2.8 Linear Transformation II

MATH 294 SPRING 1987 PRELIM 3 # 3
2.8.1 Consider the subspace of $\mathbb{C}^\infty_2$ given by all things of the form

$$\vec{x}(t) = \begin{bmatrix} a \sin t + b \cos t \\ c \sin t + d \cos t \end{bmatrix},$$

where $a, b, c$ & $d$ are arbitrary constants. Find a matrix representation of the linear transformation

$$T(\vec{x}) = D\vec{x}, \text{ where } D\vec{x} \equiv \dot{\vec{x}}.$$

carefully define any terms you need in order to make this representation. Hint: A good basis for this vector space starts something like this

$$\left\{ \begin{pmatrix} \sin t \\ 0 \end{pmatrix}, \ldots \right\}.$$

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2.8.2 The idea of eigenvalue $\lambda$ and eigenvector $\mathbf{v}$ can be generalized from matrices and $\mathbb{R}^n$ to linear transformations and their related vector spaces. If $T(\mathbf{v}) = \lambda \mathbf{v}$ (and $\mathbf{v} \neq 0$) then $\lambda$ is an eigenvalue of $T$, and $\mathbf{v}$ is its associated eigenvector. For the subspace of $x(t)$ in $\mathbb{C}^1_\infty$ with $x(0) = x(1) = 0$ find an eigenvalue and eigenvector of $T(x) = D^2x$, where $D^2x \equiv \ddot{x} - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} x$. What is the kernel of $T$?

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2.8.3 $T$ is linear transformation from $\mathbb{C}^2_\infty$ to $\mathbb{C}^2_\infty$ which is given by $T(x) = \dot{x}$

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2.8.4 Find the kernel of the linear transformation

$$T(x(t)) \equiv \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $T$ transforms $\mathbb{C}^2_\infty$ into $\mathbb{C}^2_\infty$

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2.8.5 Define $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \equiv \begin{bmatrix} x + y \\ x - z \\ y + z \end{bmatrix}$, which is a linear transformation of $\mathbb{R}^3$ into itself.

- a) Is $T$ 1-1?
- b) Is $T$ onto?
- c) Is $T$ an isomorphism?

Substantiate your answers.
2.8.6  $T$ is a linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^2$ such that

$$
T \begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix} = \begin{bmatrix}
2 \\
1
\end{bmatrix},
T \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix},
T \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
-1
\end{bmatrix}.
$$

a) Is $T$ 1-1?
b) Determine the matrix of $T$ relative to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$.

2.8.7  Consider the boundary-value problem

$X'' + \lambda X = 0, \quad 0 < x < \pi, \quad X(0) = X(\pi) = 0$,

where $\lambda$ is a given real number.

a) Is the set of all solutions of this problem a subspace of $C_\infty[0, \pi]$? Why?
b) Let $W = \text{set of all functions } X(x) \in C_\infty[0, \pi]$ such that $X(0) = X(\pi) = 0$.

Is $T \equiv D^2 - \lambda$ linear as a transformation of $W$ into $C_\infty[0, \pi]$? Why?
c) For what values of $\lambda$ is $\text{Ker}(T)$ nontrivial?
d) Choose one of those values of $\lambda$ and determine $\text{Ker}(T)$.

2.8.8  Let $W$ be the following subspace of $\mathbb{R}^3$,

$$
W = \text{Comb} \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, 
\begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}, 
\begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix}, 
\begin{bmatrix}
3 \\
0 \\
-3
\end{bmatrix}.
$$

a) Show that $\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}$ is a basis for $W$.

For b) and c) below, let $T$ be the following linear transformation $T : W \to \mathbb{R}^3$,

$$
T \begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}
$$

for those $w = \begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}$ in $\mathbb{R}^3$ which belong to $W$.

[You are allowed to use a) even if you did not solve it.]

b) What is the dimension of $\text{Range}(T)$? (Complete reasoning, please.
c) What is the dimension of $\text{Ker}(T)$? (Complete reasoning, please.
2.8. LINEAR TRANSFORMATION II

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2.8.9 Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the linear transformation given in the standard basis for \( \mathbb{R}^2 \) by

\[
T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ 0 \end{bmatrix}.
\]

a) Find the matrix of \( T \) in the standard basis for \( \mathbb{R}^2 \).

b) Show that \( \beta = \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \) is also a basis for \( \mathbb{R}^2 \).

In c) below, you may use the result of b) even if you did not show it.

c) Find the matrix of \( T \) in the basis \( \beta \) given in b). (I.e., in \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) both copies of \( \mathbb{R}^2 \) have the basis \( \beta \).

