2.5 Spaces of a Matrix and Dimension

MATH 294 SPRING 1982 PRELIM 1 # 3

2.5.1 **a)** Let $C[0,1]$ denote the space of continuous function defined on the interval $[0,1]$ (i.e. $f(x)$ is a member of $C[0,1]$ if $f(x)$ is continuous for $0 \le x \le 1$). Which one of the following subsets of $C[0,1]$ does **not** form a vector space? Find it and explain why it does not.

MATH 294 SPRING 1982 PRELIM 1 # 3

2.5.2 **a)**

i) The subset of functions f which belongs to $C[0,1]$ for which $\int_0^1 f(s)ds = 0$.

ii) The set of functions f in $C[0,1]$ which vanish at exactly one point (i.e. $f(x) = 0$ for only one x with $0 \le x \le 1$).
Note different functions may vanish at different points within the interval.

iii) The subset of functions f in $C[0,1]$ for which $f(0) = f(1)$

b) Let $f(x) = x^3 + 2x + 5$. Consider the four vector $v_1 = f(x), v_2 = f'(x), v_3 = f'', v_4 = f'''(x), (f'(x)$ means $\frac{df}{dx})$

i) What is the dimension of the space spanned by the vectors? Justify your answer.

ii) Express $x^2 + 1$ as a linear combination of the v_i's

MATH 294 SPRING 1983 PRELIM 1 # 2

2.5.3 Consider the system

$$\left.\begin{array}{ccccccccc} x & + & y & - & z & + & w & = & 0 \\ x & & & + & 3z & + & w & = & 0 \\ 2x & + & y & + & 2z & + & 2w & = & 0 \\ 3x & + & 2y & + & z & + & 3w & = & 0 \end{array}\right\}$$

a) Find a basis for the vector space of solutions to the system above. You need not prove this is a basis

b) What is the dimension of the vector space of solutions above? Give a reason.

c) Is the vector

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

a solution to the above system?

MATH 294 SPRING 1987 FINAL # 2

2.5.4 Determine whether the given vectors form a basis for S, and find the dimension of the subspace. S is the set of all vectors of the form $(a, b, 2a, 3b$ in R^4. The given set is $\{(1,0,2,0), (0,1,0,3), (1,-1,2,-3)\}$

MATH 294 FALL 1986 FINAL # 1

2.5.5 The vectors $(1,0,2,-1,3),(0,1,-1,2,4),(-1,1,-2,1,-3),(0,1,1,-2,-4)$, and $(1,4,2,-1,3)$ span a subspace S of R^5.

a) What is the dimension of S?

b) Find a basis for$S$

MATH 294 FALL 1986 FINAL # 1

2.5.6 Compute the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

MATH 294 FALL 1987 PRELIM 3 # 3

2.5.7 Find the dimension of the subspace of R^6 consisting of all linear combinations of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

MATH 293 SPRING 1990 PRELIM 1 # 3

2.5.8 Find the dimension and a basis for the following spaces

a) The space spanned by $\{(1,0,-2,1),(0,3,1,-1),(2,3,-3,1),(3,0,-6,-1)\}$

b) The set of all polynomials $p(t)$ in P^3 satisfying the two conditions

i) $fracd^3dt^3p(t) = 0$ for all t

ii) $p(t) + fracdp(t)dt = 0$ at $t = 0$

c) The subspace of the space of functions of t spanned by $\left\{e^{at}, e^{bt}\right\}$ if $a \neq b$

d) The space spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ in W, given that $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for W.

MATH 293 SPRING 1990 PRELIM 2 # 1

2.5.9 Let $A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ -1 & -2 & -5 & 4 \\ 0 & 1 & 1 & -1 \\ 1 & 4 & 6 & -2 \end{bmatrix}$ Find a basis for the **column space** of A

MATH 293 SPRING 1990 PRELIM 2 # 2

2.5.10 Consider the equation

$$Ax = b \ ; \ A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ -1 & -2 & -5 & 4 \\ 0 & 1 & 1 & -1 \\ 1 & 4 & 6 & -2 \end{bmatrix}$$

a) Solve for x given $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}$

b) Find a basis for the null space of A

c) **Without carrying out explicit calculation**, does a solution exist for any b in V^4? (No credit will be given for explicit calculation for).

