2.7 Eigen-stuff

MATH 294 FALL 1985 FINAL # 3

2.7.1 Find an angle θ, expressed as a function of a,b, and c so that the matrix product

$$\begin{pmatrix} \cos\theta & sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a diagonal matrix. In particular, what is θ if $a = c$, and what is the resulting digital matrix? (Hint: $\cos^2\theta - \sin^2\theta = \cos 2\theta$; $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$)

MATH 294 FALL 1985 FINAL # 5

2.7.2 Find all of the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & 3 \\ -1 & -2 & 0 & 4 \\ -2 & -3 & -4 & 0 \end{pmatrix}$$

Show why your answers are correct.

MATH 294 SPRING 1985 FINAL # 9

2.7.3 In general, the eigenvalues of A are (A is a real 2×2 matrix)

a) Always real.
b) Always imaginary.
c) Complex conjugates.
d) Either purely real or purely imaginary.

MATH 294 SPRING 1985 FINAL # 10

2.7.4 If A has purely real eigenvalues, then (A is a real 2×2 matrix)

a) The eigenvalues must be distinct.
b) The eigenvalues must be repeated.
c) The eigenvalues may be distinct or repeated.
d) The eigenvalues must both be zero.

MATH 294 SPRING 1985 FINAL # 11

2.7.5 If A has purely imaginary eigenvalues, then (A is a real 2×2 matrix)

a) The eigenvalues must have the same magnitude but opposite sign.
b) The eigenvalues must be repeated.
c) The eigenvalues may or may not be repeated.
d) The eigenvalues must both be zero.

MATH 294 FALL 1986 FINAL # 3

2.7.6 **a**) Find all eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}.$$

b) For any square matrix A, show that if $\det(A) \neq 0$, then zero cannot be an eigenvalue of A.

MATH 294 SPRING 1983 FINAL # 10

2.7.7 A is the matrix given below, $\vec{v}$ is an eigenvector of A. Find any eigenvalue of A.

$$A = \begin{bmatrix} 3 & 0 & 4 & 2 \\ 8 & 5 & 1 & 3 \\ 4 & 0 & 9 & 8 \\ 2 & 0 & 1 & 6 \end{bmatrix} \text{ with } \vec{v} = [\text{an eigenvector of } A] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

MATH 294 SPRING 1984 FINAL # 5

2.7.8 Let λ_1 and λ_2 be distinct eigenvalues of a matrix A and let x_1 and x_2 be the associated eigenvectors. Show that x_1 and x_2 are linearly independent.

MATH 294 FALL 1984 FINAL # 2

2.7.9 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

.

MATH 294 FALL 1984 FINAL # 5

2.7.10 Does the matrix with the zero row vector deleted have $\lambda = 0$ as an eigenvalue?

MATH 294 FALL 1986 FINAL # 4

2.7.11 **a**) Find an orthogonal matrix R such that $R^T AR$ is diagonal,where

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

.

b) Write the matrix $D = R^T AR$.

c) What are the eigenvectors and associated eigenvalues of A.

MATH 294 SPRING 1987 PRELIM 2 # 4

2.7.12 Problems (a) and (b) below concern the matrix A:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{bmatrix}$$

a) One of the eigenvalues of A is 1, what are the other(s)?
b) Find an eigenvector of A.

MATH 294 SPRING 1987 PRELIM 3 # 4

2.7.13 Find *one* eigenvalue of the matrix A below. Three eigenvectors of the matrix are given.

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 1 \\ 1 & 5 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ -1 & 3 & -1 & 5 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

The following three vectors are eigenvectors of A:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

MATH 294 SPRING 1987 FINAL # 10

2.7.14 Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ find R so that $RAR^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

MATH 294 FALL 1987 PRELIM 2 # 1

2.7.15 Find the eigenvalue and eigenvectors of the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

MATH 294 FALL 1987 PRELIM 2 # 2

2.7.16 Find the eigenvalues, eigenvectors and/or generalized eigenvectors of the matrix $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$.

MATH 294 FALL 1989 PRELIM 2 # 1

2.7.17 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

a) Show that $\lambda = 0$ is an eigenvalue of A.
b) Find a corresponding eigenvector.
c) Determine whether the system of equations

$$A\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

has a solution or not.