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2.8.10 Let \( A \) be a linear transformation from a vector space \( V \) to another vector space \( U \). Let \( (\vec{v}_1, \ldots, \vec{v}_n) \) be a basis for \( V \) and let \( (\vec{u}_1, \ldots, \vec{u}_n) \) be a basis for \( U \).

Suppose it is known that

\[
A(\vec{v}_1) = 2\vec{u}_2 \\
A(\vec{v}_2) = 3\vec{u}_3 \\
\vdots \\
A(\vec{v}_i) = (i + 1)\vec{u}_{i+1} \\
\vdots \\
A(\vec{v}_{n-1}) = n\vec{u}_n
\]

and \( A(\vec{v}_n) = 0 \) ← zero vector in \( U \).

Can you find \( A(\vec{v}) \) in terms of the \( \vec{u}_i \)'s where

\[
\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_n = \sum_{i=1}^n \vec{v}_i
\]
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2.8.11 T/F

c) If \( T : V \rightarrow W \) is a linear transformation, then the range of \( T \) is a subspace of \( V \).
d) If the range of \( T : V \rightarrow W \) is \( W \), then \( T \) is 1-1.
e) If the null space of \( T : V \rightarrow W \) is \( \{0\} \), then \( T \) is 1-1.
f) Every change of basis matrix is a product of elementary matrices.
g) If \( T : U \rightarrow V \) and \( S : V \rightarrow W \) are linear transformations, and \( S \) is not 1-1, then \( ST : U \rightarrow W \) is not 1-1.
h) Every change of basis matrix is a product of elementary matrices.
i) If \( V \) is a vector space with an inner product, (, , ) if \( \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\} \) is an orthonormal basis for \( V \), and if \( \vec{v} \) is a vector in \( V \), then \( \vec{v} = \sum_{i=1}^{n} (\vec{v}, \vec{w}_i) \vec{w}_i \).
j) If \( V \) is a vector space with an inner product, (, , ) if \( \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\} \) is an orthonormal basis for \( V \), and if \( \vec{v} \) is a vector in \( V \), then \( \vec{v} = \sum_{i=1}^{n} (\vec{v}, \vec{w}_i) \vec{w}_i \).
k) \( T : V_n \rightarrow V_n \) is an isomorphism if and only if the matrix which represents \( T \) in any basis is non-singular.

MATH 294 SPRING 1992 PRELIM 3 # 5

2.8.12

a) Find the change of basis matrices \( (B : S) \) and \( (S : B) \). If a vector \( \vec{v} \) has the representation \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\] in the standard basis, find its representation \( \beta(\vec{v}) \) in the \( B \) basis.
b) A transformation \( T \) is defined as follows: \( T\vec{v} = \) the reflection of \( \vec{v} \) across the \( x-z \) plane in the standard basis. (For reflection, in \( V_2 \) the reflection of \( a\hat{i} + b\hat{j} \) across the x axis would be \( a\hat{i} - b\hat{j} \).) Find a formula for \( T \) in the standard basis. Why is \( T \) a linear transformation?
c) Find \( T_B \), the matrix of \( T \) in the \( B \) basis.
d) Interpret \( T \) geometrically in the \( B \) basis, i.e., describe \( T_B \) in terms of rotations, reflections, etc.

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2.8.13

Let \( C^2(-\infty, \infty) \) be the vector space of twice continuously differentiable functions on \( -\infty < x < \infty \) and \( C^0(infinity, \infty) \) be the vector space of continuous functions on \( -\infty < x < \infty \).
a) Show that the transformation \( L : C^2(\infty, \infty) \rightarrow C^0(-\infty, \infty) \) defined by \( Ly = \frac{\partial^2 y}{\partial x^2} - 4y \) is linear.
b) Find a basis for the null space of \( L \). Note: You must show that the vectors you choose are linearly independent.
MATH 293  SPRING 1995  FINAL  # 2

2.8.14 Let \( P^3 \) be the vector space of polynomials of degree \( \leq 3 \), and let \( L : P^3 \to P^3 \) be given by

\[
L(p)(t) = t \frac{\partial^2 p}{\partial t^2}(t) + 2p(t).
\]

a) Show that \( L \) is a linear transformation.
b) Find the matrix of \( L \) in the basis \( (1, t, t^2, t^3) \).
c) Find a solution of the differential equation

\[
t \frac{\partial^2 p}{\partial t^2} + 2p(t) = t^3.
\]

Do you think that you have found the general solution?