MATH 293 SPRING 1990 PRELIM 2 # 3

2.5.11 **a**) Find a basis and the dimension of the column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 2 & 6 & 18 \\ -1 & 1 & 1 \\ 4 & 12 & 36 \end{bmatrix}$$

b) Find a basis for the null space of the above matrix.

MATH 293 SPRING 1990 PRELIM 2 # 4

2.5.12 Which of the following sets form a vector subspace of V_4? Explain.

a) the set of vectors of the form $(x, y, x + y, 0)$

b) the set of vectors of the form $(x, 2x, 3x, 4x)$

c) the set of vectors (x, y, z, w) such that $x + y + w = 1$

d) If the set in (b) is a subspace, find a basis for it and its dimension. In the above, x, y, z, w are any real numbers.

MATH 293 FALL 1991 PRELIM 3 # 6

2.5.13 True-False

True means always true. False means not always true.

a) The column space of a matrix is preserved under row operations

b) The column rank of a matrix is preserved under row operations.

c) For an $n \times n$ matrix, with $m \neq n$, rank plus nullity equals n.

d) The row space of a matrix A is the same vector space as the row space of the row reduced form of A.

e) If two matrices A and B have the same row space, the $A = B$.

MATH 293 FALL 1991 FINAL # 7

2.5.14 Show that the matrices A and B have the same row space:

$$A = \begin{pmatrix} 3 & 1 & 9 \\ 2 & 1 & 7 \\ 1 & 1 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 & 3 \\ 1 & -1 & -1 \\ 2 & -3 & -5 \end{pmatrix}$$

MATH 293 FALL 1991 FINAL # 7

2.5.15 Find the vector in the subspace spanned by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

which is closest to the vector

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

MATH 293 FALL 1991 FINAL # 8

2.5.16 True-False. True means always true, false means not always true. Warning! Matrices are not necessarily square.

a) The rank of A equals the rank of A^T.

b) The nullity of A equals the nullity of A^T.

MATH 293 SUMMER 1992 FINAL # 3

2.5.17 Let V be the vector space of all 2×2 matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where a_{ij},$i, j = 1, 2$, are real scalars.
Consider the set S of all 2×2 matrices of the form

$$\begin{pmatrix} a+b & a-b \\ b & a \end{pmatrix}$$

where a and b are real scalars.

a) Show that S is a subspace. Call it W.

b) Find a basis for W and the dimension of W.

MATH 293 SUMMER 1992 FINAL # 3

2.5.18 Consider the vector space V
$\{f(t) = a + b \sin t + c \cos t\}$, for all real scalars a, b and c and $0 \leq t \leq 1$
Now consider a subspace
W of V in which $\frac{df(t)}{dt} + f(t) = 0$ at $t = 0$
Find a basis for the subspace W.

MATH 293 FALL 1992 PRELIM 3 # 5

2.5.19 Fill in the blanks of the following statements.
In what follows A is an $m \times n$ matrix

a) The dimension of the row space is 2.
The dimension of the null space is 3.
The number of columns of A is ________ .

b) $Ax = b$ has a solution x if and only if b is in the ________ space of A.

c) If $Ax = 0$ and $Ay = 0$ and if C_1 and C_2 are arbitrary constants then $A(C_1x+C_2y)$ $=$ ________ .

MATH 293 FALL 1992 FINAL # 3

2.5.20 **a**) Let A be an $n \times n$ nonsingular matrix. Prove that $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$. Hint: You may use the fact that if A and B are $n \times n$ matrices $\det(AB) = \det(A)\det(B)$.

b) An $n \times n$ matrix A has a nontrivial null space. Find $\det(A)$ and explain your answer.

c) Given two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in V_3. Find a vector (or vectors) $w_1, w_2, ...$ in V_3 such that the set $\{v_1, v_2, w_1, ...\}$ is a basis for V_3.

d) Let S be the set set of all vectors of the form $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i}, \vec{j}$ and $\vec{k}$ are the usual mutually perpendicular unit vectors. Let W be the set of all vectors that are perpendicular to the vector $\vec{v}_1 = \vec{i} + \vec{j} + \vec{k}$. Is W a vector subspace of V_3? Explain your answer.