MATH 294 FALL 1989 PRELIM 3 # 1

2.7.18 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Show that $\lambda = 0$ is a double eigenvalue, and that $\lambda = 1$ is a simple eigenvalue. For b) and c) below, you may use the result of a) even if you did not show it.]
b) Find all linearly independent eigenvectors corresponding to the eigenvalues $\lambda = 0$ and $\lambda = 1$ respectively.
c) Find two linearly independent generalized eigenvectors corresponding to the double eigenvalue $\lambda = 0$.

MATH 293 SPRING 1990 PRELIM 3 # 1

2.7.19 **a**) Find the eigenvalues, eigenvectors and dimension of the subspace of eigenvectors corresponding to each eigenvalue of

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$$

b) For what value of c (if any) is $\lambda = 2$ an eigenvalue of

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & c & 1 \\ -1 & -1 & 1 \end{bmatrix}?$$

In that case find a basis for the subspace of eigenvectors corresponding to $\lambda = 2$.

MATH 294 SPRING 1990 PRELIM 3 # 2

2.7.20 **a**) There is a 2×2 matrix R such that $R^t AR$ is a diagonal matrix, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find $R^t AR$. (Hint: You needn't find R; there are two correct answers)

b) Describe the conic $v^t Av = 1$ for v in V_2 and $A = \begin{bmatrix} 4 & -1 \\ -1 & -2 \end{bmatrix}$. Explain why your answer is correct.

MATH 293 FALL 1991 FINAL # 3

2.7.21 Diagonalize the one of the following matrices which can be diagonalized:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 6 & -10 & 6 \\ 2 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

MATH 293 FALL 1991 PRELIM 3 # 5

2.7.22 Find the eigenvalues and eigenvectors of the matrix A where

$$A = \begin{pmatrix} 3 & 4 & 2 \\ -2 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Hint: $\lambda = 1$ is one eigenvalue of A.

MATH 293 FALL 1991 PRELIM 3 # 6

2.7.23 An $n \times n$ matrix always has n eigenvalues (some possibly complex), but these are not always distinct. (T/F)

MATH 293 FALL 1991 FINAL # 4

2.7.24 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

MATH 293 SPRING 1992 PRELIM 3 # 3

2.7.25 Find the eigenvalues and three linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

MATH 294 SPRING 1992 FINAL # 3

2.7.26 Consider the eigenvalue problem: Find all real numbers λ (eigenvalues) such that the differential equation $-\frac{\partial^2 w}{\partial x^2} = \lambda w, 0 < x < L$ with the boundary conditions $\frac{\partial w}{\partial x}(0) = \frac{\partial w}{\partial x}(L) = 0$ has nontrivial solutions (eigenfunctions). Given that there are no eigenvalues $\lambda < 0$, find all possible eigenvalues $\lambda \geq 0$ and corresponding eigenfunctions. You must derive your result. No credit will be given for simply writing down the answer.

MATH 294 SPRING 1992 FINAL # 3

2.7.27 A vector space V has two bases

$$B_1 : \{e^t, e^{2t}, e^{3t}\} \text{and} B_2 : e^t + e^{2t}, e^{3t}, e^{2t}$$

A linear operator $T : V \to V$ is $T = \frac{\partial}{\partial t}$

a) Find the matrix T_{B_1} which represents T in the basis B_1.

b) For the vectors $v = e^{2t}, w = \frac{\partial v}{\partial t}$, find $\beta_1(v)$ and $\beta_1(w)$ which represent these vectors in the basis B_1.

c) Noting that $T(v) = 2v$, i.e. v is an eigenvector of T with eigenvalue equal to 2, interpret the equation

$$\beta_1(w) = T_{B_1}\beta_1(v)$$

as an eigenvalue-eigenvector equation for T_{B_1}. What are the eigenvalue and eigenvector in this equation?

d) Now consider the basis B_2. Find the matrices $(B_2 : B_1)$ and $(B_1 : B_2)$.

e) Find $\beta_2 v, T_{B_2}$ and $\beta_2(w)$.

f) Is the equation $\beta_2(w) = T_{B_2}\beta_2(v)$ also an eigenvalue-eigenvector equation? If so, what are the eigenvalue and eigenvector in this case?