MATH 293  SPRING 1995  FINAL  # 3

2.8.15 Let \( V \) be the vector space of real \( 3 \times 3 \) matrices.
a) Find a basis of \( V \). What is the dimension of \( V \)?
Now consider the transformation \( L : V \to V \) given by \( L(A) = A + A^T \).
b) Show that \( L \) is a linear transformation.
c) Find a basis for the null space (kernel) of \( L \).

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2.8.16 Let \( P_2 \) be the vector space of polynomials of degree \( \leq 2 \), equipped with the inner product

\[
\langle p(t), q(t) \rangle = \int_{-1}^{1} p(t)q(t)dt
\]

Let \( T : P_2 \to P_2 \) be the transformation which sends the polynomial \( p(t) \) to the polynomial

\[
(1 - t^2)p''(t) - 2tp'(t) + 6p(t)
\]

a) Show that \( T \) is linear.
b) Verify that \( T(1) = 6 \) and \( T(t) = 4t \). Find \( T(t^2) \).
c) Find the matrix \( A \) of \( T \) with respect to the standard basis \( \epsilon = (1, t, t^2) \) for \( P_2 \).
d) Find the basis for \( Nul(A) \) and \( Col(A) \).
e) Use the Gram-Schmidt process to find an orthogonal basis \( B \) for \( P_2 \) starting form \( \epsilon \).
2.8.17 Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation that rotates every vector (starting at the origin) by \( \theta \) degrees in the counterclockwise direction. Consider the following two bases for \( \mathbb{R}^2 \):

\[
B = \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),
\]

and

\[
C = \left( \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}, \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} \right);
\]

a) Find the matrix \([T]_B\) of \( T \) in the standard basis \( B \).

b) Find the matrix \([T]_C\) of \( T \) in the basis \( C \). Does \([T]_C\) depend on the angle \( \alpha \)?

2.8.18 Consider the vector space \( V \) of 2 matrices. Define a transformation \( T : V \to V \) by \( T(A) = A^T \), where \( A \) is an element of \( V \) (that is, it is a 2 \times 2 matrix), and \( A^T \) is the transpose of \( A \).

a) Show that \( T \) is linear transformation.

b) Find an eigenvalue of \( T \) (You need only find one, not all of them). \( \text{(Hint: Search for matrices } A \text{ such that } T(A) \text{ is a scalar multiple of } A) \)

c) Find an eigenvector for the particular eigenvalue that you found in part (b).

d) Let \( W \) be the complete eigenspace of \( T \) with the eigenvalue from part (b) above. Find a basis for \( W \). What is the dimension of \( W \)?

2.8.19 Let \( T : P^2 \to P^3 \) be the transformation that maps the second order polynomial \( p(t) \) into \((1 + 2t)p(t)\),

a) Calculate \( T(1), T(t), \text{ and } T(t^2) \).

b) Show that \( T \) is a linear transformation.

c) Write the components of \( T(1), T(t), T(t^2) \) with respect to the basis \( C = \{1, t, t^2, 1 + t^3\} \).

d) Find the matrix of \( T \) relative to the bases \( B = \{1, t, t^2\} \) and \( C = \{1, t, t^2, 1 + t^3\} \).
2.8. LINEAR TRANSFORMATION II

MATH 294 FALL 1998 PRELIM 3 # 1

2.8.20 Consider the following three vectors in \( \mathbb{R}^3 \):

\[
\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]

[Note: \( \vec{u}_1 \) and \( \vec{u}_2 \) are orthogonal.]

a) Find the orthogonal projection of \( \vec{y} \) onto the subspace of \( \mathbb{R}^3 \) spanned by \( \vec{u}_1 \) and \( \vec{u}_2 \).

b) What is the distance between \( \vec{y} \) and \( \text{span}\{\vec{u}_1, \vec{u}_2\} \)?

c) In terms of the standard basis for \( \mathbb{R}^3 \), find the matrix of the linear transformation that orthogonally projects vectors onto \( \text{span}\{\vec{u}_1, \vec{u}_2\} \).

MATH 294 FALL 1998 FINAL # 4

2.8.21 Here we consider the vector spaces \( P_1, P_2, \) and \( P_3 \) (the spaces of polynomials of degree 1, 2, and 3).

a) Which of the following transformations are linear? (Justify your answer.)

i) \( T: P_1 \to P_3, T(p) \equiv t^2p(t) + p(0) \)

ii) \( T: P_1 \to P_1, T(p) \equiv p(t) + t \)

b) Consider the linear transformation \( T: P_2 \to P_2 \) defined by \( T(a_0 + a_1t + a_2t^2) \equiv (-a_1 + a_2) + (a_0 + a_1)t + (a_2)t^2 \), with respect to the standard basis of \( P_2, \beta = \{1, t, t^2\} \), is \( A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Note that an eigenvalue/eigenvector pair of \( A \) is \( \lambda = 1, v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \). Find an eigenvalue/eigenvector (or eigenfunction) pair of \( T \). That is, find \( \lambda \) and \( g(t) \) in \( P_2 \) such that \( T(g(t)) = \lambda g(t) \).

c) Is the set of vectors in \( P_2 \{3 + t, -2 + t, 1 + t^2\} \) a basis of \( P_2 \)? (Justify your answer.)