MATH 293 SPRING 1993 PRELIM 3 # 6

2.5.21 Let A be an $n \times n$ matrix. Suppose the rank of A is r, and that $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r$ are vectors in R^n such that $A\mathbf{u}_1, A\mathbf{u}_2, ..., A\mathbf{u}_r$ is a basis for $R(A)$ (col. space of A). Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-r}$ be a basis for $N(A)$ (null space of A). Then show that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-r}\}$ is a basis for R^n

MATH 293 SPRING 1993 FINAL # 2

2.5.22 **a**) Solve for y, for x near $\frac{\pi}{2}$, if $y' + y \cot x = \cos x$ and $y\left(\frac{\pi}{2}\right) = 0$

b) Find a basis for the null space of the differential operator

$$L = \frac{d^2}{dx^2} - 7\frac{d}{dx} + 12, -\infty < x < -\infty.$$

(Hint: Find as many linearly independent solutions as needed for the equation $L[y(x)] = 0$.)

MATH 293 FALL 1994 PRELIM 3 # 5

2.5.23 Answer each of the following as True or False. **If False, explain, by an example.**

a) Every spanning set of R^3 contains at least three vectors.

b) Every orthogonal set of vectors in R^5 is a basis for R^5.

c) Let A be a 3 by 5 matrix. Nullity A is at most 3.

d) Let W be a subspace of R^4. Every basis of W contains at least 4 vectors.

e) In R^n, $||cX|| = |c|||X||$

f) If A is an $n \times n$ symmetric matrix, then rank $A = n$.

MATH 293 FALL 1994 FINAL # 4

2.5.24 A basis for the null space of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is:

a. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $b.$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $c.$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $d.$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $e.$ $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

MATH 293 FALL 1994 FINAL # 8

2.5.25 If A is an n by n matrix and rank$(A) < n$. Then

a) A is non singular,
b) The columns of A are linearly independent
c) Some eigenvalue of A is zero
d) $AX = 0$ has only the trivial solution
e) $AX = B$ has a solution for every B

MATH 293 SPRING 1995 PRELIM 3 # 1

2.5.26 Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 0 & 4 & -2 \\ -1 & 3 & 1 & 7 \end{bmatrix}.$$

a) Find a basis for the range of A (i.e., the column space of A).
b) Find a basis for the null-space of A (i.e., the kernel of A).
c) Find a basis for the column space of A^T.

MATH 293 SPRING 1995 PRELIM 3 # 3

2.5.27 Let P_3 be the space of polynomials $p(t)$ of degree ≤ 3. Consider the subspace $S \subset P_3$ of polynomials that satisfy

$$p(0) + \left.\frac{dp}{dt}\right|_{t=0} = 0.$$

a) Show that S is a subspace of P_3.
b) Find a basis for S
c) What is the dimension of S.

MATH 293 FALL 1995 PRELIM 3 # 1

2.5.28 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & -1 & -2 \end{bmatrix}.$$

a) Find a basis for the row space of A.
b) Find a basis for the column space of A
c) What is the rank of A?
d) What is the dimension of the null space?

MATH 293 FALL 1995 PRELIM 3 # 3

2.5.29 Let P_3 be the space of polynomials $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ of degree ≤ 3. Consider the subset S of polynomials that satisfy

$$p''(0) + 4p(0) = 0$$

Here $p''(0)$ means, as usual, $\left.\frac{d^2p}{dt^2}\right|_{t=0}$.

a) Show that S is a subspace of P_3. Give reasons.
b) Find a basis for S.
c) What is the dimension of S? Give reasons for your answer.
Hint: What constrain, if any, does the given formula impose on the constants a_0, a_1, a_2, and a_3 of a general $p(t)$?

MATH 293 FALL 1995 FINAL # 2

2.5.30 Consider the subspace W of R^4 which is defined as

$$W = span \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a) Find a basis for W.
b) What is the dimension of W?
c) It is claimed that W is a "plane" in R^4. Do you agree? Give reasons for your answer.
d) It is claimed that the "plane" W can be described as the intersection of two 3-D regions S_1 and S_2 in R^4. The equations of S_1 and S_2 are:
$S_1 : x - u = 0$
$S_2 : ax + by + cz + du = 0$
where $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ is a generic point in R^4 and a, b, c, d are real constants.
Find one possible set of values for the constants $a, b, candd$.