MATH 293 FALL 1992 FINAL # 4

2.7.28 **a)** Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

b) Let $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$. Find a nonsingular matrix C such that $C^{-1}AC = D$ where D is a diagonal matrix. Find C^{-1} and D.

MATH 294 FALL 1992 FINAL # 6

2.7.29 Find the eigenvalues and eigenfunctions (nontrivial solutions) of the two-point boundary-value problem

$$y'' + \lambda y = 0, 0 < x < 1, (\text{assume} \lambda \geq 0)$$

$$y'(0) = y(1) = 0.$$

MATH 293 SPRING 1993 FINAL # 4

2.7.30 It is known that a 3×3 matrix A has: 1) A twice-repeated eigenvalue $\lambda_1 = 1$ with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and 2) another eigenvalue $\lambda_2 = 0$ with corresponding eigenvectors $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

a) Find a basis for the null space $Nul(B)$ of B, where B is the matrix $B = (A - I_3$, and I_3 is the 3×3 identity matrix.

b) Find a basis for the null space $Nul(A)$ of A.

c) For part a) above, $Null(B)$ is a plane in 3-dimensional space. The equation of this plane can be written in the form $Ax + By + Cz = 0$. Find A, B, and C.

d) For part b above, is $Null(A)$ a line, a plane, or something else? Please explain your answer carefully.

MATH 294 FALL 1994 FINAL # 5

2.7.31 Let $A = \begin{bmatrix} -3 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A.

b) Find a basis for the eigenspace associated with each eigenvalue. The eigenspace corresponding to an eigenvalue is the set of all eigenvectors associated with the eigenvalue, plus the zero vector.

c) Find an orthogonal matrix P and a diagonal matrix D so that $P^{-1}AP = D$. What is D?

MATH 293 FALL 1994 FINAL # 7

2.7.32 If an $n \times n$ A has n distinct eigenvalues, then

a) $\det(A)$ not zero,

b) $\det(A)$ is zero,

c) A is similar to I_n,

d) A has n linearly independent eigenvectors,

e) $A = A^T$

MATH 294 FALL 1994 FINAL # 10

2.7.33 Which of the following is an eigenvector of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$?

a. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, b. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, c. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, d. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, e. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

MATH 294 SPRING 1995 FINAL # 6

2.7.34 Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

a) Find all eigenvalues of A, and for each an eigenvector.
b) Find a matrix P such that $D = P^{-1}AP$ is diagonal.
c) Find D.

MATH 293 SPRING 1995 FINAL # 7

2.7.35 **a)** Find all eigenvalues of the matrix

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

b) For each eigenvalue find a corresponding eigenvector.
c) Are the eigenvectors orthogonal?

MATH 293 FALL 1995 PRELIM 3 # 5

2.7.36 **a)** One eigenvalue of $A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \end{pmatrix}$ is 2. Find a corresponding eigenvector.

b) Find the characteristic polynomial $\det(A - \lambda I)$ if $A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}$. Also find all the eigenvalues of A.

MATH 293 FALL 1995 FINAL # 6

2.7.37 For $A = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

a) Show that the characteristic polynomial of A is

$$-\lambda(\lambda^2 - \lambda - 6)$$

b) Find all eigenvalues of A of 3 linearly independent eigenvectors.
c) Check your solution of part (b).

MATH 293 FALL 1995 FINAL # 8

2.7.38 For $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$

a) Find the eigenvalues and 2 linearly independent eigenvectors. Show that these 2 eigenvectors are orthogonal.
b) Find a matrix P and a diagonal matrix D so that $P^{-1}AP = D$
c) If your matrix P in part (b) is not orthogonal, how can it be modified to make it orthogonal? (so that $P^{-1}AP = D$ still holds).

MATH 293 SPRING 1996 PRELIM 3 # 5
2.7.39 Let

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}.$$

a) Find the characteristic polynomial of B.
b) Find the eigenvalues of B. Hint: one eigenvalue is 2.
c) Find eigenvectors corresponding to the eigenvalues other than 2.

