subspace of $\Re^5$? Why?