MATH 293 SPRING 1996 PRELIM 3 # 4

2.5.31 Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & -1 & 2 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

a) Find a basis for the null space of A. What is the dimension of the null space of A?

b) Let$\mathbf{x} = (0, \frac{1}{2}, 1, 0)$. We know that $A\mathbf{x} = \mathbf{0}$. True or false:

i) $\mathbf{x}$ is a trivial solution to $A\mathbf{x} = \mathbf{0}$.

ii) $\mathbf{x}$ is in the solution space of $A\mathbf{x} = \mathbf{0}$.

iii) $\mathbf{x}$ is in the null space of A.

iv) $\{\mathbf{x}\}$ is a basis for the null space of A.

c) The vector

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is one vector in a basis for the column subspace of A. Find another vector $\vec{v}$ in a basis for the column subspace of A such that $\{\vec{v}, \vec{w}\}$ is linearly independent.

d) What is the rank of A? How do you know?

MATH 293 SPRING 1996 FINAL # 16

2.5.32 The vector space of all polynomials of degree six or less has dimension:

a) 5

b) 6

c) 7

d) 8

e) none of the above

MATH 293 SPRING 1996 FINAL # 21

2.5.33 A basis for the null space of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is

a) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

e) none of the above

MATH 293 SPRING 1996 FINAL # 23

2.5.34 Suppose A is a matrix with 6 columns and 4 rows. Which of the following must be true?

a) The null space of A has dimension ≥ 2 and the rank of A is 4.

b) The null space of A has dimension ≤ 4 and the rank of A is 2.

c) The null space of A has dimension ≤ 2 and the rank of A is ≤ 4.

d) The null space of A has dimension ≥ 2 and the rank of A is ≤ 4.

e) None of the above

MATH 294 SPRING 1997 FINAL # 2

2.5.35 (All parts are independent problems)

a) If the $\det A = 2$. Find the $\det A^{-1}, \det A^T$

b) From $PA = LU$ find a formula for A^{-1} in terms of P, L and U. Assume P, L, U, A are invertible $n \times n$ matrices.

c) Find the rank of matrix A.

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$$

d) Find a 2×2 matrix E such that for *every* 2×2 matrix A, the second row of EA is equal to the sum of the first two rows of A, e.g. if

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $EA = \begin{bmatrix} 1 & 2 \\ 3+1 & 4+2 \end{bmatrix}$

e) Write down a 2×2 matrix P which projects every vector onto the x_2 axis. Verify that $P^2 = P$.

MATH 294 SPRING 1997 FINAL # 7

2.5.36 Suppose A is a 6 row by 7 column matrix for which $nulA = Span\{\vec{x}_0\}$ for some $\vec{x}_0 \neq \vec{0}$ in $\Re^7$. Which of the following are always TRUE of A? (NO Justification is necessary.) Express your answer as e.g: TRUE: a,b,c,d; FALSE: e

a) The columns of A are linearly dependent.
b) The linear transformation $\vec{x} \to A\vec{x}$ is onto.
c) $A\vec{x} = \vec{0}$ has only the trivial solution.
d) The columns of A form a basis for $\Re^6$.
e) The columns of A span all of $\Re^6$.

MATH 294 FALL 1997 PRELIM 2 # 1

2.5.37 Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 2 & 0 & 2 \\ 4 & 3 & 1 & 2 \\ -2 & 0 & -2 & 2 \end{pmatrix}.$$

a) Find a basis for the null space N of A. What is the dimension of N?
b) Find a basis for the column space C of A. What is the dimension of C?
c) Find a basis for the row space R of A. What is the dimension of R?

MATH 294 FALL 1997 FINAL # 2

2.5.38 Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ -2 & -1 & 3 & 2 \end{pmatrix}$$

Find bases for the null space of A and the column space of A. What are the dimensions of these two vector spaces?