MATH 293 SPRING 1996 PRELIM 3 # 6
2.7.40 Let C be a 2-by-2 matrix. Suppose

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are eigenvectors for C with eigenvalues 1 and 0, respectively. Let

$$\vec{x} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

Find $C^{100}\vec{x}$.

MATH 293 SPRING 1996 FINAL # 14
2.7.41 Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \Re$. A complex eigenvector of A is:

a) $\begin{bmatrix} -2 \\ i \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ -i \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ i \end{bmatrix}$ d) $\begin{bmatrix} -i \\ 2 \end{bmatrix}$ e) none of the above

MATH 294 # 5
2.7.42

MATH 293 SPRING 1996 FINAL # 15
2.7.43 Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then
a) A has three linearly independent eigenvectors with eigenvalue 2.
b) The eigenspace corresponding to the eigenvalue 2 has a basis consisting of exactly one eigenvector.
c) The eigenspace corresponding to the eigenvalue 2 has dimension 2.
d) A is not diagonalizable.
e) None of the above.

MATH 293 SPRING 1996 FINAL # 38
2.7.44 The only matrix with 1 as an eigenvalue is the identity matrix. (T/F)

MATH 293 SPRING 1996 FINAL # 39
2.7.45 If A is an $n \times n$ matrix for which $A = PDP^{-1}$, D diagonal, then A cannot have n linearly independent eigenvectors. (T/F)

MATH 293 SPRING 1996 FINAL # 40
2.7.46 If x is an eigenvector of a matrix A corresponding to the eigenvalue λ, then $A^3\vec{x} = \lambda^3\vec{x}$. (T/F)

MATH 293 SPRING 1997 PRELIM 2 # 2
2.7.47 Let $A = \begin{bmatrix} 9 & 0 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

a) Find the characteristic polynomial $\det(A - \lambda I)$ of A, and find all the eigenvalues. (*hint*: $\lambda - 9$ is one factor of the polynomial.

b) Find an eigenvector for each eigenvalue.

MATH 294 SPRING 1997 FINAL # 5
2.7.48 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

a) Find the characteristic polynomial of A. Verify that the eigenvalues of A are: 0, 1, 2

b) For each eigenvalue, find a basis for the corresponding eigenspace.

c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

d) Show that the columns of P form an orthogonal basis for $\Re^3$.

e) Find $A^{10}\vec{x}$ where $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

MATH 294 FALL 1997 PRELIM 2 # 4
2.7.49 The following information is known about a 3×3 matrix A. (Here e_1, e_2, e_3 is the standard basis for $\Re^3$).

i) $Ae_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$,

ii) $e_1 + e_2$ is an eigenvector of A with corresponding eigenvalue 1.

iii) $(e_2 + e_3)$ is an eigenvector of A with corresponding eigenvalue 2.

Find the matrix A.

MATH 294 FALL 1997 PRELIM 3 # 3

2.7.50 Given an $n \times n$ matrix A with n linearly independent eigenvectors, it is possible to find a square root of A (that is, an $n \times n$ matrix $\sqrt{A}$) with $\left(\sqrt{A}^2 = A\right)$) by using the following method:
(1) Find $D = P^{-1}AP$, where D is a diagonal matrix (and P is some suitable matrix).
(2) Find $\sqrt{D}$ by taking the square roots of the entries on the diagonal. (This might involve complex numbers).
(3) The square root of A is then $\sqrt{A} = P\sqrt{D}P^{-1}$.
Use this method to find $sqrtA$ for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

MATH 294 FALL 1997 FINAL # 3

2.7.51 Let

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of A. Is A diagonalizable?

MATH 294 FALL 1997 PRELIM 3 # 4

2.7.52 **a**) Let

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For which values of θ is this matrix diagonalizable?

b) Let A be 2×2 matrix with characteristic polynomial $(\lambda - 1)^2$. Suppose that A is diagonalizable. Find a matrix A with these properties. Now, find *all* possible matrices A with these properties. Justify your answer!