MATH 293 SPRING 1997 PRELIM 2 # 4

2.8.22 Let \( M \) be the transformation from \( P^n \) to \( P^n \) such that

\[ Mp(t) = \frac{1}{2} [p(t) + p(-t)] (t \text{ real}) \]

a) If \( n = 3 \) find the matrix of this transformation with respect to the basis \( \{1, t, t^2, t^3\} \).

b) Let \( N = I - M \). What is \( Np(t) \) in terms of \( p(t) \)? Show that \( M^2 = MM = M, MN = MN = 0 \)

MATH 294 FALL 1987 PRELIM 2 # 3 MAKE-UP

2.8.23 a) If \( A \) is an \( n \times n \) matrix with \( \text{rank}(A) = r \), then what is the dimension of the vector space of all solutions of the system of linear equations \( A\vec{x} = \vec{0} \)?

b) What is the dimension of the kernel of the linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) which has \( A \) for its matrix in the standard basis.
2.8.24 Show that if $T : V \to W$ is a linear transformation from $V$ to $W$, and $\ker(T) = \vec{0}$, then $T$ is 1-1. (Recall: $\ker(T) = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}$.)

2.8.25 Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation.

a) If $T \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $T \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, what is $T \begin{pmatrix} -9 \\ 26 \end{pmatrix}$?

b) What are $T \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T \begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

c) What is the matrix of $T$ in the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$, and the standard basis for $\mathbb{R}^4$?

2.8.26 a) Find a basis for $\ker(L)$, where $L$ is linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by

$$L(\vec{x}) = \begin{pmatrix} 1 & 2 & -4 & 3 \\ 1 & 2 & -2 & 2 \\ 2 & 4 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

c) What is the dimension of $\ker(L)$?

d) Is the vector $\vec{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ in $\text{range}(L)$? (Justify your answer.) If so, find all vectors $\vec{x}$ in $\mathbb{R}^4$ which satisfy $L(\vec{x}) = \vec{y}$.

2.8.27 Let $P$ be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

a) Find a basis for $\ker(P)$.

b) Find a basis for $\text{range}(P)$.

c) Find all vectors $\vec{x}$ in $\mathbb{R}^3$ such that $P\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

d) Find all vectors $\vec{x}$ in $\mathbb{R}^3$ such that $P\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
2.8. LINEAR TRANSFORMATION II

MATH 293 SPRING 1995 PRELIM 3 # 4
2.8.28 Let $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which represent orthogonal projection onto the line $\ell_\theta$ forming angle $\theta$ with the x-axis.

a) Find the matrix $T$ of $L_\theta$ (with respect to the standard basis of $\mathbb{R}^2$).

b) Is $L_\theta$ invertible. Explain your answer geometrically.

c) Find all the eigenvalues of $T$.

MATH 294 FALL 1998 PRELIM 2 # 1
2.8.29 The unit square $OBCD$ below gets mapped to the parallelogram $OB'C'D'$ (on the $x_1 - x_3$ plane) by the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ shown.

Problems (b) - (e) below can be answered with or without use of the matrix $A$ from part (a).

a) Is this transformation one-to-one? For this and all other short answer questions on this test, some explanation is needed.)

b) What is the null space of $A$?

c) What is the column space of $A$?

d) Is $A$ invertible? (No need to find the inverse if it exists.)
Consider the homogeneous system of equations $B \vec{x} = \vec{0}$, where

$$
B = \begin{bmatrix}
0 & 1 & 0 & -3 & 1 \\
2 & -1 & 0 & 3 & 0 \\
2 & -3 & 0 & 0 & 4
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}, \quad \text{and} \quad \vec{0} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

a) Find a basis for the subspace $W \subset \mathbb{R}^5$, where $W = \text{set of all solutions of } B \vec{x} = \vec{0}$.

b) Is $B$ 1-1 (as a transformation of $\mathbb{R}^5 \rightarrow \mathbb{R}^3$)? Why?

c) Is $B : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ onto? Why?

d) Is the set of all solutions of $B \vec{x} = \begin{bmatrix}
3 \\
0 \\
0
\end{bmatrix}$ a subspace of $\mathbb{R}^5$? Why?