MATH 294 SPRING 1998 PRELIM 3 # 1

2.5.39 The matrix A is row equivalent to the matrix B:

$$A \equiv \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \equiv B$$

a) Find a basis for $RowA$, $ColA$, and $NulA$.
b) To what vector spaces do the vectors in $RowA$, $ColA$, and $NulA$ belong?
c) What is the rank of A?

MATH 294 SPRING 1998 FINAL # 3

2.5.40 Given that the matrix B is row equivalent to the matrix A where

$$A \equiv \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix} \text{ and } B \equiv \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find rank A and dim Null A.

b) Determine bases for Col A and Null A.

c) Determine a value of c so that the vector $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ c \end{bmatrix}$ is in Col A

d) For this value of c, write the general solution of $A\vec{x} = \vec{b}$.

MATH 294 FALL 1997 PRELIM 2 # 3

2.5.41 Let W be the subspace of $\Re^4$ defined as

$$W = span\left(\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 4 \end{pmatrix}\right).$$

a) Find a basis for W. What is the dimension of W?

b) It is claimed that W can be described as the intersection of two linear spaces S_1 and S_2 in $\Re^4$. The equations of S_1 and S_2 are

$$S_1 : x - y = 0$$

and

$$S_2 : ax + by + cz + dw = 0,$$

where a,b,c,d are real constants that must be determined. Find one possible set of values of a,b,c and d.

MATH 294 FALL 1997 PRELIM 3 # 1

2.5.42 Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

a) Find an orthofonal basis for the null space of A.

b) Find a basis for the orthogonal complement of $NulA$, i.e. find $(NulA)^{\perp}$.

MATH 294 FALL 1997 PRELIM 3 # 2

2.5.43 Let $A = [\vec{v}_1 \vec{v}_2]$ be a 1000×2 matrix, where $\vec{v}_1, \vec{v}_2$ are the columns of A. You aren't given A. Instead you are given only that

$$A^T A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Find an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ of the column space of A. Your formulas for $\vec{u}_1$ and $\vec{u}_2$ should be written as linear combinations of $\vec{v}_1, \vec{v}_2$. (Hint: what do the entries of the matrix $A^T A$ have to do with dot products?

MATH 294 FALL 1998 PRELIM 2 # 4

2.5.44 The reduced echelon form of the matrix $A = \begin{bmatrix} 3 & 3 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 1 & 0 & 1 & -2 \\ 0 & -3 & 2 & -1 \end{bmatrix}$ is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

a) What is the rank of A

b) What is the dimension of the column space of A?

c) What is the dimension of the null space of A?

d) Find a solution to $A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

e) What is the row space of A?

f) Would any of your answers above change if you changed A by randomly changing 3 of its entries in the 2nd, third, and fourth columns to different small integers and the corresponding reduced echelon form for B was presented? (yes?, no?, probably?, probably not?, ?)

MATH 294 FALLL 1998 FINAL # 5

2.5.45 Consider $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ -1 & 2 & 5 & 8 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The augmented matrix of this system is $\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ -1 & 2 & 5 & 8 & 1 \end{bmatrix}$ which is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

a) What are the rank of A and dim nul A? (Justify your answer.)
b) Find bases for col A, row A, and nul A.
c) What is the general solution $\vec{x}$ to $A\vec{x} = \vec{b}$ with the given A and $\vec{b}$?
d) Select another $\vec{b}$ for which the above system has a solution. Give the general solution for that $\vec{b}$.

MATH 294 SPRING 1999 PRELIM 2 # 4

2.5.46 Let A be a matrix where all you know is that it is 5×7 and has rank 3.

a) Define new matrices from A as follows:

- C has as columns a basis for Col A,
- M has as columns a basis for Nul A^T, and
- $T = [CM]$.

Is this enough information to find the size (number of rows and columns) of T?
i) if yes, find the number of rows and columns and justify your answer, or
ii) if no, explain what extra information is needed to find the size of T?

b) Are there any two non-zero vectors $\vec{u}$ and $\vec{v}$ for which:

- $\vec{u}$ is in Col A,
- $\vec{v}$ is in Nul A^T, ***and***
- $\vec{v}$ is a multiple of $\vec{u}$?

i) if yes, why?
ii) if no, why?, or

c) if it depends on information not given, what information? How would that information help?