MATH 294 FALL 1997 PRELIM 3 # 5
2.7.53 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

a) Find the characteristic polynomial of A. Verify that the eigenvalues of A are: 0, 1, 2
b) For each eigenvalue, find a basis for the corresponding eigenspace.
c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$
d) Show that the columns of P form an orthogonal basis for $\Re^3$.
e) Find $A^{10}\vec{x}$ where $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

MATH 294 SPRING 1998 PRELIM 1 # 2
2.7.54 Find the values of λ (eigenvalues) for which the problem below has a non-trivial solution. Also determine the corresponding non-trivial solutions (eigenfunctions.)

$$y'' + \lambda y = 0 \text{ for } 0 < x < 1$$

$$y(0) = 0, y'(1) = 0.$$

(Hint: λ must be positive for non-trivial solutions to exists. You may assume this.)

MATH 293 SPRING 1998 PRELIM 2 # 5
2.7.55 Consider the matrix A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Find *all* the eigenvalues of A and find a corresponding eigenvector for *each* eigenvalue. (Hint: 1 is an eigenvalue.)

MATH 294 SPRING 1998 PRELIM 3 # 4
2.7.56 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

a) Find the eigenvalues and eigenvectors of A.
b) Diagonalize A. That is, give P and D such $A = PDP^{-1}$.
c) Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be the standard basis vectors of $\Re^2$. Map $\vec{e}_1 \to P\vec{e}_1$ and $\vec{e}_2 \to P\vec{e}_2$ and sketch $P\vec{e}_1$ and $P\vec{e}_2$.
d) Give a geometric interpretation of $\vec{x} \to P\vec{x}$.

MATH 294 SPRING 1998 PRELIM 3 # 5

2.7.57 True or false? Justify each answer.

a) In general, if a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis of V.

b) A linearly independent set in a subspace H is a basis for H.

c) An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d) If an $n \times n$ matrix A is diagonalizable, it is invertible.

MATH 294 FALL 1998 PRELIM 3 # 3

2.7.58 **a)**

b) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 10 & -4 & 6 & 5 \\ -4 & 8 & 4 & -6 \\ 6 & 4 & 10 & -1 \\ 5 & -6 & -1 & 5 \end{bmatrix}$. Find an eigenvalue of A.

c) Consider the 20×20 matrix that is all zeros but for the main diagonal. The main diagonal has the numbers 1 to 20 in order. Precisely describe as many eigenvalues and eigenvectors of this matrix as you can.

d) If possible, diagonalize the matrix $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. Explicitly evaluate any relevant matrices (if any inverses are needed the can be left in the form $[\ ]^{-1}$).

MATH 294 FALL 1998 FINAL # 6

2.7.59 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

a) Find orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ of A. [Hint: do not go on to parts d-e below until you have double checked that you have found two orthogonal unit vectors that are eigenvectors of A.]

b) Use the eigenvectors above to diagonalize A.

c) Make a clear sketch that shows the standard basis vectors $\{\vec{e}_1, \vec{e}_2\}$ of $\Re^2$ and the eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ of A.

d) Give a geometric interpretation of the change of coordinates matrix, P, that maps coordinates of a vector with respect to the eigen basis to coordinates with respect to the standard basis.

e) Let $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Using orthogonal projection express $\vec{b}$ in terms of $\{\vec{v}_1, \vec{v}_2\}$ the eigenvectors of A.

MATH 293 SPRING 1993 FINAL # 6

2.7.60 **a)** Write a matrix $A = S^{-1}AS$ such that A is diagonal, if

$$A = \begin{pmatrix} 6 & -10 & 6 \\ 2 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

b) What is matrix S?

MATH 293 SPRING 1993 FINAL # 6

2.7.61 **a)** Write a matrix $\Lambda = S^{-1}AS$ such that Λ is diagonal, if

$$A = \begin{pmatrix} 6 & -10 & 6 \\ 2 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

b) What is the matrix S?

MATH 293 SPRING 1998 PRELIM 2 # 5

2.7.62 Consider the matrix A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Find all the eigenvalues of A and find a corresponding eigenvector for each eigenvalue. (Hint: 1 is an eigenvalue.)