MATH 294 SPRING 1999 PRELIM 2 # 1

2.5.47 **a)** What is the null space of $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$?

b) What is the column space of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & \pi \\ 3 & 4 & \sqrt{2} \end{bmatrix}$?

d) Are the column of $A = \begin{bmatrix} 1 & 2 & \pi \\ 3 & 4 & \sqrt{2} \\ 3 & 4 & \sqrt{2} \end{bmatrix}$ linearly independent (hint: no long row reductions are needed)?

e) What is the row space of $A = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$?

MATH 293 SUMMER 1992 PRELIM 7/21 # 5

2.5.48 Consider the space P of all polynomials of degree ≤ 3 of the type $\{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3\}$ for all scalars a_0, a_1, a_2, a_3 and $0 \leq t \leq 1$. Now consider a subspace W of P where, for any $p(t) \in W$, we also have

$$\int_0^1 p(t)dt = 0$$

$$\left.\frac{dp}{dt}\right|_{t=0} = 0$$

a) Find a basis for W.
b) What is the dimension of W?

UNKNOWN UNKNOWN UNKNOWN # ?

2.5.49 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

a) Find the vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that a solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ exists.
b) Find a basis for the column space $\mathcal{R}(A)$ of A.
c) It is claimed that $\mathcal{R}(A)$ is a plane on $\Re^3$. If you agree, find a vector $\vec{n}$ in $\Re^3$ that is normal to this plane. Check your answer.
d) Show that $\vec{n}$ is perpendicular to each of the columns of A. Explain carefully why this is true.

MATH 294 FALL 1997 PRELIM 3 # 1 PRACTICE
2.5.50 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & -1 & -2 \end{bmatrix}$$

a) Find a basis for the column space C of A. What is the dimension of C?
b) Find a basis for the column space N of A. What is the dimension of N?
c) Let $W = span \left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right)$. Is W orthogonal to N? Please justify your answer by showing your work.

MATH 293 SPRING ? PRELIM 2 # 1
2.5.51 a) Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 2 & 7 & 8 \\ 2 & 0 & 2 & 4 \end{bmatrix}$$

b) Find a basis for the column space of A in (a).

MATH 293 SPRING ? PRELIM 2 # 2
2.5.52 a) If A and B are 4×4 matrices such that

$$AB = \begin{pmatrix} 2 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

show that the column space of A is at least three dimensional.

b) Find A^{-1} if $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

MATH 293 SPRING ? FINAL # 2
2.5.53 a) Find a basis for the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

b) Find the rank of A and a basis for its column space, noting that $A = A^T$.
c) Construct an orthonormal basis for the row space of A.

MATH 293 SPRING ? FINAL # 6

2.5.54 Give a definition for addition and for scalar multiplication which will turn the set of all pairs $(\vec{u}, \vec{v})$ of vectors, for $\vec{u}, \vec{v}$ in V_2, into a vector space V.

a) What is the zero vector of V?

b) What is the dimension of V?

c) What is a basis for V?

MATH 293 SPRING ? FINAL # 3

2.5.55 **a**) Give all solutions of the following system in vector form.

$$\begin{aligned} 6x_1 + 4x_3 &= 1 \\ 5x_1 - x_2 + 5x_3 &= -1 \\ x_1 + 3x_3 &= 2 \end{aligned}$$

b) What is the null space of the matrix of coefficients of the unknowns in a)?

MATH 293 SPRING ? FINAL # 4

2.5.56 Let W be the subspace of V_4 spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 3 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} -8 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

a) Find the dimension and a basis for W.

b) Find an orthogonal basis for W.

MATH 293 UNKNOWN FINAL # 5

2.5.57 **a**) Let A be an $n \times n$ matrix. Show that if $A\vec{x} = \vec{b}$ has a solution then $\vec{b}$ is a linear combination of the column vectors of A.

b) Let A be a 4×4 matrix whose column space is the span of vectors $\vec{v} = (v_1, v_2, v_3, v_4)^T$, satisfying $v_1 - 2v_2 + v_3 - v_4 = 0$. Let $\vec{b} = (1, b_2, b_3, 0)^T$. Find all values of b_2, b_3 for which the matrix equation $A\vec{x} = \vec{b}$ has a solution.