MATH 294 SPRING 1999 PRELIM 2 # 2b

2.7.63 The matrix $A = \begin{bmatrix} 1 & 3 & 4 & 2.718 \\ 3 & 5 & \pi & 8 \\ \sqrt{5} & 3 & 4 & 1 \\ 3 & 4 & 6 & 7 \end{bmatrix}$ has 4 distinct eigenvalues (no eigenvalue is equal to any of the others. The eigenvectors of A make up the 4 columns of a matrix P. Does the equation $P\vec{x} = \begin{bmatrix} 2.718 \\ \pi \\ \sqrt{5} \\ 4 \end{bmatrix}$ have

a) no solution (why?), or
b) exactly one solution (why?), or
c) exactly two solutions (why?), or
d) an infinite number of solutions (why?), or
e) it depends on information not given (what information?, how would that information tell you the answer?)?

MATH 294 SPRING 1999 PRELIM 2 # 2c

2.7.64 The matrix $A = \begin{bmatrix} 66 & -52 & 8 & -4 \\ -52 & 83 & -26 & -24 \\ 8 & -26 & 54 & -52 \\ -4 & -24 & -52 & 22 \end{bmatrix}$ has four eigenvalues $\lambda_1 = -30, \lambda_2 = 30, \lambda_3 = 90, \lambda_4 = 135$. Some four vectors $\vec{v}_i$ satisfy $A\vec{v}_i = \lambda_i \vec{v}_i$ (for i = 1,2,3,4). This is all you are told about the vectors $\vec{v}_i$. The vectors $\vec{v}_i$ make up, in the order given, the columns of a matrix P. If this is sufficient to answer each of the questions below, then answer the questions, if not explain why you need more information. No credit for unjustified correct answers. [Hint: massive amounts of arithmetic are not needed for any of the three parts].

a) What is the element in the third row and second column of $P^T P$?

b) What is the element in the third row and third column of $P^{-1}AP$?

c) What is the element in the second row and second column of $P^T P$?

MATH 294 SPRING 1999 PRELIM 2 # 3

2.7.65 A couch potato spends *all* of his/her time either smoking a cigarette ('C') *or* eating a bag of fries ('F') *or* watching a TV show ('T'). Since there is no smoking inside and no food allowed in the living room, he/she only does one thing at a time.

- After a cigarette there is a 50% chance he/she will light up again, but right after smoking he/she never eats, (If you think this is ambiguous please reread the initial statement,)
- After eating a bag of fries he/she has a 50% chance of going out for a smoke, a 25% chance of eating another bag of fries, and a 25% chance of turning the TV on.
- After watching TV show he/she only wants to eat.

On average he/she watches 300 TV shows a month. **On average, how many cigarettes does he/she smoke in a month?**

MATH 293 UNKNOWN FINAL # 8

2.7.66 **a**) Let A be a nonsingular $n \times n$ matrix, $X, B n \times n$ matrices. Solve the equation $\left[A \times A^T\right]^T - B^T$ for X and show that X is symmetric.

b) Let $\vec{u} = [u_1, \ldots, u_n]^T$ and $C = I - \vec{u}\vec{u}^T$. Express the entries c_{ij} of the matrix C in terms of the u_i. Show that C has zero as an eigenvalue provided that $||\vec{u}|| = 1$. Determine the corresponding <u>unit</u> eigenvector. (Hint: Do not attempt to evaluate the characteristic polynomial of C. Use instead the definition of eigenvalue and eigenvector.)

MATH 293 SPRING ? FINAL # 4

2.7.67 **a**) Determine the real numbers a,b,c,d,e,f, given that $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ d & e & f \end{pmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

b) What are the corresponding eigenvalues?

MATH 294 FALL 1987 PRELIM 3 # 1 PRELIM

2.7.68 **a**) Find the general solution of the system $\vec{x}' = A\vec{x}$ if

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

b) How many independent eigenvectors can we find for the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}?$$

MATH 293 SUMMER 1992 FINAL # 2

2.7.69 Given

$$A = \frac{1}{2}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

a) Find all the eigenvalues of A.

b) Find all linearly independent eigenvectors of A.

c) Can A be diagonalized by a change of basis? If so, let $D = (B:S)^{-1}A(B:S)$ where D is diagonal.

Find $(B:S)$ and D.